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LEGENDRE'S ELLIPTIC INTEGRAL, ASSOCIATED HYPERGEOMETRIC TRANSFORMATION FORMULAS AND EXPLICIT EVALUATIONS

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Еліптичний інтеграл Лежандра, формули гіпергеометричних перетворень і явні вирази для гіпергеометричних функцій

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Анотація. Досліджуються наслідки формули гіпергеометричного перетворення Міньярі Скарпелло і Рітеллі [*Open J. Math. Sci.* **2** 84–92 (2018)] виведеної з результатів, отриманих Лежандром у 1825 році, а також суттєво розширюється область її застосування. Отримуються явні вирази для гіпергеометричних функцій ${}_2F_1$ з різними наборами параметрів і аргументів. Встановлено зв'язок з особливими значеннями еліптичних інтегралів першого роду в теорії сингулярних величин. З'ясовуються взаємозв'язки з гіпергеометричним перетворенням Рамануджана та його розрахунками явних виразів для деяких функцій ${}_2F_1$.

Legendre's elliptic integral, associated hypergeometric transformation formulas and explicit evaluations

D. Ritelli, M.A. Shpot

Abstract. We study the implications of a hypergeometric transformation formula by Mingari Scarpello and Ritelli [*Open J. Math. Sci.* **2** 84–92 (2018)] derived from results given by Legendre in 1825 and essentially extend its validity range. Applications include closed-form evaluations of hypergeometric functions ${}_2F_1$ with different sets of parameters and arguments. Relations to special values of elliptic integrals of first kind in the singular value theory are established. Connections with a hypergeometric transformation due to Ramanujan and his explicit determinations of certain ${}_2F_1$ functions are discussed.

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Еліптичний інтеграл Лежандра, формули гіпергеометричних перетворень і явні вирази для гіпергеометричних функцій

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1. Introduction

The present article discusses the consequences and connections, that are in many ways surprising, of the formula (8) below, proposed by Mingari Scarpello and Ritelli (MSR) in [1]

This formula, as reported in [1, (13)], comes from a reformulation of the elliptic integral

$$Z^{\,|} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt[3]{1 - c^2 \sin^2 \varphi}} \tag{1}$$

studied by Legendre [2, Chap. XXVII, § III], in terms of the Gauss hypergeometric function ${}_2F_1(\frac{1}{3}, \frac{1}{2}; 1; \cdot)$. Legendre, through a series of ingenious transformations of variables one which is nowadays identified as Cauchy-Schlömilch transformation[1] succeeded to represent (1) as a complete elliptic integral of the first kind $\boldsymbol{K}$ defined traditionally as

$$\boldsymbol{K}(k) := \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}\,, \qquad 0 < k < 1. \tag{2}$$

We do not give the details of the procedure followed by Legendre in [2] here, as they are presented in modern terms and notations in [1]. Instead of that we briefly recall his final result, while in Appendix, we give a quite precise translation of the relevant section of the original text by Legendre.

Introducing the complementary modulus b' via $b'^2 + c^2 = 1$ and defining the parameter $n = \sqrt[3]{b'}$ Legendre obtained[2]

$$Z^{\,|} = \frac{3^{3/4}}{\sqrt{\mu' \sqrt{\mu\mu'}}}\, \boldsymbol{K}(k') \tag{3}$$

where

$$\{\mu, \mu'\} = 1 \mp n + n^2 \qquad \text{and} \qquad k'^2 = \frac{1}{2} - \frac{\sqrt{3}}{4} \cdot \frac{\mu'^2 - 3n^2}{\mu' \sqrt{\mu\mu'}}\,. \tag{4}$$

[1]See [3] for some relevant historical references. One of the main ingredients of Legendre's calculation has been the transformation $x \to z$ given by $x^2 + n^2 = x\,z$ in notation of [2, p. 180].

[2]This is a slight reformulation of the Legendre's result $Z^{\,|} = \frac{3}{\sqrt{\mu'\lambda}} F^{\,|} k'$ given in p. 181 of [2] where we took into account that $\lambda^2 = 3\mu\mu'$ and $F^{\,|} k \equiv \boldsymbol{K}(k)$. The values μ, μ' and k' in (4) are precisely those of Legendre. In order to adhere to MSR notation in the following, we renamed the Legendre's b to b'.

In terms of n, the factor in front of $\boldsymbol{K}(k')$ reads

$$c(n) := \frac{3^{3/4}}{\sqrt{\mu'\sqrt{\mu\mu'}}} = \frac{3^{3/4}}{\sqrt{(1+n+n^2)\sqrt{1+n^2+n^4}}}\,, \tag{5}$$

while the argument k' of the elliptic integral $\boldsymbol{K}$ in (3) is defined by

$$a(n) := k'^2 = \frac{1}{2} - \frac{\sqrt{3}}{4}\cdot\frac{1+2n+2n^3+n^4}{(1+n+n^2)\sqrt{1+n^2+n^4}}\,. \tag{6}$$

