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THE TANGENT GROUPS OF A LIE GROUP AND GAUGE INVARIANCE IN CLASSICAL MECHANICS

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Групи, дотичні до групи Лі, та калібрувальна інваріянтність у класичній механіці

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Анотація. Вивчається група Лі, елементи та групові операції якої є джетовими продовженнями елементів та групових операцій іншої групи Лі. З допомогою контактних перетворень означена дія такої групи на диференційовному многовиді та дотичній до нього в'язці. Виявилось, що лагранжіян, симетричний відносно перетворень з дотичної групи Лі, описує динамічну систему з в'язями першого класу.

The tangent groups of a Lie group and gauge invariance in classical mechanics

Y.Yaremko

Abstract. A tangent Lie group with elements and group operations which are (jet) prolongations of those corresponding to another Lie group is examined. An action of such extended Lie group on differentiable manifold and its tangent bundle is defined by using contact transformations. It turned out that a tangent Lie symmetrical Lagrangian describes the dynamical system with first-class constraints. The infinitesimal symmetries originated from a tangent Lie group of symmetries are investigated.

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Introduction

While examining the structure of Euler-Lagrange equations originated from degenerate Lagrangian, Gitman and Tyutin [1] proved that a necessary and sufficient condition of the existence of relationships involving motion equations is the invariance of action integral with respect to a coordinate transformation which is specified by some time-dependent parameters (gauge transformation). The number of these parameters is equal to the number of above relationships. In ref.[2] it was showed that this transformation is an invertible contact transformation. The adjective "contact" means that it leave an infinite Cartan distribution invariant [3]. Such a coordinate substitution is discussed on both Lagrangian and Hamiltonian levels in ref.[2]. This is an essential prerequisite to study the situation in frame of the canonical formalism for higher-derivative theories by using of the Dirac-Bergmann theory of constraints.

In ref.[4] an action of a Lie group G on the total space of a trivial bundle $(\mathcal{Q},\pi,I\!R)$ was considered. The vector fields generated by an action of G on $\mathcal{Q}$ are lifted to the first jet manifold $J^1\pi$. Non-autonomous Lagrangian $L : J^1\pi \to I\!R$ is supposed to be an invariant under the action of G on $J^1\pi$. It means that L satisfies the system of first-order differential equations in partial derivatives originated from the requirement of invariance of action integral with respect to the infinitesimal transformations of $J^1\pi$. Further the group parameters are postulated to be the time-dependent functions, so that an infinitesimal transformation becomes a gauge transformation. In ref.[4] the method was developed which permits to construct gauge invariant Lagrangian function starting with "standard" $G-$invariant Lagrangian mentioned above. The former has to satisfy the system with double number of differential equations because the additional group parameters, namely first-order derivatives of the original ones, are appeared. Gauge invariance of the theory is achieved due to additional dynamical variables. More exactly, an original configuration manifold is substituted by the new manifold with higher dimension.

Therefore, if we study the symmetry of dynamical system with first-class constraints, it is better to suppose that group parameters are (time-dependent) gauge variables. According to ref.[5], if we have a Lie group G with multiplication map $\mu : G\times G \to G$, then its tangent bundle TG is also a Lie group with multiplication $T\mu$ (see also ref.[3, pg.115]). Such a group is called the tangent group of an original Lie group. The tangent bundle of order k, T^kG of G, is endowed with a group structure too [5]. In this manner the left action of G on differentiable manifold,

say Q, may be lifted to an action of tangent group on tangent bundle of Q. Thus, several powerful tools from Lie group theory can be applied to above symmetry problem.

In ref.[6] a Hamiltonian system having a Lie group G as configuration manifold is considered. Phase manifold is $G\times g^*$ where g^* denotes the dual space to Lie algebra g of group G. The authors built $G-$invariant dynamics; the explicit expression for momentum mapping [7,8] is obtained. In Section 5 such a dynamical system will be examined in the framework of Lagrangian formalism.

The present paper is organized as follows. In Section 1 we recall in detail the theory of tangent groups necessary to make the text self-contained. The approach different from viewpoint given in ref.[5] is elaborated. In Section 2 we define the actions of tangent groups on differentiable manifold and its tangent bundle by using contact transformations. In Section 3 we study a tangent Lie group of symmetries of a Lagrangian system and its Hamiltonian counterpart. In Section 4 we show how the existence of a tangent Lie group of symmetries induces infinitesimal symmetries. Finally, in Appendix A we apply the results to tangent Euclidean group of space translations and space rotations.

1. Second-order tangent group of a Lie group

In this Section we construct a Lie group with elements and group operations which are jet prolongations of those corresponding to another Lie group. We obtain the explicit expressions for the constants of structure of new group by using the Maurer-Cartan equation. Left-invariant vector fields are classified in the usual framework of theory of lifts of vector fields to tangent bundles.

Let $(\mathcal{G},\gamma,M)$ be a bundle with one-dimensional base M. The total space of γ is $(\mathcal{R}+1)$-dimensional manifold $\mathcal{G}$. We deal with a local trivialisation of γ around $t\in I$ where diffeomorphism $o_t : \gamma^{-1}(I) \to I\times G$ is defined. Here $I\subset M$ is an open interval, and a typical fibre G of γ is an $\mathcal{R}$-dimensional Lie group.

In adapted coordinate system (U,g), which is constructed from local trivialisations using coordinate systems on the base space and the typical fibre, projection $\gamma : \mathcal{G} \to M$ relates the point $(t,g_\alpha)\in U\subset\gamma^{-1}(I)$, with the point t in the time interval I. Greek symbols $\nu_\alpha(a) := (g_\alpha\circ\nu)(t), \lambda_\beta(b) := (g_\alpha\circ\lambda)(t)\,, \nu,\lambda\in\Gamma_I(\gamma)$ etc. denote the local coordinates of group elements a,b etc. in manifold G. These coordinates are chosen so that $\varepsilon_\kappa(e)=0$, $\kappa=1,\ldots,\mathcal{R} := \overline{1,\mathcal{R}}$, for identity element e. We also use the induced coordinate systems (U^1,g^1) and (U^2,g^2) on the 1-st and

2-nd order jet bundles γ_1 and γ_2, respectively. Greek indices are meant to run from 1 to $\mathcal{R}$ throughout the paper; the summation convention is used for dummy indices.

Starting with a group multiplication $\varphi : G \times G \to G$, we define a bundle morphism [3] from fibred product bundle $\gamma \times_M \gamma$ to bundle γ by the map $\bar{\varphi} : \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$. We suppose that the projection of $\bar{\varphi}$ is an identity map id_M, so that locally this map may be written as

$$\begin{aligned} t &= t\,, \\ \eta_\alpha &= \varphi_\alpha(\lambda_\beta(b), \nu_\gamma(a))\,, \end{aligned} \tag{1.1}$$

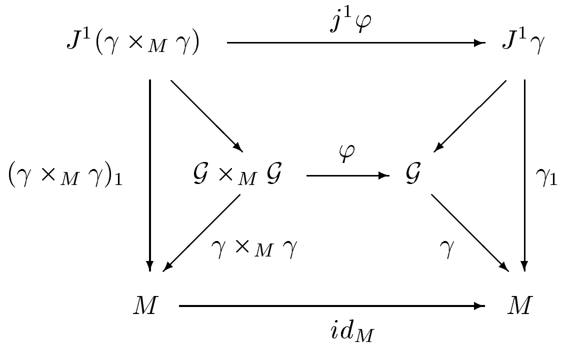
where smooth function $\varphi(b,a)$ determines the group multiplication. To find the "multiplication law" for the first-order derivative coordinates, we construct the first prolongation of a bundle morphism $(\bar{\varphi}, id_M)$. Following ref.[3], we obtain the map $j^1\bar{\varphi} : J^1\gamma \times_M \gamma \to J^1\gamma$ defined by

$$\begin{aligned} j^1\bar{\varphi}(j^1_t(\lambda \times_M \nu)) &= j^1_t(\bar{\varphi} \circ \lambda \times_M \nu) \\ &= j^1_t \eta\,. \end{aligned} \tag{1.2}$$

Here symbol $\lambda \times_M \nu$ denotes a fibre product of the local sections $\lambda, \nu \in \Gamma_I(\gamma)$ defined by $(\lambda \times_M \nu)(t) = (\lambda(t), \nu(t))$. The resultant section $\eta \in \Gamma_I(\gamma)$ is equal to $\bar{\varphi}(\lambda \times_M \nu)$. (As usual [3], a local section of γ with domain I, say $\eta : I \to \mathcal{G}$, is an inverse to γ map, $\gamma \circ \eta = id_I$, given by $t \mapsto (t, \eta_\alpha(t))$, where $\eta_\alpha = g_\alpha \circ \eta$, and $j^1_t\eta$ is the 1-st jet of this section at a point $t \in I$.) We obtain $\mathcal{R}$ expressions in local coordinates

$$\begin{aligned} \eta^1_\alpha &= d_T \varphi_\alpha(\lambda_\beta(b), \nu_\gamma(a)) \\ &= \lambda^1_\beta \frac{\partial \varphi_\alpha(b,a)}{\partial \lambda_\beta(b)} + \nu^1_\gamma \frac{\partial \varphi_\alpha(b,a)}{\partial \nu_\gamma(a)}\,, \end{aligned} \tag{1.3}$$

in addition to the relations (1.1) (d_T is the Tulczyjew differential operator [9]). The following commutative diagram summarizes the situation:

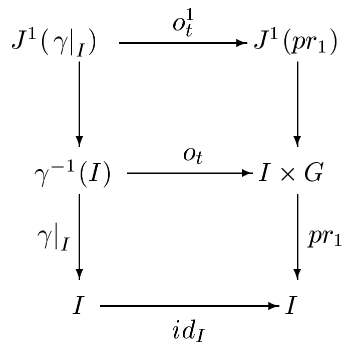


Similarly we prolong a bundle morphism $(\bar{\phi}, id_M) : \gamma \to \gamma$ which is based on the group inversion $\phi : G \to G$.