On the other hand, as it was done in [1], the Legendre's integral $Z^{\rceil}$ can be directly expanded in powers of c^2 which leads to the identifications

$$Z^{\rceil} = \frac{\pi}{2}\,{}_2\mathrm{F}_1\Big(\frac{1}{2},\frac{1}{3};1;c^2\Big) = \frac{\pi}{2}\,{}_2\mathrm{F}_1\Big(\frac{1}{2},\frac{1}{3};1;1-b'^2\Big) = \frac{\pi}{2}\,{}_2\mathrm{F}_1\Big(\frac{1}{2},\frac{1}{3};1;1-n^6\Big).$$

Representing the complete elliptic integral of first kind in (3) in terms of Gauss ${}_2\mathrm{F}_1$ function via

$$\boldsymbol{K}(k) = \frac{\pi}{2}\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;k^2\Big) \tag{7}$$

we obtain, following [1],

$${}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;1-b^6\Big) = c(b)\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;a(b)\Big) \tag{8}$$

where the functions $c(b)$ and $a(b)$ are defined by (5) and (6), respectively, with the replacement $n \mapsto b$.

It is quite interesting to note that in the original calculation by Legendre, the value of k'^2 *from* (6) *is associated with a sine squared of some angle, namely,* $k'^2 = \sin^2\frac{\theta'}{2} = \frac{1}{2}(1-\cos\theta')$.

On the other hand, we shall find in the following that for certain values of b_n*, the function* $a(b_n)$ *will correspond to* x_n*, the squares of special singular moduli* k_n *labelled by integer numbers* n *in the singular value theory. This implies that for the cases of* $n = 3, 6, 9, 12, 15, 24, 27, 75$ *we should be able to associate the values of* x_n *with appropriate trigonometric functions of* θ'_n*. It would be interesting to find out which angles* θ'_n *correspond to associated singular moduli* k_n*. An instance of such relation is the known formula* (12).

We are not sure that the connections of this kind are present in the singular value theory and in associated calculations using Jacobi theta functions or lattice sums.

It is conceivable that the MSR relation (8) extends the family of hypergeometric transformations due to Ramanujan, following from his theories of elliptic functions to alternative bases — see [4], [5, Chap. 33], and [6][3]. From [4, p. 4184, Theorem 5.6], [5, p. 112, Theorem 5.6], [6, (2.21)], we reproduce perhaps the most famous relation:

$$\textit{If}\qquad \alpha(p) := \frac{p^3(2+p)}{1+2p} \qquad \textit{and} \qquad \beta(p) := \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\,, \tag{9}$$

then, for $0 \le p < 1$,

$${}_2F_1\Big(\frac{1}{3},\frac{2}{3};1;\beta(p)\Big) = \gamma(p)\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;\alpha(p)\Big) \quad \textit{where} \;\; \gamma(p) := \frac{1+p+p^2}{\sqrt{1+2p}}\,. \tag{10}$$

The relation (10) involves the same functions (of different arguments) and has the same form as that in (20), which is obtained directly from (8) via a quadratic transformation.

2. Structural properties of equation (8)

In [1, Theorem 2.2], it is asserted that the region of validity for the hypergeometric transformation (8) is $0 < b < 1$. However, below we shall show that the relation (8) has a hidden symmetry with respect to interchanges $b \leftrightarrow b^{-1}$ and is valid indeed for all non-negative real numbers $b \in \mathbb{R}_+$. Moreover, we check that (8) is correct for certain complex values of b.

Although in [1, Theorem 2.2] the values of b are constrained to the interval $0 < b < 1$, it is interesting to look at the function $a(b)$, which is the argument of the Gauss function on the right of (8), without this restriction. It has a very interesting shape displayed in Fig. 2, LEFT.

First of all, we observe that the function $a(b)$ is constrained in the interval $[0,1]$ for all $-\infty < b < \infty$. Thus, the function ${}_2F_1(\frac{1}{2},\frac{1}{2};1;a(b))$ converges (absolutely) for all b except of $b = -1$, for which $a(-1) = 1$ and it diverges ($b = -1$ corresponds to the maximum of the curve $a(b)$ in Fig. 2, LEFT and to the infinite peak of ${}_2F_1(\frac{1}{2},\frac{1}{2};1;a(b))$ in Fig. 2, RIGHT).

Let us write the function $a(b)$ in the form

$$a(b) = \frac{1}{2} - \frac{\sqrt{3}}{4}\,r(b) \qquad \text{where} \qquad r(b) := \frac{b^4+2b^3+2b+1}{(b^2+b+1)\sqrt{b^4+b^2+1}}\,. \tag{11}$$

[3]The last reference presents a concise list of relevant hypergeometric identities in pp. 522–524.

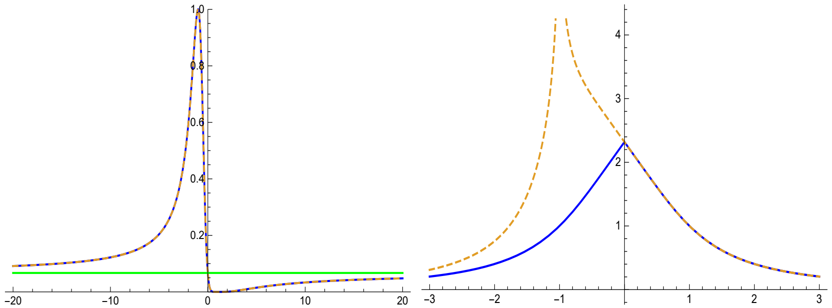

Figure 1. LEFT: Coinciding functions $a(b)$ (blue dots) and $a(b^{-1})$ (dashed yellow); asymptotics $a(\pm\infty) = (2-\sqrt{3})/4$ — green horizontal line. RIGHT: ${}_2F_1$ functions in (8): ${}_2F_1(\frac{1}{3},\frac{1}{2};1;1-b^6)$ (blue) and $c(b)\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;a(b))$ (dashed yellow) — coincide only when $b \geq 0$.

It is straightforward to see that $r(b) = 1$ when $b = 1$ and in the both asymptotic limits $b \to \infty$ and $b \to -\infty$. We have $r(0) = r(\pm\infty) = 1$, and hence, $a(0) = a(\pm\infty) = (2-\sqrt{3})/4$ (see Fig. 2, LEFT).

Moreover, we notice that the value $a(0) = a(\pm\infty) = (2-\sqrt{3})/4 = \sin^2\frac{\pi}{12} = k_3^2$, where (taking into account that $2 \pm \sqrt{3} = (\sqrt{3}\pm 1)^2/2$)

$$k_3 = \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} \tag{12}$$

is the third ($n = 3$) singular modulus of the elliptic integral $\boldsymbol{K}(k_3)$. Here we refer to the singular value theory of elliptic integrals of the first kind going back to seminal work of Selberg and Chowla [7], Watson[4], and Ramanujan [5].

The equality of the values $a(0)$ and $a(\infty)$ is a special instance of a more general symmetry: The function $r(b)$, and hence $a(b)$ are invariant under the inversion of the parameter b, $b \leftrightarrow 1/b$. We have $a(b) = a(b^{-1})$ for any real $b \in]-\infty,\infty[$, while for all non-negative $b \in [0,\infty[$, the function $a(b)$ is constrained in the interval $[0,(2-\sqrt{3})/4]$. The treatment of the relation (8) with $b > 1$ will be considered in the section 4.1.

Further, the coefficient $c(b)$ on the right of (8) obeys the transformation rule $c(b) = b^{-2}c(b^{-1})$ under the inversion of b.

Finally, we notice the different symmetries of the left- and right-hand sides in (8) — see Fig. 2, RIGHT. On the left of (8) the function is even,

[4]G. N. Watson published a series of six papers on singular moduli in 1932–1937, starting with [8].

while on the right the symmetry with respect to $b \leftrightarrow -b$ is broken. Thus, while the both functions coincide for $b \in [0,\infty[$, the equality(8) cannot hold for $b < 0$ as it stands.

The necessary condition for the validity of (8) is that the ${}_2F_1$ functions on the both sides of this equation converge. Thus, given that $b \geq 0$, this necessary condition is

$$|1-b^6| \leq 1\,, \qquad \text{whence} \qquad b \in [\,0, 2^{1/6}]\,. \tag{13}$$

We can simply do a mirror reflection of the $b \geq 0$ branch of the function $c(b)\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;a(b))$ with respect to ordinate, by changing $b \to -b$ in it, to find its counterpart for negative values of b and provide an exact matching of both sides of (8). More explicitly, the symmetric version of (8) for even functions on both sides of this equation would read

$${}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;1-b^6\Big) = c(|b|)\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;a\big(|b|\big)\Big) \tag{14}$$

where we merely replaced b by its absolute value $|b|$ on the right.

However, we still remain with an interesting question related to an inverse problem:

If we have the function $G_2(b) := c(b)\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;a(b))$ and consider it with negative b and $a(b) > x_3 = (2-\sqrt{3})/4$ (dashed yellow curve in the second quadrant of Fig. 2, RIGHT), how should we modify the left-hand side of equation (8) to express the function G_2 in terms of ${}_2F_1$ functions with different sets of parameters?[5]

As we shall see in subsequent sections, with $b > 0$ we have access via (8) to arguments $x_n \equiv a(b_n)$ and associated special values of elliptic integral of the first kind of singular moduli $k_n = \sqrt{x_n}$ with $n \geq 3$. Finding an answer to the above question would provide us with a counterpart of the MSR relation capable of giving access to the basic cases $n = 1$ and $n = 2$ of the singular value theory of elliptic integrals.

Our final note is, that with regard to the above question, there is a subtlety: We encounter different situations depending of whether the MSR relation is expressed in terms of b (the same as n in notation of Legendre) or in terms of b' (which appeared as $b'^2 = 1-c^2$ just above the equation (3) and has been originally denoted as b by Legendre).[6]

[5]See also the remark after the equation (18).

[6]In this respect the notation b in the MSR equation is unfortunate because it is in conflict with the original b of Legendre with another meaning.

In the case of the MSR relation in the form (8), matching the both sides of equation at $b<0$ is achieved by the simple change $b \mapsto |b|$ implied by the symmetry (see Fig. 2, RIGHT and (14)).