Let's sum it up. A bundle morphism may be described as a map from the total space of one bundle to the total space of another bundle which does not mix the fibers. It means that any fibre of one bundle is mapped into a fibre of another bundle. We start with a bundle $(\mathcal{G}, \gamma, M)$ whose typical fibre G is a Lie group. A typical fibre of the fibred product bundle $\gamma \times_M \gamma$ is the Cartesian product of the typical fibre of γ with itself, i.e. $G \times G$. Since first jet bundle $(J^1(\gamma \times_M \gamma), (\gamma \times_M \gamma)_1, M)$ is then nothing but the fibred product bundle $\gamma_1 \times_M \gamma_1$ of a bundle $(J^1\gamma, \gamma_1, M)$ with itself [3], the typical fibre of γ_1 is endowed with a group structure. Group operations of multiplication and inversion are the restrictions on this typical fibre of the first prolongations $j^1\bar{\varphi}$ and $j^1\bar{\phi}$ of the bundle morphisms $\bar{\varphi}$ and $\bar{\phi}$, respectively.

If γ is the trivial bundle $(I\!\!R \times G, pr_1, I\!\!R)$ then tangent manifold TG is a typical fibre of γ_1. It is also Lie group with multiplication $T\varphi$ and inversion $T\phi$. Following ref.[5], we say then TG is the tangent group of a Lie group G.

According to ref.[3], the first prolongation $j^1(o_t, id_I)$ is the local trivialisation of γ_1 around $t \in I$:



For open interval $I \subset M$ we have $J^1(\gamma|_I) \cong \gamma_1^{-1}(I)$ (see ref.[3, Lemma 4.1.14]). If I contains $t = 0$ the first jet manifold $J^1(pr_1)$ is diffeomorphic to $I \times TG$. On the assumption of the typical fibres of γ_1 are diffeomorphic, we construct the local trivialisation $o^1_t : \gamma_1^{-1}(I) \to I \times TG$. Therefore, if g_α are coordinate functions on G, then $(g_\alpha, {g_\alpha}^1)$ is the coordinate system on a typical fibre TG. We denote j^1a, j^1b etc. the elements of TG with coordinates $(\nu_a lpha(t), \nu_a lpha^1(t))$ and $(\lambda_\beta(t), \lambda^1_\beta(t))$, respectively. As it follows from eqs.(1.3), the identity element j^1e has zero-valued derivative coordinates: ${\varepsilon_\kappa}^1(e) = 0$ for all $\kappa = \overline{1, \mathcal{R}}$.

In analogy with $j^1\bar{\varphi}$ we construct the second order prolongation of bundle morphism $(\bar{\varphi}, id_M)$. Following ref.[3], we obtain the map $j^2\bar{\varphi} : J^2(\gamma \times_M \gamma) \to J^2\gamma$ defined by $j^2\bar{\varphi}(j^2_t(\lambda \times_M \nu)) = j^2_t\bar{\varphi}(\lambda \times_M \nu)$. On the local level we have the relations $\eta^2_\alpha = d^2_T\varphi_\alpha(b,a)$ together with eqs.(1.1) and (1.3). Thus we construct the second order (jet) prolongation of a Lie group G over M. Its multiplication $j^2\bar{\varphi}|T^2G$ and inversion $j^2\bar{\phi}|T^2G$ are derived from corresponding group operations of an original Lie group G. The identity j^2e of a Lie group T^2G has zero-valued coordinates.

Now we consider the embedding $\iota_1 : \mathcal{G} \to J^1\gamma$, locally given by $(t,\nu_\alpha) \mapsto (t,\nu_\alpha,0)$. The submanifold $\iota_1(\mathcal{G}) \subset J^1\gamma$ is a slice [10] of the coordinate system (U^1, g^1). It is interesting to study a relative inclusion $i_1 : G \to TG$ where G and TG are meant as the "simultaneous" fibers of $\mathcal{G}$ and $J^1\gamma$, respectively. One easily proves that i_1 is a group homomorphism and (G, i_1) is a closed subgroup [10] of a Lie group TG. Similarly we construct a closed subgroup $i_2(G) \subset T^2G$ where the inclusion map $i_2 : G \to T^2G$ is related with the embedding

$$\begin{aligned} \iota_2 \quad : \quad & \mathcal{G} \to J^2\gamma\,, \\ & (t,\nu_\alpha) \mapsto (t,\nu_\alpha,0,0)\,. \end{aligned} \tag{1.4}$$

Note that target projections $\gamma_{1,0} : J^1\gamma \to \mathcal{G}$ and $\gamma_{2,0} : J^2\gamma \to \mathcal{G}$ induce the group homomorphisms $\tau_{1,0} : TG \to G$ and $\tau_{2,0} : T^2G \to G$, respectively. First-jet projection $\gamma_{2,1} : J^2\gamma \to J^1\gamma$ induces homomorphism $\tau_{2,1}$ from group T^2G to group TG.

Therefore, an original Lie group is a Lie subgroup and a submanifold of its own first- and second-order tangent groups [5]. (More exactly, we consider the slices $i_1(G) \subset TG$ and $i_2(G) \subset T^2G$ on which all the derivative coordinates are equal to zero.) Moreover, the constants of structure of these tangent Lie groups are determined by the structure constants of G. To demonstrate it we study the space $\mathcal{X}_L(T^2G)$ of all left invariant vector fields on T^2G and its dual space $\mathcal{X}^*_L(T^2G)$.

Taking into account an exclusive role of Tulczyjew differential operator in prolongation algorithm, we deal with the dual space $\mathcal{X}^*_L(T^2G)$. We write the local expressions for canonical left invariant one-forms [10,7,11] which constitute the basis for $\mathcal{X}^*_L(T^2G)$ at a point j^2a:

$$\theta^{\gamma k} = \left[\frac{\partial}{\partial {\nu_\beta}^j(a)} d^k_T\varphi_\gamma(b,a)\right]_{b=a^{-1}} d{\nu_\beta}^j\,. \tag{1.5}$$

The exterior derivatives of the $\theta^{\gamma k}$ are given by the Maurer-Cartan equation

$$d\theta^{\gamma k} = -\frac{1}{2}C^\Gamma_{\mathrm{AB}}\theta^{\alpha i}\wedge\theta^{\beta j}\,. \tag{1.6}$$

We use multi-index notation in structure constants $\{C^\Gamma_{\mathrm{AB}}\}$ of T^2G, where multi-indices A, B and Γ are the 2-tuples of natural numbers, e.g. $\mathrm{A} = (\alpha i)$. Small roman indices run from 0 to 2. Particularly, for subgroup $i_2(G) \subset T^2G$ we have

$$d\theta^\gamma = -\frac{1}{2}c^\gamma_{\alpha\beta}\theta^\alpha\wedge\theta^\beta\,, \tag{1.7}$$

where $\{c^\gamma_{\alpha\beta}\}$ are structure constants of original Lie group G (zero-valued roman indices are omitted). Left-invariant one-forms θ^α are given by eqs.(1.5) if integer k is equal to zero.

Tulczyjew operator d_T is the derivation of type d_* of zero degree which acts on the 0-forms as a total time derivative. Having used the commutation $d \circ d_T = d_T \circ d$ and the expressions $d_T{\nu_\beta}^i = {\nu_\beta}^{i+1}$, after short calculations we establish the following relations between "higher-order" one-forms (1.5) and original ones:

$$\theta^{\gamma 1} = d_T\theta^\gamma\,, \qquad \theta^{\gamma 2} = d^2_T\theta^\gamma\,. \tag{1.8}$$

Thanks to commutation of Tulczyjew operator with an exterior derivative and positively signed Leibniz' rule for wedge product [9] we arrive at

$$\begin{aligned} d\theta^{\gamma 1} &= -\frac{1}{2}c^\gamma_{\alpha\beta}\theta^{\alpha 1}\wedge\theta^\beta - \frac{1}{2}c^\gamma_{\alpha\beta}\theta^\alpha\wedge\theta^{\beta 1}\,, \\ d\theta^{\gamma 2} &= -\frac{1}{2}c^\gamma_{\alpha\beta}\theta^{\alpha 2}\wedge\theta^\beta - c^\gamma_{\alpha\beta}\theta^{\alpha 1}\wedge\theta^{\beta 1} - \frac{1}{2}c^\gamma_{\alpha\beta}\theta^\alpha\wedge\theta^{\beta 2}\,. \end{aligned} \tag{1.9}$$

When comparing these expressions with Maurer-Cartan equations (1.6) we deduce the constants of structure $\{C^\Gamma_{\mathrm{AB}}\}$. It is convenient to write them as the following block matrices:

$$\hat{C}^{(\gamma 0)} = \begin{bmatrix} \hat{c}^\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{C}^{(\gamma 1)} = \begin{bmatrix} 0 & \hat{c}^\gamma & 0 \\ \hat{c}^\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\hat{C}^{(\gamma 2)} = \begin{bmatrix} 0 & 0 & \hat{c}^\gamma \\ 0 & 2\hat{c}^\gamma & 0 \\ \hat{c}^\gamma & 0 & 0 \end{bmatrix}. \tag{1.10}$$

Here symbol $\hat{c}^\gamma$ denotes the skew-symmetric matrix $\|c^\gamma_{\alpha\beta}\|$ with fixed integer γ.

Similarly we obtain the structure constants of a Lie group TG:

$$\hat{C}^{(\gamma 0)} = \begin{bmatrix} \hat{c}^\gamma & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{C}^{(\gamma 1)} = \begin{bmatrix} 0 & \hat{c}^\gamma \\ \hat{c}^\gamma & 0 \end{bmatrix}. \tag{1.11}$$

The basis for $\mathcal{X}_L(T^2G)$ consists of the left invariant vector fields [10,7], say $X_B^{(2)}$, locally given by

$$\begin{aligned} X^{(2)}_{(\beta 0)} &= L^\alpha_\beta \frac{\partial}{\partial \nu_\alpha} + d_T(L^\alpha_\beta)\frac{\partial}{\partial {\nu_\alpha}^1} + d_T^2(L^\alpha_\beta)\frac{\partial}{\partial {\nu_\alpha}^2}\,, \\ X^{(2)}_{(\beta 1)} &= L^\alpha_\beta \frac{\partial}{\partial {\nu_\alpha}^1} + 2d_T(L^\alpha_\beta)\frac{\partial}{\partial {\nu_\alpha}^2}\,, \\ X^{(2)}_{(\beta 2)} &= L^\alpha_\beta \frac{\partial}{\partial {\nu_\alpha}^2}\,. \end{aligned} \tag{1.12}$$

Here $L^\alpha_\beta(\nu(a))$ are the components of the left invariant vector fields X_β which form the basis for $\mathcal{X}_L(G)$.