On the other side, if we express the same relation in terms of b' (which is the content of [1, (27),(28)]), namely

$$ {}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;1-b'^2\Big) = \frac{3^{3/4}\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;a\big(b'^{1/3}\big)\Big)}{\sqrt{(1+b'^{1/3}+b'^{2/3})\sqrt{1+b'^{2/3}+b'^{4/3}}}}\,, \tag{15}$$

we see that with $b'<0$ there appear a dependence on *complex number* $b'^{1/3}$ in the right-hand side, the function ${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;a\left(b'^{1/3}\right)\right)$ becomes complex-valued. At the same time, the even function on the left remains unchanged. There is no simple means to repair the equality as in the previous case. The situation with $b'<0$ and complex $b'^{1/3}$ requires a special consideration.

3. Simplest special cases

Special cases related to simple specific values of the parameter b and the function $a(b)$ are:

- $b=-1$, $a(-1)=1$: Having a zero parametric excess $s=1-\frac{1}{2}-\frac{1}{2}$, the ${}_2F_1$ function on the right of (8) diverges, the Gauss theorem[7] for the unit argument does not apply. At the same time, on the left, the ${}_2F_1$ function of zero argument trivially becomes 1. The formula (8) does not hold for $b=-1$, as well as for any other negative b.
- $b=0$, $a(0)=(2-\sqrt{3})/4$: On the left-hand side of (8), the function ${}_2F_1(\frac{1}{3},\frac{1}{2};1;1)$ is exactly evaluated by the Gauss theorem[7]. On the right, the argument of ${}_2F_1(\frac{1}{2},\frac{1}{2};1;a(0))$ belongs to the set of "exceptional" points of the singular value theory of elliptic integrals of the first kind. We have $a(0)=x_3=k_3^2=(2-\sqrt{3})/4$, where $k_3=\sin\frac{\pi}{12}=(\sqrt{3}-1)/2\sqrt{2}$. And k_3 is the third singular value of the modulus k of $\boldsymbol{K}(k)$ from the special set $\{k_n|n\in\mathbb{N}\}$, for which $\boldsymbol{K}(k_n)$ are expressible in terms of products of Euler's Gamma functions [7,9,10].

Thus we have

$$ {}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;1\Big) = \frac{3}{2^{4/3}\pi^2}\,\Gamma^3\Big(\frac{1}{3}\Big),$$

while $\quad c(0)=3^{3/4}\quad$ and $\quad {}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;x_3\Big) = \frac{3^{1/4}}{2^{4/3}\pi^2}\,\Gamma^3\Big(\frac{1}{3}\Big).$

It is easy to see that the equation (8) holds for $b=0$, and this point has to be included into its range of validity (cf. [1, p.88, top]).

We see that in the simplest possible special case $b=0$ belonging to the validity range of (8) just a simple application of the Gauss theorem to the function ${}_2F_1(\frac{1}{3},\frac{1}{2};1;1)$ immediately yields the known singular value of the elliptic integral $\boldsymbol{K}(k_3)$ on the right! This is a non-trivial value which has been obtained in the past several times using different techniques [11, p. 555], [7, p.102], [9,12] while the problem of calculating the related integrals goes back to Legendre [2, Chap. XI]. Though Legendre has discovered the relation $\boldsymbol{K}(k_3')=\sqrt{3}\,\boldsymbol{K}(k_3)$ [2, p.60] (cf. (30)), he was apparently not aware of the explicit value of $\boldsymbol{K}(k_3)$.

- $b=1$, $a(1)=0$: Here the both ${}_2F_1$ functions in (8) degenerate to 1, and, with $c(1)=1$, the equation is trivially fulfilled. The value $b=1$ also belongs to the validity range of (8) (cf. [1, p.88, top]). In fact, at $b=1$ the function $a(b)$ has a very flat minimum, quite symmetric in a close proximity to $b=1$. Accordingly, the elliptic integral on the right of (8) has also a very similar shape in the vicinity of $b=1$.

When we escape from the interval $0<b<1$ and go through $b=1$ to $b_+>1$, the argument of ${}_2F_1(\frac{1}{3},\frac{1}{2};1;1-b^6)$ becomes negative and there is still numerical evidence that the relation (8) remains valid. With $1<b\le 2^{1/6}$, this function converges, while for $b>2^{1/6}$ it is defined via analytic continuation. This analytic continuation is provided by the right-hand side of (8): The function ${}_2F_1(\frac{1}{2},\frac{1}{2};1;a(b))$ is convergent and real-valued when $b>2^{1/6}$, since its argument $a(b)<1$ in this range. The coefficient $c(b)$ is also real and positive. Their product defines the real-valued function ${}_2F_1(\frac{1}{3},\frac{1}{2};1;1-b^6)$ with arguments less than -1.For example, a direct substitution of $b=3^{1/3}$ into (8) leads to the relation

$$ {}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;-8\Big) = \frac{3}{2}\left(\frac{5\cdot 3^{1/3}-7}{6}\right)^{\frac{1}{4}} {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{16-\sqrt{6\big[32+35\cdot 3^{1/3}(5\cdot 3^{1/3}-7)\big]}}{32}\right).$$

- $b=\infty$, $a(\infty)=(2-\sqrt{3})/4$: Taking into account that $\lim_{b\to\infty}a(b):=x_3=k_3^2$ (see (12)) and $c(b\to\infty)\simeq 3^{3/4}b^{-2}$ we can verify the correctness of the relation (17) from the next section in the asymptotic limit $b\to\infty$ precisely in the same way as in the case $b=0$ just above.

[7]
$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(s)}{\Gamma(c-a)\Gamma(c-b)}\,,\qquad c\neq 0,-1,-2,\ldots,\quad s:=c-a-b,\quad \operatorname{Re}s>0.$$

This is a simplest manifestation of the symmetry with respect to interchanges $b \leftrightarrow b^{-1}$ noticed in sec. 2. In the next section we are going to deduce a more generic statement related to this symmetry.

4. Alternative forms of (8)

In this section we transform the Gauss function ${}_2F_1(\frac{1}{3}, \frac{1}{2}; 1; 1-b^6)$ on the left-hand side of (8) in order to produce some alternative versions of this equation.

4.1. Linear Pfaff transformation

First of all, we use the linear Pfaff transformation [13, p. 60, (4)] [14, p. 68, (2.2.6)], [15, 7.3.1.3], [16, (15.8.1)]

$$ {}_2F_1(\alpha, \beta; \gamma, z) = (1-z)^{-\alpha}\, {}_2F_1\Big(\alpha, \gamma-\beta; \gamma; \frac{z}{z-1}\Big), \quad |z| < 1, \quad \left|\frac{z}{1-z}\right| < 1, \tag{16} $$

and obtain the following alternative representation of (8):

$$ {}_2F_1\Big(\frac{1}{3}, \frac{1}{2}; 1; 1-b^{-6}\Big) = b^2 c(b)\, {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; a(b)\Big). \tag{17} $$

There are not much modifications but they are important: in the argument of the Gauss function on the left, b^6 is replaced by b^{-6}, and this allows consideration of $b > 1$. The additional factor b^2 that appears in front of $c(b)$ moderates the asymptotic behavior $c(b) \sim b^{-2}$ at large b and provides a finite $b \to \infty$ limit on the right.

When $b \in [1, 2^{1/6}]$, the linear transformation (16) transforms negative arguments of the original function $1 - b^6 \in [-1, 0]$ to positive arguments of the transformed function $1-b^{-6} \in [0, \frac{1}{2}]$. The both functions converge.

For $b > 2^{1/6}$, there is a mapping of $1-b^6 < -1$ to $1-b^{-6} \in]\frac{1}{2}, 1]$. Thus, the original function ${}_2F_1(\frac{1}{3}, \frac{1}{2}; 1; 1-b^6)$ with "large" negative arguments is analytically continued to a convergent function ${}_2F_1(\frac{1}{3}, \frac{1}{2}; 1; 1-b^{-6})$ which appears in the modified version (17) of (8).

The validity of equation (8) is thus confirmed in the whole region $b \in [0, \infty[$, which essentially extends the validity range $0 < b < 1$ originally announced in [1, Theorem 2.2].

In summary, we formulate the following theorem:

For any $b \in \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$, the function ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; a(b)\right)$ is the same for two different[8] values b and b^{-1} and relates to the hypergeometric function ${}_2F_1(\frac{1}{3}, \frac{1}{2}; 1; 1-b^6)$ and its Pfaff transform ${}_2F_1(\frac{1}{3}, \frac{1}{2}; 1; 1-b^{-6})$, respectively, via

$$ {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; a(b)\Big) = \begin{cases} [c(b)]^{-1}\, {}_2F_1\Big(\frac{1}{3}, \frac{1}{2}; 1; 1-b^6\Big), & 0 \leq b \leq 2^{1/6}; \\ \big[b^2 c(b)\big]^{-1}\, {}_2F_1\Big(\frac{1}{3}, \frac{1}{2}; 1; 1-b^{-6}\Big), & b \geq 2^{-1/6}. \end{cases} \tag{18} $$

With indicated restrictions on the parameter b, all hypergeometric series in (18) are convergent.

In the overlapping region $b \in [2^{-1/6}, 2^{1/6}]$, the both functions ${}_2F_1(\frac{1}{3}, \frac{1}{2}; 1; 1-b^6)$ and ${}_2F_1(\frac{1}{3}, \frac{1}{2}; 1; 1-b^{-6})$ are convergent simultaneously.

For negative b, the statement (18) can be augmented by the equation (14) where the function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; a(b))$ has been forced to be even by the replacement $b \mapsto |b|$.

At the same time, the question put in the end of the section 2 remains open and means: If we consider the negative values of b in ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; a(b))$ without changing b to its absolute value, what should be the right-hand side of the equation (18) in order for it to be satisfied with $b \in]-\infty, 0[$ and $(2-\sqrt{3})/4 < a(b) < 1$?

4.2. Quadratic transformations

Applying the quadratic transformation [17, p. S.120, (46)], [15, 7.3.1.68]

$$ {}_2F_1(\alpha, \beta; 2\beta, z) = (1-z)^{-\alpha/2}\, {}_2F_1\Big(\alpha, 2\beta-\alpha; \beta + \frac{1}{2}; -\frac{(1-\sqrt{1-z})^2}{4\sqrt{1-z}}\Big) \tag{19} $$

to the left-hand side of (8) we obtain the relation

$$ {}_2F_1\Big(\frac{1}{3}, \frac{2}{3}; 1; -\frac{(1-b^3)^2}{4b^3}\Big) = b\, c(b)\, {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; a(b)\Big) \tag{20} $$

valid for *negative* arguments[9] $z(b) = -4b^{-3}(1-b^3)^2$ of the function ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$ corresponding to positive values of b.