Vector field $X^{(2)}_{(\beta 0)}$ is the 2-lift of corresponding one X_β to tangent bundle T^2G, i.e. $X^{(2)}_{(\beta 0)} = X^{(2,2)}_\beta$ (see ref.[12]). The former belongs to the basis of a space $\mathcal{X}_L(i_2(G)) \subset \mathcal{X}_L(T^2G)$. The others $X^{(2)}_{(\beta 1)}$ and $X^{(2)}_{(\beta 2)}$ are intimately connected with the 1-st and 0-th lifts [12] of X_β to T^2G, respectively. Namely, we have $X^{(2)}_{(\beta 1)} = J_1 X^{(2,2)}_\beta$ and $X^{(2)}_{(\beta 2)} = (1/2)(J_1)^2 X^{(2,2)}_\beta$ where J_1 is the canonical almost tangent structure [13] of order 2 on T^2G.

Let $X^{(1)}_{(\beta i)}$, $i = 0, 1$ be the canonical left-invariant vector fields on $\mathcal{X}_L(TG)$. If $\tau^2_1 : T^2G \to TG$ is the canonical projection, then $X^{(2)}_{(\beta i)}$ and $X^{(1)}_{(\beta i)}$ are τ^2_1–related, i.e. $T\tau^2_1(X^{(2)}_{(\beta i)}) = X^{(1)}_{(\beta i)}$. Each of homomorphism of groups, mentioned in this Section, corresponds the Lie algebra homomorphism which describes its effect on left invariant vector fields, as well as the mapping which relates the dual algebras. The former are then nothing but the differential of originating group homomorphism and the latter is precisely the transpose of this differential [10].

2. An action of the tangent Lie group on a smooth manifold

In this Section we define an action of the first prolongation of a Lie group G on an $\mathcal{N}$-dimensional differentiable manifold Q. We lift it to the action of the second prolongation of this group on tangent bundle TQ.

When considering the group parameters as the constants, an action of a Lie group G on Q, say $\Phi : G \times Q \to Q$, lifts to an action $\Phi^1 : G \times TQ \to TQ$ of G on TQ as follows [7,8,12]: $(\Phi^1)_a : TQ \to TQ$ where $(\Phi^1)_a = T\Phi_a$ for any fixed $a \in G$. Treatment of group parameters via the time-dependent variables makes the notion of lift of a group action quite different from mentioned above. We construct it by analogy with the algorithm of group prolongation developed in previous Section.

Let us consider a bundle $(\mathcal{Q}, \pi, M)$ whose typical fibre is the above smooth manifold Q. It is a bundle over the same base space M as a bundle γ. Let (I', Q, o'_t) be a local trivialisation of π around $t \in I' \subset M$. We denote (V, q) an adapted coordinate system which is constructed from local trivialisations. So projection $\pi : \mathcal{Q} \to M$ relates the point $(t, q_a) \in V \subset \pi^{-1}(I')$ with point t in the time interval I'. Small Latin indices a, b, c are meant to run from 1 to $\mathcal{N}$ throughout the paper.

We construct the fibred product bundle $\gamma \times_M \pi$ where the total space $\mathcal{G} \times_M \mathcal{Q}$ consists of the "simultaneous" points of the Cartesian product $\mathcal{G} \times \mathcal{Q}$. Let $x : I' \to \mathcal{Q}$ be a local section of π with domain I' given by $t \mapsto (t, x_a(t))$, where $x_a = q_a \circ x$. The fibre product coordinates may be defined in the following manner[3]: if (t, g_α) is an adapted coordinate system on $U \subset \mathcal{G}$ and (t, q_a) is an adapted coordinate system on $V \subset \mathcal{Q}$ where $\gamma(U) \cap \pi(V) \neq \emptyset$, then we may take (t, g_α, q_a) as an adapted coordinate system on

$$(U \cap \gamma^{-1}(\pi(V))) \times_M (V \cap \pi^{-1}(\gamma(U))) \subset \mathcal{G} \times_M \mathcal{Q}\,. \tag{2.1}$$

Since $\gamma \times_M \pi$ has the properties of a bundle whose typical fibre is the Cartesian product of the typical fibres of γ and π, we may define a bundle morphism $(\Phi, id_M) : \gamma \times_M \pi \to \pi$ by using of the map $\Phi : G \times Q \to Q$. We determine an action of the tangent group TG of a Lie group G on the typical fibre TQ of first-jet bundle $(J^1\pi, \pi_1, M)$ by means of the first prolongation of (Φ, id_M), namely $(j^1\Phi, id_M) : \gamma_1 \times_M \pi_1 \to \pi_1$. In local coordinates the map

$$\begin{aligned} j^1\Phi \;:\; & J^1\gamma \times_M J^1\pi \to J^1\pi\,, \\ & (j^1_t\nu, j^1_t y) \mapsto j^1_t x\,, \end{aligned} \tag{2.2}$$

is written as

$$\begin{aligned} t &= t\,, \\ x_a &= f_a(\nu_\alpha, y_b)\,, \\ {x_a}^1 &= d_T f_a\,. \end{aligned} \tag{2.3}$$

We use the induced coordinate system (V^1, q^1) on the 1-st order jet bundle π_1. So, $j^1_t y$ and $j^1_t x$ are the 1-jets of the sections $y, x \in \Gamma_{I'}(\pi)$ at a point $t \in I'$. Eqs.(2.3), excluding the identity for the base variable, describe on the local level the desired map $T\Phi : TG \times TQ \to TQ$. We call that $T\Phi$ is a total first-order lift of an action $\Phi : G \times Q \to Q$ over base M, or 1-st $M-$lift of Φ in short. The word "total" has a technical meaning which is elucidated below.

For fixed 1-st jet $j^1_t\nu$, $\nu \in \Gamma_I(\gamma)$, the map $(T\Phi)_{j^1 a}$ is a transformation of manifold TQ determined by the element $j^1 a \in TG$.

The action of TG on TQ induces a Lie algebra homomorphism of $Lie(TG) := T_{j^1 e}(TG)$ into vector space $\mathcal{X}(TQ)$. To each vector field $\xi^{(1)}_{(\alpha i)} := X^{(1)}_{\alpha i}(j^1 e)$, $i = 0, 1$, we assign the vector field $Y^{(1)}_{(\alpha i)}$ on TQ:

$$Y^{(1)}_{(\alpha 0)} = Y^b_\alpha \frac{\partial}{\partial y_b} + d_T(Y^b_\alpha)\frac{\partial}{\partial {y_b}^1} \quad (a)\,, \quad Y^{(1)}_{(\alpha 1)} = Y^b_\alpha \frac{\partial}{\partial {y_b}^1} \quad (b)\,. \qquad (2.4)$$

Symbol Y^b_α denotes the component of the fundamental vector field [5] corresponding to $\xi_\alpha \in Lie(G)$. Actually $\{Y^{(1)}_{(\alpha i)} | \alpha = \overline{1,\mathcal{R}}; i = 0, 1\}$ is a Lie subalgebra of the set $\mathcal{X}(TQ)$ of all vector fields on TQ.

Let us compare these results with standard situation where coordinates of the group elements are meant to be the constants. In such a case the transformations of TQ are generated by fundamental vector fields which are complete lifts of their prototypes, acting on Q [12]. Since $\dim TG = 2 \dim G$, we have double number of infinitesimal generators, namely $Y^{(1)}_{(\alpha 0)}$ and $Y^{(1)}_{(\alpha 1)}$, which are then nothing but the complete and vertical lifts [11] of the original one Y_α. Therefore, it is reasonable to say that we deal with the *total* 1-st lift of an action of G on Q.

We may lift an action of TG on Q to the action T^2G on TQ in similar circumstances: if $(\Phi^{(1,0)}, id_M)$ is a bundle morphism from $\gamma_1 \times_M \pi$ to π and $(j^1\Phi^{(1,0)}, id_M)$ is its first prolongation, then we restrict the latter to the "simultaneous" fibres. We realize this scheme for the base which is a real line $I\!R$. It means that global trivialisation of γ is allowed, so that diffeomorphism $o : \mathcal{G} \to I\!R \times G$ may be defined. Naturally, we suppose that π is a trivial bundle too.

We define now a C^∞ map

$$\begin{aligned} \Phi^{(1,0)} \quad : \quad & TG \times Q \to Q\,, \\ & (j^1_0\nu, y(0)) \mapsto x(0)\,, \end{aligned} \qquad (2.5)$$

which is an action of a Lie group TG on manifold Q on the left [11,7]. The bracketed and separated by comma integers $(1, 0)$ up to capital letter Φ are associated with the orders of tangent bundles over G and Q, respectively. The curve $y : I\!R \to Q$ runs across a point $y(0) \in V$ with coordinates $\{y_a | a = \overline{1,\mathcal{N}}\}$ and the curve $x : I\!R \to Q$ passes through a point $x(0) \in V$ with coordinates $\{x_a | a = \overline{1,\mathcal{N}}\}$. Locally the map (2.5) may be written as:

$$x_a = f_a(\nu_\alpha, {\nu_\alpha}^1, y_b)\,. \qquad (2.6)$$

An action of TG on Q induces a Lie algebra homomorphism of the Lie algebra $Lie(TG)$ into vector space $\mathcal{X}(Q)$ [7]. To each vector field $\xi^{(1)}_{(\alpha i)}$, $i = 0, 1$, we assign the following fundamental vector field on Q:

$$Y^{(1,0)}_{(\alpha i)} = \left.\frac{\partial f_a({\nu_\alpha}^i(j^1 a), y_b)}{\partial {\nu_\alpha}^i}\right|_{j^1 a = j^1 e} \frac{\partial}{\partial y_a}\,. \qquad (2.7)$$

Each of them is the infinitesimal generator of an 1-parameter group of transformations of Q.

The map (2.5) lifts to the left action $\Phi^{(2,1)} : T^2G \times TQ \to TQ$ of group T^2G on tangent bundle TQ by composition of the 1-st jet prolongation $T\Phi^{(1,0)} : T(TG \times Q) \to TQ$ with the canonical embedding

$$\begin{aligned} i_{1,0} \quad : \quad & T^2G \times TQ \to T(TG \times Q)\,, \\ & (j^2_0\nu, j^1_0 y) \mapsto j^1_0(j^1\nu, y)\,. \end{aligned} \qquad (2.8)$$

The first-jet $j^1_0 y$ is represented in TV by $(y_b, {y_b}^1)$ where $y_b = (q_b \circ y)(0)$, and ${y_b}^1 = (dq_b \circ y/dt)(0)$. In local coordinates we obtain the following transformational law for first-order derivative coordinates: ${x_a}^1 = d_T f_a$, where ${x_a}^1 = (dq_a \circ x/dt)(0)$.

The fundamental vector fields which correspond to $\xi^{(2)}_{(\alpha i)} \in Lie(T^2G)$ may be expressed in terms of both complete and vertical lifts [11] of vector fields (2.7):

$$\begin{aligned} Y^{(2,1)}_{(\alpha 0)} &= (Y^{(1,0)}_{(\alpha 0)})^c\,, \quad Y^{(2,1)}_{(\alpha 1)} = (Y^{(1,0)}_{(\alpha 1)})^c + (Y^{(1,0)}_{(\alpha 0)})^v\,, \\ Y^{(2,1)}_{(\alpha 2)} &= (Y^{(1,0)}_{(\alpha 1)})^v\,. \end{aligned} \qquad (2.9)$$

In general, an invertible first-order contact transformation, locally given by

$$\begin{aligned} x_a &= f_a(\nu_\beta, {\nu_\beta}^1, y_b)\,, \quad \eta_\gamma = \varphi_\gamma(\lambda_\beta, \nu_\alpha)\,; \\ {x_a}^s &= d^s_T f_a\,, \quad \eta^s_\gamma = d^s_T\varphi_\gamma\,, \quad s > 0\,, \end{aligned} \qquad (2.10)$$

can be derived from differentiable maps (2.5) and (1.1) by using of the algorithm developed in ref.[2]. (The coordinates λ^s_β are meant to be the constants for all values of non-negative index s.) In this way we avoid the need to prolong a Lie group G up to order higher than 2. Such a generalization allows to use the results of ref.[2] which are concerned with the effect of a change of variables of type (2.10) on the dynamics of Lagrangian system.

3. Tangent Lie symmetries and presymplectic mechanical systems

In this Section we consider a tangent Lie group of symmetries of the autonomous Lagrangian systems. We find two kinds of Lagrangian functions which are invariant under the action of tangent group. Dynamical systems with first-class constraints are obtained in both cases.

Let $L : TQ \to I\!\!R$ be a Lagrangian function, α_L on TQ the Poincaré-Cartan 1-form and E_L on TQ the energy function associated with L defined, respectively, by

$$\alpha_L = \sum_{a=1}^{\mathcal{N}} \hat{p}_a dx_a \, , \qquad E_L = \sum_{a=1}^{\mathcal{N}} \hat{p}_a {x_a}^1 - L \, , \tag{3.1}$$

where $\hat{p}_a = \partial L/\partial {x_a}^1$ are original momenta. Now we study the effect of transformation $T\Phi : TG \times TQ \to TQ$, locally given by eqs.(2.3), on dynamics of this Lagrangian system.

First of all we examine the situation where L is not invariant under the action $T\Phi$. Having carried out the transformation (2.3) in Lagrangian L, we construct the Lagrangian function $\tilde{L} : T(G \times Q) \to I\!\!R$:

$$\tilde{L}(\nu_\beta, {\nu_\beta}^1, y_b, {y_b}^1) = L(f_a(\nu_\beta, y_b), d_T f_a) \, . \tag{3.2}$$

In this paper we indicate the initial Lagrangian function, motion equations etc., by the adjective "original" and those transformed by coordinate substitution of type (2.3) or (2.10) by the adjective "new". New objects will be marked by "tilde". We call "motion" the solution of motion equations.

It can be easily proven that there are the following relations between the expressions for original and new Euler-Lagrange equations:

$$\left[\frac{\delta \tilde{S}}{\delta y_b}\right] = \frac{\partial f_a}{\partial y_b}\left[\frac{\delta S}{\delta x_a}\right]_{T\Phi} \text{(a)} \, , \qquad \left[\frac{\delta \tilde{S}}{\delta \nu_\beta}\right] = \frac{\partial f_a}{\partial \nu_\beta}\left[\frac{\delta S}{\delta x_a}\right]_{T\Phi} \text{(b)} \, . \tag{3.3}$$

It follows from regularity of matrix $||\partial f_a/\partial y_b||$ that new motions have to satisfy the original Euler-Lagrange expressions for motion equations, transformed by coordinate substitution $T\Phi$. Therefore, new motions are connected with original ones by relations $x_a = f_a(\nu_\alpha, y_b)$. Hence solutions of new motion equations are obtained modulo arbitrary time-dependent functions $\nu_\alpha(t)$. The set of dynamical variables is divided into two subsets: (i) those whose evolution is determined by given initial

conditions, and (ii) those whose time development is completely arbitrary. It immediately follows that new Lagrangian $\tilde{L}$ is singular, whereas the original one L is non-degenerate.

New Lagrangian is invariant by TG, i.e. the structure of $\tilde{L}$ does not change under the coordinate transformation $(\Psi^1)_{j^1c} : TG \times TQ \to TG \times TQ$, locally given by

$$\begin{aligned} y_b &= f_b(\delta_\beta, z_c) \, , \quad \eta_\gamma = \varphi_\gamma(\nu_\alpha, \delta_\beta) \, ; \\ {y_b}^1 &= d_T f_b \, , \quad \eta^1_\gamma = d_T \varphi_\gamma \, . \end{aligned} \tag{3.4}$$

Here δ_β and ${\delta_\beta}^1$ are coordinates of the element $j^1c \in TG$. By analogy with the scheme developed in ref.[4], an invariance of the theory is achieved by extension of an original configurational manifold Q to the Cartesian product $G \times Q$, so that group variables are included to the set of independent Lagrangian variables.

Let us suppose that the original Lagrangian $L : TQ \to I\!\!R$ is invariant under the action $T\Phi$, i.e. $L \circ T\Phi = L$. Such a symmetrical Lagrangian must be singular. Indeed, the solutions of corresponding Euler-Lagrange equations (see eq.(3.3a) and eq.(3.3b) where left-hand side is vanished) are obtained modulo arbitrary time-dependent functions $\nu_\alpha(t)$.

It is advantageous to study an infrequent case of action TG on Q (see eqs.(2.5) and (2.6)). Any formula which will be deduced below is applicable (after trivial simplification) for the description of the above situation where an action of G on Q, and its lift to an action of TG on TQ, and dynamics originated from TG-invariant Lagrangian function are considered.

For sake of simplisity, we deal with the contact transformation obtained from (2.10) by taking the $\lambda_\beta \to 0$ limits. Since an original Lagrangian is lower-derivative, we are interesting in left action $\Phi^{(2,1)} : T^2G \times TQ \in TQ$. Having carried out it in Lagrangian L, we construct the higher-derivative Lagrangian function $\tilde{L} : T^2G \times TQ \to I\!\!R$:

$$\tilde{L}(\nu_\beta, {\nu_\beta}^1, {\nu_\beta}^2, y_b, {y_b}^1) = L(f_a(\nu_\beta, {\nu_\beta}^1, y_b), d_T f_a) \, . \tag{3.5}$$

Cartesian product $T^2G \times TQ$ is the subbundle of tangent bundle $T(TG \times Q)$.

Following ref.[2], we write the relations between the expressions for original and new Euler-Lagrange equations in the form

$$\begin{aligned} \left[\frac{\delta \tilde{S}}{\delta y_b}\right] &= \frac{\partial f_a}{\partial y_b}\left[\frac{\delta S}{\delta x_a}\right]_{\Phi^{(2,1)}} \text{(a)} \, , \\ \left[\frac{\delta \tilde{S}}{\delta \nu_\beta}\right] &= \frac{\partial f_a}{\partial \nu_\beta}\left[\frac{\delta S}{\delta x_a}\right]_{\Phi^{(2,1)}} - d_T \frac{\partial f_a}{\partial {\nu_\beta}^1}\left[\frac{\delta S}{\delta x_a}\right]_{\Phi^{(2,1)}} \text{(b)} \, . \end{aligned} \tag{3.6}$$

As one might exspect, new motions are obtained modulo arbitrary time-dependent functions $\nu_\alpha(t)$ and $\nu_\alpha{}^1(t)$. Let us construct a Hamiltonian system $(T^*(TG\times Q),\tilde\omega,\tilde H)$ due to Ostrogradski-Legendre transformation [13].

According to ref.[2], new Ostrogradski generalized momenta

$$\hat\pi_b=\frac{\partial\tilde L}{\partial y_b{}^1}\ \text{(a)}\,,\quad \hat\eta_{\beta,1}=\frac{\partial\tilde L}{\partial\nu_\beta{}^2}\ \text{(b)}\,,$$
$$\hat\eta_{\beta,0}=\frac{\partial\tilde L}{\partial\nu_\beta{}^1}-d_T\frac{\partial\tilde L}{\partial\nu_\beta{}^2}\ \text{(c)}\,, \tag{3.7}$$

are linked together by the following relationships:

$$\hat\pi_b=\hat p_a\frac{\partial f_a}{\partial y_b}\ \text{(a)}\,,\quad \hat\eta_{\beta,1}=\hat p_a\frac{\partial f_a}{\partial\nu_\beta{}^1}\ \text{(b)}\,,$$
$$\hat\eta_{\beta,0}=\hat p_a\frac{\partial f_a}{\partial\nu_\beta}+\frac{\partial f_a}{\partial\nu_\beta{}^1}\left[\frac{\delta S}{\delta x_a}\right]_{\Phi^{(2,1)}}\ \text{(c)}\,, \tag{3.8}$$

where $\hat p_a=\partial L/\partial x_a{}^1$ are original momenta. Following ref.[2], we rewrite the new Poincaré-Cartan one-form

$$\tilde\alpha_L=\hat\pi_b dy_b+\hat\eta_{\beta,0}d\nu_\beta+\hat\eta_{\beta,1}d\nu_\beta{}^1 \tag{3.9}$$

as follows

$$\tilde\alpha_L=\alpha_L+\frac{\partial f_a}{\partial\nu_\beta{}^1}\left[\frac{\delta S}{\delta x_a}\right]_{\Phi^{(2,1)}}d\nu_\beta\,. \tag{3.10}$$

New energy function

$$\tilde E_L=\hat\pi_b y_b{}^1+\hat\eta_{\beta,0}\nu_\beta{}^1+\hat\eta_{\beta,1}\nu_\beta{}^2-\tilde L \tag{3.11}$$

is expressed in similar form

$$\tilde E_L=E_L+\frac{\partial f_a}{\partial\nu_\beta{}^1}\left[\frac{\delta S}{\delta x_a}\right]_{\Phi^{(2,1)}}\nu_\beta{}^2\,. \tag{3.12}$$