It is remarkable that in distinction to (8) and (18), the equation (20) is manifestly invariant under the inversion of b: It is easy to see that $z(b) = z(b^{-1})$ (see Figure 4.2) and $bc(b) = b^{-1}c(b^{-1})$, as well as $a(b) = a(b^{-1})$ which we noticed before. It is useful to express the original

[8] Except for $b = 1$, when b and b^{-1} coincide.

[9] The similar Ramanujan-related formula (10) holds, by contrary, for strictly positive arguments of ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$.

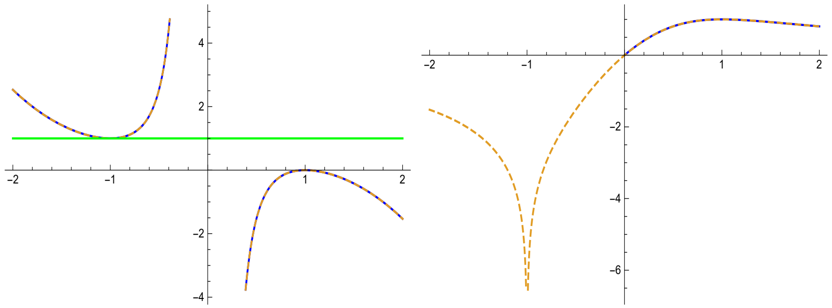

Figure 2. LEFT: Blue-yellow curve: $z(b) = -4b^{-3}(1-b^3)^2$ and $z(b^{-1}) = -4b^3(1-b^{-3})^2$, green horizontal line: $z = 1$. RIGHT: Functions in (20): ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; -4b^{-3}(1-b^3)^2)$ — blue, $b\, c(b)\, {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; a(b))$ — dashed yellow.

equation (8) in terms of this function because in the following we will be interested in its special evaluations.

We recall that the function ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$ is an important ingredient of Ramanujan's, theories of elliptic functions to alternative bases from his notebooks [5] and attracted essential attention in subsequent works — see p. 3. The formula (20) appears to have precisely the same general form as the transformation (10) of Ramanujan.

In a following publication we shall also introduce other relations of similar kind, where the function ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$ appears with different functional dependence in its argument.

Using the quadratic transformation [17, p. S.120, (47)], [15, 7.3.1.69]

$$
\begin{aligned}
{}_2F_1(\alpha, \beta; 2\beta; z) =& \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2a} \\
& {}_2F_1\left(\alpha, \alpha-\beta+\frac{1}{2}; \beta+\frac{1}{2};\ \left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right)
\end{aligned}
\tag{21}
$$

we obtain

$$
{}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; \left(\frac{1-b^3}{1+b^3}\right)^2\right) = \left(\frac{1+b^3}{2}\right)^{2/3} c(b)\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; a(b)\right). \tag{22}
$$

Here appears a "supersymmetric" ${}_2F_1$ function with equal numerator parameters, $\alpha = \beta = 1/3$, and a strictly positive, for any b, argument obeying the symmetry $b \leftrightarrow 1/b$. However, the argument $(1-b^3)^2(1+b^3)^{-2} > 1$ for $b < 0$ and thus we remain, as before, within the region

$b \geq 0$. Again, similarly as (20), the relation (22) is manifestly symmetric with respect to inversions $b \leftrightarrow b^{-1}$.

In fact, the function ${}_2F_1(\frac{1}{3}, \frac{1}{3}; 1; z)$ from the last equation can be alternatively obtained by applying the linear transformation (16) to the function ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$ in (20).

We also observe that the right-hand side of the equation (22) has no divergence for negative b in contrast to the previous cases — cf. Figs. 4.2, 4.2 and 2.

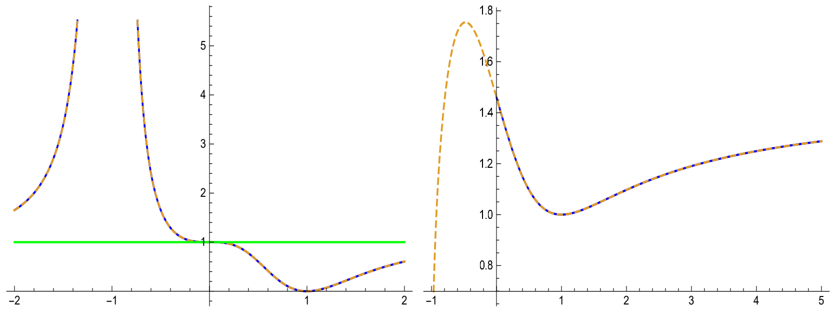

Figure 3. LEFT: Blue-yellow curve: $(1-b^3)^2/(1+b^3)^2$ and $(1-b^{-3})^2/(1+b^{-3})^2$, green horizontal line: $z = 1$. RIGHT: Functions involved in (22): ${}_2F_1(\frac{1}{3}, \frac{1}{3}; 1; (1-b^3)^2/(1+b^3)^2$ — blue, $2^{-2/3}(1+b^3)^{2/3}\, c(b)\, {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; a(b))$ — dashed yellow.