Here α_L is Poincaré-Cartan one-form and E_L is the energy function associated with original Lagrangian L (see eqs.(3.1)). All the expressions (3.10), (3.12), and (3.8c) contain the terms which are proportional to the expressions for original motion equations, transformed by coordinate substitution $\Phi^{(2,1)}$. These terms vanish. Therefore, we obtain a presymplectic dynamical system [14,13,11]. Indeed, new canonical two-form

$$\tilde\omega=dy_b\wedge d\pi_b+d\nu_\beta\wedge d\eta_{\beta,0}+d\nu_\beta{}^1\wedge d\eta_{\beta,1} \tag{3.13}$$

can be written as $\tilde\omega=dx_a\wedge dp_a$, so that original canonical coordinates are just required by generalized Darboux theorem [11]. On the local level we have the set of first-class constraints obtained by excluding the original momenta from relations

$$\pi_b=p_a\frac{\partial f_a}{\partial y_b}\ \text{(a)}\,,\ \eta_{\beta,1}=p_a\frac{\partial f_a}{\partial\nu_\beta{}^1}\ \text{(b)}\,,\ \eta_{\beta,0}=p_a\frac{\partial f_a}{\partial\nu_\beta}\ \text{(c)}\,, \tag{3.14}$$

together with Hamiltonian $\tilde H(y_b,\pi_b,\nu_\alpha,\nu_\alpha{}^1)$ which is constructed from the original one $H(x_a,p_a)$ by using of eqs.(2.6) and (3.14a). Not even having a Hamiltonian explicitly, we are sure that constraints which include momenta $\eta_{\beta,1}$ are primary but those which contain momenta $\eta_{\beta,0}$ — secondary:

$$\Phi^{(1)}_{\beta1}=\eta_{\beta,1}-\pi_c\bar f^c{}_a\frac{\partial f_a}{\partial\nu_\beta{}^1}\approx0\ \text{(a)}\,,$$
$$\Phi^{(2)}_{\beta0}=\eta_{\beta,0}-\pi_c\bar f^c{}_a\frac{\partial f_a}{\partial\nu_\beta}\approx0\ \text{(b)}\,. \tag{3.15}$$

We should only take into consideration the relationship (3.7c) between zeroth-order momentum and time-derivative of first-order momentum which is then nothing but a stationarity condition for primary constraint. The matrix $\bar f^c{}_a$ is inverted to the matrix $\partial f_a/\partial y_b$.

It can be easily proven that all Poisson brackets $\{\Phi^{(i)}_{\beta r},\Phi^{(j)}_{\alpha k}\}$ are identically equal to zero. The canonical transformation

$$x_a=f_a(y_b,\nu_\alpha,\nu_\alpha{}^1)\,,\quad p_a=\pi_c\bar f^c{}_a(y_b,\nu_\alpha,\nu_\alpha{}^1)\,, \tag{3.16a}$$

$$\lambda_\alpha{}^r=\nu_\alpha{}^r\,,\quad \theta_{\alpha,r}=\eta_{\alpha,r}-\pi_c\bar f^c{}_a\frac{\partial f_a}{\partial\nu_\beta{}^r}\,;\quad r=0,1\,, \tag{3.16b}$$

allows to eliminate the redundant degrees of freedom and leads to the non-constraint dynamics with the Hamilton function $H(x_a,p_a)$. It can be obtained from the non-singular original Lagrangian by a simple Legendre transformation. It is evident, that final constrained manifold [14,13,11] is the original phase manifold T^*Q.

The relations (3.16a) define an action of a Lie group TG on T^*Q. Corresponding infinitesimal transformations are generated by the vector fields

$$Y^{(1,0)\,*}_{(\alpha0)}=Y^b_{(\alpha0)}\frac{\partial}{\partial y_b}-\pi_b\frac{\partial Y^b_{(\alpha0)}}{\partial y_c}\frac{\partial}{\partial\pi_c}\,,$$
$$Y^{(1,0)\,*}_{(\alpha1)}=Y^b_{(\alpha1)}\frac{\partial}{\partial y_b}-\pi_b\frac{\partial Y^b_{(\alpha1)}}{\partial y_c}\frac{\partial}{\partial\pi_c}\,, \tag{3.17}$$

which are complete lifts of the vector fields (2.7) to phase manifold T^*Q (see ref.[12]). These generators constitute the basis of Lie subalgebra of the set of all vector fields on T^*Q which is homomorphic to $Lie(TG)$.

We consider now Lagrangian $L : TQ \to I\!R$ which is invariant under the action $\Phi^{(2,1)} : T^2G \times TQ \to TQ$. Since group variables are not contained in the new Lagrangian $\tilde{L}$, then generalized momenta $\hat{\eta}_{\beta,1}$ and $\hat{\eta}_{\beta,0}$ vanish identically (see eqs.(3.7b) and (3.7c)). Using it in eqs.(3.9)–(3.12) and (3.1) we show that energy function, Poincaré-Cartan one-, and two-forms admit the map $\Phi^{(2,1)}$. For a careful consideration of such a dynamical system we use the results of this Section so far as presymplectic manifold $(T^*(TG \times Q), \tilde{\omega})$ are concerned.

Left-hand sides in eqs.(3.14b) and (3.14c) vanish identically. Usage of eqs. (3.14a) allows to exclude the original momenta p_a from their right-hand sides. Obtained expressions

$$\Phi^{(1)}_{\beta 1} = \pi_c \bar{f}^c{}_a \frac{\partial f_a}{\partial \nu_\beta{}^1} \approx 0\,,\; \Phi^{(2)}_{\beta 0} = \pi_c \bar{f}^c{}_a \frac{\partial f_a}{\partial \nu_\beta} \approx 0\,, \tag{3.18}$$

are the constraints on $T^*(TG \times Q)$, whereas Hamiltonian H does not contain the pairs $(\nu_\alpha{}^i, \eta_{\alpha,i})$ of canonically conjugated group variables explicitly. Taking $\nu_\alpha \to 0$ limits we find the constraints on the original phase manifold T^*Q:

$$\Phi^{(1)}_{(\alpha 1)} = p_a Y^a_{(\alpha 1)}(x) \approx 0 \;\; \text{(a)}\,,\; \Phi^{(2)}_{(\alpha 0)} = p_a Y^a_{(\alpha 0)}(x) \approx 0 \;\; \text{(b)}\,, \tag{3.19}$$

(we put $y_a = x_a$ within that range of accuracy). Taking into account the commutation relations for vector fields (2.7) (more exactly, ensuing properties of their components $Y^a_{(\alpha i)}$), we compute Poisson brackets of functions $\Phi^{(1,2)}_{(\alpha i)}$:

$$\begin{aligned} &\{\Phi^{(1)}_{(\alpha 1)}, \Phi^{(1)}_{(\beta 1)}\} = 0\,, \quad \{\Phi^{(2)}_{(\alpha 0)}, \Phi^{(1)}_{(\beta 1)}\} = c^\gamma_{\alpha\beta} \Phi^{(1)}_{(\gamma 1)}\,, \\ &\{\Phi^{(2)}_{(\alpha 0)}, \Phi^{(2)}_{(\beta 0)}\} = c^\gamma_{\alpha\beta} \Phi^{(2)}_{(\gamma 0)}\,. \end{aligned} \tag{3.20}$$

As it would be expected, we have first class constraint manifold $K(Q, \omega)$ locally characterized by functions (3.19) which are in K in involution with respect to Poisson brackets.

Hence, there are two quite different classes of prolonged Lie symmetrical Lagrangians. Class I appears to be more natural, it consists of the singular functions which do not depend on group variables explicitly. Class II is entirely novel; here degenerate Lagrangians contain the group variables. An example of two-body dynamics which is invariant under the action of the tangent Euclidean group is studied in Appendix A in order to illustrate the classification.

An important point is that we find the relations linking together the expressions for Euler-Lagrange equations in both above cases. The essence of presented scheme is the usage of an invertible contact transformation [2] which has the form of change of variables depending on derivatives (see eqs.(2.10) and (3.4)). The structure of such coordinate substitution has an influence upon the structure of the set of first-class constraints which appear on the Hamiltonian level. In particular, the highest order of derivatives determines number of the needed steps in Dirac-Bergmann scheme to establish all the constraints. So, if we deal with contact transformation of type (3.4), the primary constraints are arised only. There are the relations of type (3.15b) or (3.19b). In ref.[1] a theorem was proven which establishes the intimately connection between an existence of the relations involving Euler-Lagrange equations and an invariance of corresponding action integral under the action of specific gauge transformation. The results derived in this Section are in excellent agreement with this theorem.

4. Infinitesimal symmetries and constants of motion

In this Section we describe two different kinds of the infinitesimal symmetries arising from the actions of the tangent groups of a Lie group G on a manifold Q and its tangent bundle TQ.

The vector field $Y_\alpha \in \mathcal{X}(Q)$, generated by an action of G on Q, is an infinitesimal symmetry of the Lagrangian $L : TQ \to I\!R$ if the conditions $Y^c_\alpha L = 0 (\alpha = \overline{1, \mathcal{R}})$ are satisfied [12]. Here fundamental vector field $Y^c_\alpha \in \mathcal{X}(TQ)$ is a complete lift [11] of Y_α to tangent bundle TQ. The group parameters are considered as the constants. On the hypothesis that the group parameters are time-dependent functions, the total lift of an action of G on Q to the action of TG on TQ is available (see Section 2). In such a case both the complete lift (2.4a) and the vertical lift (2.4b) of vector field Y_α are the infinitesimal symmetries of L. An invariant Lagrangian function satisfies the following system of first-order differential equations in partial derivatives:

$$Y^{(1)}_{(\alpha i)} L = 0\,, \quad i = 0, 1\,, \quad \alpha = \overline{1, \mathcal{R}}\,. \tag{4.1}$$

Following ref.[12], we write the constants of motion

$$\alpha_L(Y^{(1)}_{(\alpha 0)}) = \hat{\pi}_b Y^b_\alpha\,, \tag{4.2}$$

associated with vector fields $Y^{(1)}_{(\alpha 0)}$. The others $\alpha_L(Y^{(1)}_{(\alpha 1)})$ are trivial.