One could try to do some similar transformations on the right-hand side of (8), however, it is not clear which ones could lead to some esential improvement or simplification.

5. Evaluation of elliptic integrals

(i) As a special case of the formula by Spiegel (1962) [18, I.] (see also [15, 7.3.9.35])

$$
{}_2F_1\left(\frac{1}{2}, p; 3p; \frac{3}{4}\right) = \left(\frac{16}{27}\right)^p \frac{\Gamma(p)\Gamma(3p)}{\Gamma^2(2p)} \tag{23}
$$

we obtain the evaluation

$$
{}_2F_1\left(\frac{1}{3}, \frac{1}{2}; 1; \frac{3}{4}\right) = \frac{\Gamma^3(\frac{1}{3})}{2^{2/3}\pi^2} \tag{24}
$$

and use it in the equation (8). Equating the argument of $1-b^6$ with $3/4$ we obtain

$$b_{3/4} = 2^{-1/3} \tag{25}$$

(the remaining five solutions are outside the validity range $b \geq 0$ of (8) being either negative or complex).

For this special value of b, the argument of the elliptic integral on the right of (8) is[10]

$$a(2^{-1/3}) := x_{27} = \frac{1}{4}\left(2 - \sqrt{3 - 100 \cdot 2^{1/3} + 80 \cdot 2^{2/3}}\right) \tag{26}$$

and its coefficient c is given by

$$c(2^{-1/3}) = \sqrt{6}\,(8 - 5 \cdot 2^{2/3})\,. \tag{27}$$

Thus we obtain the evaluation

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2} - \frac{1}{4}\sqrt{3 + 5(2^{7/3} - 5)2^{7/3}}\right) = \frac{\Gamma^3(\frac{1}{3})}{2^{4/3}\sqrt{3}\,(2^{7/3}-5)^{1/4}\,\pi^2}\,, \tag{28}$$

or, in terms of the elliptic integral K,

$$\boldsymbol{K}(k_{27}) = \frac{\Gamma^3(\frac{1}{3})}{\pi\sqrt{3}\,(2^{7/3}-5)^{1/4}\,2^{7/3}}\,, \tag{29}$$

where $\qquad k_{27} = \sqrt{x_{27}} = \dfrac{1}{2}\sqrt{2 - \sqrt{3 + 5(2^{7/3} - 5)2^{7/3}}}\,.$

To check whether this result belongs to the set of the so-called singular values of K we perform a numerical evaluation of the combination

$$\left(\frac{\boldsymbol{K}(k')}{\boldsymbol{K}(k)}\right)^2 \stackrel{?}{=} n \quad \text{where, as usual,} \quad k' = \sqrt{1-k^2}\,, \tag{30}$$

with $k := k_{27}$. In case this combination equals $n \in \mathbb{N}$, we have to deal with the nth singular value $\boldsymbol{K}(k_n)$ as a function of the special modulus k_n.

For the elliptic integral in (29) we obtain thus $n = 27$ and compare it with the entry $K[27]$ in the table of singular values given in [10, Appendix A.3]. Equations (28) and (29) are alternative representations of this entry given by

$$\begin{aligned} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x_{27}\right) &= \frac{(2^{2/3}+2)^2\Gamma^3(\frac{1}{3})}{12 \cdot 2^{2/3}3^{1/4}\,\pi^2}\,, \\ V_{27} \equiv 4x_{27}(1-x_{27}) &= \frac{1}{4} + 25 \cdot 2^{1/3} - 20 \cdot 2^{2/3}. \end{aligned} \tag{31}$$

Indeed, with our explicit expression for x_{27} from (26) we reproduce the value V_{27} in (31). Also, our result (28) is equivalent to that in (31) up to the identification

$$\frac{(2^{2/3}+2)^2}{2^{4/3}\,3^{3/4}} = \frac{1}{(2^{7/3}-5)^{1/4}} \tag{32}$$

confirmed analytically by Mathematica [19].

(ii) Another interesting evaluation of ${}_2F_1(\frac{1}{3}, \frac{1}{2}; 1; z)$ is due to Ebisu [20, (E$''$.1)]:

$${}_2F_1\left(\frac{1}{3}, \frac{1}{2}; 1; \frac{1}{5}\right) = \frac{2^{\,2/3}\,3\,\sqrt{5}}{20\pi^2}\,\Gamma^3\left(\frac{1}{3}\right). \tag{33}$$

Again, solving the equation $1 - b^6 = 1/5$ we obtain the relevant solution

$$b_{1/5}^6 = \frac{4}{5} \quad \text{and} \quad b_{1/5} = \frac{2^{1/3}}{5^{1/6}}\,. \tag{34}$$

With this value of b,

$$a(b_{1/5}) = x_{75} = \frac{1}{2} - \frac{5 + 4\sqrt{5} + 2 \cdot 10^{1/3}\left(1+\sqrt{5}\right)}{4\left(5 + 2^{1/3}\,5^{5/6} + 10^{2/3}\right)}\,\sqrt{15 - 3 \cdot 10^{2/3}}\,, \tag{35}$$

and the coefficient c is

$$c(b_{1/5}) = \frac{3^{3/4}\,\sqrt{5}\,(5 - 10^{2/3})^{1/4}}{\sqrt{5 + 2^{1/3}\,5^{5/6} + 10^{2/3}}}\,. \tag{36}$$