In general, "standard" $G-$invariant Lagrangian function is not invariant by a Lie group TG. Indeed, starting with Lagrangian $L : TQ \to I\!R$ which satisfies infinitesimal symmetry conditions $Y^c_\alpha L = 0$, we arrive at the new Lagrangian $\tilde{L} := L \circ T\Phi$ defined on $T(G \times Q)$:

$$\tilde{L}(\nu_\beta, {\nu_\beta}^1, y_b, {y_b}^1) = L(y_b, {y_b}^1 + \omega^\beta_b(\nu_\alpha, y_c){\nu_\beta}^1)\,. \tag{4.3}$$

The multiplier ω^β_b is equal to $\bar{f}_b{}^a \partial f_a/\partial\nu_\beta$ where functions $f_a(\nu_\alpha, y_b)$ determine an action $\Phi : G \times Q \to Q$ and matrix $\bar{f}_b{}^a$ is inverted to matrix $\partial f_a/\partial y_b$. Both the Lagrangian (4.3) and the Lagrangian (3.2) are the same type, so the same scheme of constraint reduction is applicable here.

Now we consider the dynamical system based on the Lagrangian which is invariant under the transformation $(\Phi^{(2,1)})_{j^2a} : TQ \to TQ$ determined by fixed element $j^2a \in T^2G$. Vector fields (2.9) are infinitesimal symmetries of this Lagrangian. The symmetry conditions have the form of the following system of differential equations for the Lagrangian function:

$$Y^{(2,1)}_{(\alpha i)} L = 0\,, \quad i = 0,1,2\,, \quad \alpha = \overline{1,\mathcal{R}}\,. \tag{4.4}$$

Following ref.[12] we write the corresponding constants of the motion:

$$\alpha_L(Y^{(2,1)}_{(\alpha 0)}) = \hat{\pi}_b Y^b_{(\alpha 0)}\,, \quad \alpha_L(Y^{(2,1)}_{(\alpha 1)}) = \hat{\pi}_b Y^b_{(\alpha 1)}\,. \tag{4.5}$$

The remaining ones, namely $\alpha_L(Y^{(2,1)}_{(\alpha 2)})$, are equal to zero identically.

It is interesting to find Hamiltonian counterparts of these constants of the motion. According to scheme developed in ref.[12] for non-degenerate theories, we should construct the vector fields on T^*Q being Legendre-related [8] with the infinitesimal generators $Y^{(2,1)}_{(\alpha 0)}$ and $Y^{(2,1)}_{(\alpha 1)}$. Since we deal with singular Lagrangian, Legendre transformation is not diffeomorphism and this recipe is not applicable here. We use the vector fields (3.17) which are generated by an action of a Lie group TG on phase manifold T^*Q. There are then nothing but the complete lifts [12] of infinitesimal generators (2.7) to T^*Q. With the supposition of these generators are Noether symmetries [12] of a Hamiltonian function, we construct the constants of the motion which are precisely the constraints (3.18).

Solving of infinitesimal symmetry conditions (4.4) can help to find a special local coordinate system, i.e. coordinate system which allows to separate explicitly the regular part of theory (where the momenta are, as in standard non-singular theory, independent functions of velocity variables). Prolonged Lie symmetric Lagrangian depends on group invariants

I_k, i.e. functions such that $Y^{(1,0)}_{(\alpha i)} I_k = 0$, and their 1-lifts [11] I^1_k. These invariants should be choosen as independent variables and inserted into the set of special local coordinates.

An invariance of transformed Lagrangian (3.5) means, that it admits the transformations $(\Psi^{(2,1)})_{j^2c} : T^2G \times TQ \to T^2G \times TQ$ locally written as

$$\begin{aligned} y_b &= f_b(\delta_\beta, {\delta_\beta}^1, z_c)\,, \quad \eta_\gamma = \varphi_\gamma(\nu_\alpha, \delta_\beta)\,; \\ {y_b}^1 &= d_T f_b\,, \quad \eta^1_\gamma = d_T\varphi_\gamma\,, \quad \eta^2_\gamma = d^2_T\varphi_\gamma\,. \end{aligned} \tag{4.6}$$

Taking the element j^2c with coordinates $(\delta_\beta, {\delta_\beta}^1, {\delta_\beta}^2)$ in neighborhood of identity j^2e we obtain an infinitesimal transformation which is generated by the vector fields $Z^{(2,1)}_{(\alpha i)} := X^{(2)}_{(\alpha i)} - Y^{(2,1)}_{(\alpha i)}$. Putting (1.12) and (2.9) in one-form (3.9), we obtain the following constants of the motion:

$$\begin{aligned} \tilde{\alpha}_L(Z^{(2,1)}_{(\alpha 0)}) &= -\hat{\pi}_b Y^b_{(\alpha 0)} + \hat{\eta}_{\beta,0} L^\beta_\alpha(\nu) + \hat{\eta}_{\beta,1} d_T L^\beta_\alpha(\nu)\,, \\ \tilde{\alpha}_L(Z^{(2,1)}_{(\alpha 1)}) &= -\hat{\pi}_b Y^b_{(\alpha 1)} + \hat{\eta}_{\beta,1} L^\beta_\alpha(\nu)\,. \end{aligned} \tag{4.7}$$

There is enough evidence to prove the identity $\tilde{\alpha}_L(Z^{(2,1)}_{(\alpha 2)}) = 0$.

Usage of the infinitesimal symmetry conditions $Z^{(2,1)}_{(\alpha i)} L = 0$ permits to formulate the problem of how to establish the structure of gauge invariant Lagrangian as the problem of solving of the system of homogeneous first-order differential equations in partial derivatives. So, in ref.[4] the problem of how to construct $TG-$invariant Lagrangian starting with "standard" $G-$invariant Lagrangian is redused to the solving of the system of differential equations $Z^{(1)}_{(\alpha i)} L = 0, i = 0,1$. Here $Z^{(1)}_{(\alpha i)}$ is equal to $X^{(1)}_{(\alpha i)} - Y^{(1)}_{(\alpha i)}$ where vector field $X^{(1)}_{(\alpha i)}$ belongs to the basis of a Lie algebra $Lie(TG)$ at a point $j^1a = (\nu_\alpha, {\nu_\alpha}^1)$ and infinitesimal generator $Y^{(1)}_{(\alpha i)}$ is given by eqs.(2.4). Lagrangian satisfying this conditions has the structure of function (4.3) where $\nu_\alpha \to 0$ limits are taken. In general, the functions of type (3.2) and (3.5) are the unique solutions of this kind symmetry conditions.

5. Invariant dynamics on a tangent Lie group

Let G be an R-dimensional Lie group and TG its tangent bundle which is also Lie group. The elements of TG are the sections of this bundle, i.e. the vector fields on G. To appreciate this statement we consider the curves at $a, b \in G$ which are the differentiable maps $\nu : I\!R \to G$ and

$\lambda : I\!R \to G$ with $\nu(0) = a$ and $\lambda(0) = b$, respectively. If there exists a coordinate neighborhood (U, g) of a with local coordinates (g_α), then coordinates $\nu_\alpha = (g_\alpha \circ \nu)(0)$ and ${\nu_\alpha}^1 = (d(g_\alpha \circ \nu)/dt)(0)$ of group element $j_0^1\nu := j^1 a$ defines [3] the tangent vector $v_a = {\nu_\alpha}^1(\partial/\partial g_\alpha)_a \in T_aG$. Similarly, the group element $j^1 b = (\lambda_\beta, {\lambda_\beta}^1)$ where $\lambda_\beta = (g_\beta \circ \lambda)(0)$ and ${\lambda_\beta}^1 = (d(g_\beta \circ \lambda)/dt)(0)$ is the vector $v_b \in T_bG$. Tangent group multiplication $T\varphi : TG \times TG \to TG$ defined as

$$(\nu_\alpha, {\nu_\alpha}^1) \cdot (\lambda_\beta, {\lambda_\beta}^1) = (\varphi_\gamma(\nu, \lambda), d_T\varphi_\gamma(\nu, \lambda)), \qquad (5.1)$$

maps the vectors $v_a \in T_aG$, $v_b \in T_bG$ into the vector $v_{ab} \in T_{ab}G$ with components $d_T\varphi_\gamma(\nu, \lambda)$.

Let us suppose that Lagrangian function $L : TG \to I\!R$ is invariant by G. In other words L is invariant under the lifted action $\varphi^1 : G \times TG \to TG$ of G on TG which is defined as follows [6,7]:

$$(\varphi^1)_a : TG \to TG, \qquad (\varphi^1)_a = Tl_a, \qquad (5.2)$$

where $l_a : G \to G$ is the left translation by $a \in G$. Locally the tangent map Tl_a of l_a is determined by $Tl_a(v_b) = dl_a(b)(v_b)$. Differential $dl_a(b) : T_bG \to T_{ab}G$ of l_a at b can be written by means of TG-multiplication law: $dl_a(b)(v_b) = i_1(a) \cdot j^1 b$, where $i_1(a) := (\nu_\alpha, 0) \in i_1(G) \subset TG$ and $j^1 b = v_b$. Zero in expressions of type $(\nu_\alpha, 0)$ denotes an $\mathcal{R}$-dimensional row here and below in this Section. Using this formula, we rewrite the local expression for linear isomorphism $T_eG \to \mathcal{X}_L(G)$ as the TG-product of $i_1(a)$ on vector $\xi = x^\beta(\partial/\partial g_\beta)_e$ belonging to the Lie algebra $\mathcal{G} := T_eG$:

$$\begin{aligned} dl_a(e)(\xi) &= (\nu_\alpha, 0) \cdot (0, x^\beta) \\ &= (\nu_\alpha, x^\beta L^\alpha_\beta(a)). \end{aligned} \qquad (5.3)$$

Left-invariant vector field X_β with components $L^\alpha_\beta(a)$ corresponds to the basic vector $(\partial/\partial g_\beta)_e = (0, \delta_{\beta\gamma}) := \xi_\beta$.

In ref.[6] the conveniance of usage the left trivialisation [5]

$$\begin{aligned} Tl \ &: \ G \times \mathcal{G} \to TG \\ &\quad (a, X) \mapsto dl_a(e)(X) \end{aligned} \qquad (5.4)$$

in examination of mechanics on a Lie group is exhibited (see also [7]). We interpret this map as the possibility to write any $j^1 a \in TG$ as follows:

$$(\nu_\alpha, {\nu_\alpha}^1) = (\nu_\alpha, 0) \cdot (0, {\nu_\alpha}^1 V^\beta_\alpha(a)), \qquad (5.5)$$

where matrix $\hat{V} := \|V^\beta_\alpha\|$ is inverse to the matrix $\hat{L} := \|L^\beta_\alpha\|$. We denote ${\nu_\alpha}^1 V^\beta_\alpha(a)$ by X^β_a. Via the identification $(\nu_\alpha, 0) \cdot (0, X^\beta_a) := (a, X_a)$ the group structure on TG looks [5] as the semidirect product of G and its Lie algebra $\mathcal{G}$ with respect to adjoint representation of G on $\mathcal{G}$:

$$(b, X_b).(a, X_a) = (ba, Ad(a^{-1})X_b + X_a). \qquad (5.6)$$

Here $Ad(a^{-1})X_b = i_1(a^{-1}) \cdot X_b \cdot i_1(a)$.

Since second factor in product (5.5) does not change under the multiplication on any $b \in i_1(G)$, (left) $G-$invariant Lagrangian is a differentiable function $L \in \Lambda^0(\mathcal{G})$:

$$L = L({\nu_\alpha}^1 V^\beta_\alpha(\nu)). \qquad (5.7)$$

This function satisfies the infinitesimal symmetry conditions $Y^{(1)}_{\beta 0} L = 0$ where $Y^{(1)}_{\beta 0}$ is the complete lift of right-invariant vector field $Y_\beta \in \mathcal{X}_R(G)$. The latter can be written as the product of basic vector $\xi_\beta \in \mathcal{G}$ on $i_1(a)$: $Y_\beta(a) = (0, \delta_{\beta\gamma}) \cdot (\nu_\alpha, 0)$ (cf.eq.(5.3)). The constant of motion from $Y^{(1)}_{\beta 0} \in \mathcal{X}_R(i_1(G))$ is [12]:

$$\hat{\alpha}_L(Y^{(1)}_{\beta 0}) = \hat{p}_\alpha R^\alpha_\beta(\nu), \qquad (5.8)$$

where $\hat{\alpha}_L = \hat{p}_\alpha d\nu_\alpha$ is the Poincaré-Cartan one-form associated with L, $\hat{p}_\alpha = \partial L/\partial {\nu_\alpha}^1$ is momentum, and $R^\alpha_\beta(\nu)$ are the components of the vector $Y_\beta(a)$.

Following similar procedure as above, we could also construct right-invariant Lagrangian on TG, i.e. the function $L : TG \to I\!R$ which is invariant under the action of G on TG on the right. This action is defined by using the tangent map Tr_a of right translation by $a \in G$. Differential $dr_a(b)(v_b)$ is then nothing but the product of $j^1 b$ on $i_1(a)$. The splitting $(\nu_\alpha, {\nu_\alpha}^1) = (0, {\nu_\alpha}^1 W^\beta_\alpha(a)) \cdot (\nu_\alpha, 0)$ allows the right trivialisation $Tr : \mathcal{G} \times G \to TG$, so that tangent group TG is isomorphic to the semidirect product $\mathcal{G} \times_{Ad} G$ (see ref.[5]). Symbols $W^\beta_\alpha(a)$ denote the elements of matrix $\hat{W}$ which is inverted to the matrix $\hat{R} := \|R^\beta_\alpha(a)\|$. Whence we obtain the explicit expression for right-invariant Lagrangian

$$L^R = L^R({\nu_\alpha}^1 W^\beta_\alpha(\nu)), \qquad (5.9)$$

satisfying the infinitesimal symmetry conditions $X^{(1)}_{\beta 0} L = 0, \beta = \overline{1, \mathcal{R}}$. Vector field $X^{(1)}_{\beta 0} \in \mathcal{X}_L(i_1(G)) \subset \mathcal{X}_L(TG)$ is the complete lift of $X_\beta \in \mathcal{X}_L(G)$. Corresponding "right" constants of the motion are:

$$\hat{\alpha}_L(X^{(1)}_{\beta 0}) = \hat{p}_\alpha L^\alpha_\beta(\nu). \qquad (5.10)$$

The left action of a Lie group G on its cotangent bundle T^*G is defined [6] by using the inverse dual map to the differential $dl_a(b)$ of l_a at b. Fundamental vector field corresponding to $\xi_\beta \in \mathcal{G}$ is a complete lift of $Y_\beta \in \mathcal{X}_R(G)$ to T^*G [12]. In local coordinates, we have

$$Y_\beta^{c*} = R_\beta^\alpha \frac{\partial}{\partial \nu_\alpha} - p_\alpha \frac{\partial R_\beta^\alpha}{\partial \nu_\gamma} \frac{\partial}{\partial p_\gamma}, \tag{5.11}$$

where (ν_α, p_α) are the induced coordinates on T^*G. We consider the Hamiltonian system (T^*G, ω, H) where ω is the canonical symplectic form on T^*G and H is a function on T^*G. Having used dual trivialisation

$$\begin{aligned} T^*l \quad &: \quad T^*G \to G \times \mathcal{G}^* \\ & \upsilon_a \mapsto (a, dl_a^*(e)(\upsilon_a)), \end{aligned} \tag{5.12}$$

which determines so-called *body coordinates* [7], we find the left-invariant Hamiltonian:

$$H = H(p_\alpha L_\beta^\alpha(\nu)). \tag{5.13}$$

All the vector fields $Y_\beta^{c*} \in \mathcal{X}(T^*G)$ are Noether symmetries [12] of this Hamiltonian, i.e. $Y_\beta^{c*} H = 0$. On the contrary, differentiable function satisfying these symmetry conditions is defined on $\Lambda^0(\mathcal{G}^*)$.

Following ref.[7], we write the momentum mapping

$$\begin{aligned} J^L \quad &: \quad T^*G \to \mathcal{G}^* \\ & \upsilon_a \mapsto dr_a^*(e)(\upsilon_a) \end{aligned} \tag{5.14}$$

for the symplectic left action of G on T^*G. Taking into account ref.[8] we obtain its Lagrangian counterpart $\bar{J}^L : T^*G \to \mathcal{G}^*$ defined as $\bar{J}^L = J^L \circ Leg$. Here $Leg : TG \to T^*G$ is the Legendre transformation corresponding to the Lagrangian (5.7). Since J^L is an Ad^*-equivariant mapping for the action of G on T^*G [7], then $\hat{J}^L : \mathcal{G} \to C^\infty(T^*G)$ is a homomorphism of $\mathcal{G}$ to the Lie algebra of functions (namely, $Leg-$transformed constants of motion (5.8)) under the Poisson bracket. The reduction theorem can be applied on the Lagrangian level[8] as well as the Hamiltonian level[7]. The dimension of reduced space is equal to $\mathcal{R} - \dim G_\mu$ where G_μ is the isotropy subgroup of G under the co-adjoint action Ad^* at the one-form $\mu \in \mathcal{G}^*$ which is the regular value of momentum mapping.

We can construct the right-invariant Hamiltonian on T^*G in similar circumstances: if $T^*r : T^*G \to \mathcal{G}^* \times G$ is the dual (right) trivialisation which determines *space coordinates* [7], then H^R depends on the ($Leg-$transformed) "left" constants of motion (5.8) whereas the momentum mapping $J^R : T^*G \to \mathcal{G}^*$ for the symplectic right action of

G on T^*G is given by using the differential of dual left translation: $\upsilon_a \mapsto dl_a^*(e)(\upsilon_a)$.

It is obvious that does not exist $TG-$invariant Lagrangian defined on tangent bundle TG. We consider (left) TG_0-invariant function $L : TG \to I\!\!R$ where G_0 is $\mathcal{R}_0-$dimensional Lie subgroup of a Lie group G. According to Section 3 the set of first-class constraints

$$\Phi_\rho^{(1)} = p_\kappa Y_\rho^\kappa(\nu) \approx 0 \tag{5.15}$$

is appeared on the Hamiltonian level (cf.(3.19b)). Here Y_ρ^κ, $\rho, \kappa = \overline{1, \mathcal{R}_0}$, are the components of right-invariant vector field $Y_\rho \in \mathcal{X}_R(G_0)$. Therefore, $TG_0 - invariant$ canonical two-form on TG is presymplectic and we have a presymplectic dynamical system in such a case. Since matrix with elements Y_ρ^κ is regular the constraints (5.15) are reduced to trivial form $p_\kappa \approx 0$. Hence T^*G/T^*G_0 is the final constrained manifold.

Conclusions

By "sewing on" a Lie group of an one-dimensional manifold we join the geometrical theory of jet bundles with the theory of Lie groups. The language of jets allows to describe the derivative of maps, so we apply the notion of jet prolongation to the group operations and group elements. The reason is that Lie groups are differentiable manifolds in which the group operations are smooth. As a result we obtain a tangent Lie group being the jet prolongation of an original Lie group. Obvious generalizations, of course, concern the order of group prolongation as well as the dimension of a (base) manifold. On the other hand, it is interesting to study those aspects of the theory which explain the inner structure of tangent Lie groups or homomorphisms induced by the canonical jet projections, for instance, a clear geometric interpretation of the homogeneous manifolds [10] TG/G and T^2G/G.

By using above scheme we define an action of the first prolongation of a Lie group on a differentiable manifold Q and we lift it to an action of the second prolongation of this group on tangent bundle TQ. The requirement of an invariance of Lagrangian function under the action of prolonged Lie group leads to degeneracy of this Lagrangian. There are two types of singular Lagrangians of this kind: (i) these depending on the group variables, and (ii) those which do not contain the group variables explicitly. On the Hamiltonian level we have gauge constrained theories in both above cases. Does it mean, that any dynamical system supplemented with the first-class constraints is connected with a prolonged Lie group? One of the ways of continuing the present work would be the investigation of this problem.

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Appendix A. Tangent group of the group of rotations and translations of Euclidean space and invariant two-body problem

Let E be the product manifold $SO(3)\times I\!R^3$ supplemented with a group structure by setting $(O_1,\boldsymbol{\nu}_1)\cdot(O_2,\boldsymbol{\nu}_2)=(O_1O_2,O_1\boldsymbol{\nu}_2+\boldsymbol{\nu}_1)$, $O\in SO(3)$, $\boldsymbol{\nu}\in I\!R^3$. Tangent manifold TE is again a Lie group. Its multiplication law is written as follows:

$$\begin{aligned}&(O_1,\boldsymbol{\nu}_1;\dot O_1,\dot{\boldsymbol{\nu}}_1)\cdot(O_2,\boldsymbol{\nu}_2;\dot O_2,\dot{\boldsymbol{\nu}}_2)=\\ =\;&(O_1O_2,O_1\boldsymbol{\nu}_2+\boldsymbol{\nu}_1;\dot O_1O_2+O_1\dot O_2,\dot O_1\boldsymbol{\nu}_2+O_1\dot{\boldsymbol{\nu}}_2+\dot{\boldsymbol{\nu}}_1)\,,\end{aligned}\quad\text{(A1)}$$

where $\dot O_a:=d_TO_a$ and $\dot{\boldsymbol{\nu}}_a:=d_T\boldsymbol{\nu}_a$. An identity element $j^1e\in TE$ is $(1_3,0;0_3,0)$ and the element $(O^t,-O^t\boldsymbol{\nu};\dot O^t,-\dot O^t\boldsymbol{\nu}-O^t\dot{\boldsymbol{\nu}})$ is an inverse to $(O,\boldsymbol{\nu};\dot O,\dot{\boldsymbol{\nu}})\in TE$.

Lie group E is the group of rotations and translations of Euclidean space $I\!R^3$, if we identify the element $(O,\boldsymbol{\nu})$ with the affine motion $\mathbf{x}\mapsto O\mathbf{x}+\boldsymbol{\nu}$ of $I\!R^3$. Similarly, the identification of $(O,\boldsymbol{\nu};\dot O,\dot{\boldsymbol{\nu}})\in TE$ with the transformation $(\mathbf{x},\dot{\mathbf{x}})\mapsto(O\mathbf{x}+\boldsymbol{\nu};O\dot{\mathbf{x}}+\dot O\mathbf{x}+\dot{\boldsymbol{\nu}})$ of $TI\!R$ implies that Lie group TE is the group of rotations and translations of tangent space $TI\!R^3$.

Let $Q:=I\!R^3\times I\!R^3/\{0\}$ be 6-dimensional configuration manifold of two point particles spanned by position variables $x_{ai}(a=1,2;i=1,2,3)$. The group E acts on Q by the action

$$\begin{aligned}&E\times Q\to Q\,,\\ &((O,\boldsymbol{\nu}),\mathbf{x}_a)\mapsto(O\mathbf{x}_a+\boldsymbol{\nu})\,.\end{aligned}\quad\text{(A2)}$$

We lift this action to the tangent bundle TQ. The action $TE\times TQ\to TQ$ given by

$$((O,\boldsymbol{\nu};\dot O,\dot{\boldsymbol{\nu}}),\mathbf{x}_a,\mathbf{x}_a{}^1)\mapsto(O\mathbf{x}_a+\boldsymbol{\nu};O\mathbf{x}_a{}^1+\dot O\mathbf{x}_a+\dot{\boldsymbol{\nu}})\,,\quad\text{(A3)}$$

is obtained. We find lower-derivative Lagrangian $L:TQ\to I\!R$ which admits the transformations belonging to the first-order prolongation TE of Euclidean group E of space translations and space rotations.

A symmetrical Lagrangian has to satisfy the infinitesimal symmetry conditions $Y^{rot}_{(ks)}L=0$ and $Y^{tr}_{(ks)}L=0$, $s=0,1$. Vector fields

$$Y^{rot}_{(k0)}=\epsilon^i{}_{kj}x_{aj}\frac{\partial}{\partial x_{ai}}+\epsilon^i{}_{kj}\dot x_{aj}\frac{\partial}{\partial\dot x_{ai}}\,,\quad Y^{rot}_{(k1)}=\epsilon^i{}_{kj}x_{aj}\frac{\partial}{\partial\dot x_{ai}}\,,\quad\text{(A4)}$$

$$Y^{tr}_{(k0)}=\sum_{a=1}^{2}\frac{\partial}{\partial x_{ak}}\,,\qquad Y^{tr}_{(k1)}=\sum_{a=1}^{2}\frac{\partial}{\partial\dot x_{ak}}\,,\quad\text{(A5)}$$

generate the infinitesimal rotations and translations of TQ. Invariant Lagrangian has the following form

$$L(\mathbf{x}_a,\dot{\mathbf{x}_a})=L(\mathbf{r}^2,(\mathbf{r}\dot{\mathbf{r}}))\,,\quad\text{(A6)}$$

where $r:=|\mathbf{r}|=|\mathbf{x}_1-\mathbf{x}_2|$ is a distance between point particles. There are six zero-valued constants of motion

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2\,,\quad\mathbf{J}=[\mathbf{x}_1,\mathbf{p}_1]+[\mathbf{x}_2,\mathbf{p}_2]\,,\quad\text{(A7)}$$

associated with vector fields $Y^{tr}_{(k0)}$ and $Y^{rot}_{(k0)}$, respectively (see eqs.(4.2)). Here symbols p_{ai} denote the functions $\partial L/\partial\dot x_{ai}$ defined on tangent bundle TQ. On the Hamiltonian level p_{ai} are canonical momenta and all the relations (A5) are primary first-class constraints.

It is convenient to use center-of-mass variables

$$\mathbf{R}=\frac{m_1\mathbf{x}_1+m_2\mathbf{x}_2}{m_1+m_2}\,,\quad\mathbf{r}=\mathbf{x}_1-\mathbf{x}_2\quad\text{(A8)}$$

with further applying of spherical coordinates $r_1=r\cos\phi\sin\theta$, $r_2=r\sin\phi\sin\theta$, and $r_3=r\cos\theta$ in internal subspace of Q. In this coordinate system Lagrangian function (A4) has the form $L(r,\dot r)$. Hence, trajectory of the distance variable r is determined by given initial condition as in standard Cauchy problem only; time evolution of the others $\{R_1,R_2,R_3,\phi,\theta\}$ is arbitrary.

It is interesting to compare this dynamical system with one determined by Galileo-invariant Lagrangian, say, leading to Newtonian mechanics:

$$L=\sum_{a=1}^{2}\frac{m_a\dot{\mathbf{x}}_a^2}{2}-\frac{\alpha}{\mathbf{r}}\,.\quad\text{(A9)}$$

In spite of constants of motion (A5), the total momentum $\mathbf{P}$ and the angular momentum $\mathbf{J}$ of this dynamical system are arbitrary constants. In general, the trajectory of center-of-mass variable $\mathbf{R}$ is rectilinear. Putting $\mathbf{P} = \mathbf{0}$ we select own inertial frame. It does not lead to restrictions on the dynamics of system. But the choice $\mathbf{J} = \mathbf{0}$ means the very specific case when trajectories of both particles lie on the same straight line.

Having carried out the substitution

$$\begin{aligned} x_{ai} &= O_{ik} y_{ak} + \nu_i \,, \\ \dot{x}_{ai} &= O_{ik} \dot{y}_{ak} + \dot{O}_{ik} y_{ak} + \dot{\nu}_i \,, \end{aligned} \tag{A10}$$

in Lagrangian (A9) we arrive at the singular function

$$\tilde{L} = \sum_{a=1}^{2} \frac{m_a}{2} \left[\dot{y}_{ai} + O_{ki} \dot{O}_{kl} y_{al} + O_{ki} \dot{\nu}_k \right]^2 - \frac{\alpha}{r} \,, \tag{A11}$$

where $r = |\mathbf{y}_1 - \mathbf{y}_2|$ is a distance between particles. It is invariant under the action of transformation (A10). Corresponding to this transformed Lagrangian Hamiltonian function

$$H = \sum_{a=1}^{2} \frac{\boldsymbol{\pi}_a^2}{2m_a} + \frac{\alpha}{r} \tag{A12}$$

is supplemented with the set of primary constraints:

$$\begin{aligned} \Phi^{(1)}_{\nu k} &= \eta_k - \sum_{a=1}^{2} \pi_{ai} O_{ki} \approx 0 \,, \\ \Phi^{(1)}_{\lambda j} &= \xi_j - \sum_{a=1}^{2} \pi_{ai} O_{ki} \frac{\partial O_{kl}}{\partial \lambda_j} y_{al} \approx 0 \,. \end{aligned} \tag{A13}$$

Momenta η_k and ξ_j are canonically conjugated to group parameters ν_k and λ_j, respectively. All the relations (A13) are first-class constraints. The time evolution of variables ν_k and λ_j is completely arbitrary. By fixing their time-dependence we choose the concrete non-inertial frame for description of dynamics of our Galileo-invariant system.

References

1. D.M.Gitman and I.V.Tyutin, *Canonical Quantization of Fields with Constraints*, Nauka, Moscow, 1986 *(in Russian)*.
2. Y.Yaremko, Contact transformations in Wheeler-Feynman electrodynamics, *Ann. Inst. H.Poincaré*, vol. **66**, 3, 1997, p.293–322.

3. D.J.Saunders, *The Geometry of Jet Bundles*, London Math. Soc., Lecture Notes Series **142**, Cambridge Univ. Press, 1989.
4. V.I.Tretyak, *On the gauge transformations in classical mechanics*, Prepr. ITP-91-16Y, Kyiv (Ukraine), 1991 *(in Ukrainian)*.
5. I.Kolář, P.W.Michor, and J.Slovák, *Natural Operations in Differential Geometry*, Springer–Verlag, Berlin Heidelberg, 1993.
6. C.Blazquez and J.Llosa, Classical spin particles: a kinematical approach, *Rep. Math.Phys.*, vol. **30**, 2, 1991, p. 131–148.
7. R.Abraham and J.Marsden, *Foundations of Mechanics*, 2nd ed, Benjamin, New York, 1978.
8. M. de León, P. Pitanga, and P.R.Rodrigues, Symplectic reduction of higher-order Lagrangian systems with symmetry, *J.Math.Phys.*, vol. **35**, 12, 1994, p. 6546–6556.
9. W.M.Tulczyjew, Sur la différentiellee de Lagrange, *C.R.Acad.Sci. Paris*, t.**280**, Sér. A, 1975, p.1295–1298.
10. F.W.Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics **94**, Springer–Verlag, 1983.
11. M.de León and P.R.Rodrigues, *Methods of Differential Geometry in Analytical Mechanics,* North–Holland Math. Studies, Ser. **158**, Amsterdam, 1989.
12. M. de León and D.M. de Diego, Symmetries and constants of the motion for higher-order Lagrangian systems, *J.Math.Phys.*, vol. **36**, 8, 1995, p. 4138–4161.
13. M. de León and P.R.Rodrigues, *Generalized Classical Mechanics and Field Theory,* North–Holland Math. Studies, Ser. **112**, Amsterdam, 1985.
14. M.J.Gotay, J.M.Nester, and G.Hinds, Presymplectic manifolds and the Dirac-Bergmann theory of constraints, *J.Math.Phys.,* vol.**19**, 11, 1978, p. 2388–2399.

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Групи, дотичні до групи Лі, та калібрувальна інваріянтність у класичній механіці

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