With these data, we obtain from (8) the evaluation

$$\begin{aligned} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x_{75}\right) = \frac{2^{-1/3}}{10\pi^2}\left(\frac{3}{5}\right)^{1/4} &\left(5 + 2 \cdot 10^{1/3} + 10^{2/3}\right)^{1/4} \\ &\sqrt{5 + 2^{1/3}\,5^{5/6} + 10^{2/3}}\;\Gamma^3\left(\frac{1}{3}\right). \end{aligned} \tag{37}$$

[10]The appearance of the notation x_{27} here will become evident quite soon.

In terms of the elliptic integral K,

$$\boldsymbol{K}(k_{75}) = \frac{\Gamma^3(\frac{1}{3})}{20\pi 2^{1/3}}\Big(\frac{3}{5}\Big)^{1/4}\big(5{+}2{\cdot}10^{1/3}{+}10^{2/3}\big)^{1/4}\sqrt{5+2^{1/3}\,5^{5/6}+10^{2/3}}\,, \tag{38}$$

where $k_{75} = \sqrt{x_{75}}$ with x_{75} defined in (35).

Performing the numerical check (30) we see that the last two results indeed correspond to the case $n = 75$ of the singular value theory and has to be compared with the entry $K[75]$ of the table [10, Appendix A.3] where we find only the value of $K[75]$ but no information on k_{75} or x_{75}. In [5, p. 192] and in the Table of singular moduli by Petrović [21] we find a rather involved value of G_{75},

$$G_{75} = 3\cdot 2^{5/12}\Big(\frac{\sqrt{5}+1}{2}\,10^{1/3} + \frac{\sqrt{5}-1}{2}\,4^{1/3}\cdot 5^{1/6} - \sqrt{5} - 1\Big)^{-1}, \tag{39}$$

a special representative of the so-called class invariants G_n related to k_n via non-trivial relation

$$k_n = \frac{G_n^{-12}}{\sqrt{2}\sqrt{1+\sqrt{1-G_n^{-24}}}} = \frac{1}{\sqrt{2}}\sqrt{1-\sqrt{1-G_n^{-24}}}\,. \tag{40}$$

Hence it seems that our expression for $k_{75} = \sqrt{a(b_{1/5})}$ with $a(b_{1/5})$ from (35) represents the simplest known explicit expression for the singular modulus k_{75}. We have checked, with the help of Mathematica [19], that our result (35) indeed follows from (40) using (39).

From the value of $K[75]$ in [10, Appendix A.3] we infer

$${}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;x_{75}\Big) = \Big(5{+}2{\cdot}10^{1/3}{+}\big(5{+}3\sqrt{5}\,\big)10^{-1/3}\Big)\frac{\Gamma^3(\frac{1}{3})}{5\cdot 2^{4/3}\,3^{3/4}\pi^2} \tag{41}$$

This result agrees with our expression in(37) provided that

$$\frac{5+2\cdot 10^{1/3}+\big(5+3\sqrt{5}\,\big)10^{-1/3}}{3\cdot 5^{-1/4}\sqrt{5+2^{1/3}\cdot 5^{5/6}+10^{2/3}}\,(5+2\cdot 10^{1/3}+10^{2/3})^{1/4}} = 1 \tag{42}$$

which is true, according to a calculation with Mathematica [19].

(iii) For our next example we introduce the *golden ratio* (see e. g. [22])

$$\phi = \frac{1+\sqrt{5}}{2} \quad \text{and its inverse} \quad \phi^{-1} = \frac{\sqrt{5}-1}{2}\,. \tag{43}$$

The value ϕ and its negative inverse $-\phi^{-1} = (1-\sqrt{5})/2$ are solutions to the quadratic equation $x^2 - x - 1 = 0$.

The entry (xxiii) on p. 53 of Ebisu's book [23] represents an explicit evaluation of ${}_2F_1(a, 1-a; 2-\frac{3}{2}\,a; -\phi^{-1})$. With $a = 2/3$ it reduces to

$${}_2F_1\Big(\frac{1}{3},\frac{2}{3};1;-\phi^{-1}\Big) = \frac{5^{5/6}}{3}\,\phi^{-1}\,\frac{\Gamma(\frac{1}{3})\Gamma(\frac{4}{5})\Gamma(\frac{6}{5})}{\Gamma(\frac{7}{15})\Gamma(\frac{2}{3})\Gamma(\frac{13}{15})\Gamma(\frac{4}{3})}\,. \tag{44}$$

Eventually, we could use the formula (8) in the form (20) to proceed, but it is interesting to transform the parameters of ${}_2F_1$ appearing in (44) to that of the original equation (8). This can be achieved by means of the quadratic transformation [15, 7.3.1.49]

$$\begin{aligned}{}_2F_1\Big(\alpha,\beta;\frac{\alpha{+}\beta{+}1}{2};z\Big) &= \big(\sqrt{1-z}+\sqrt{-z}\,\big)^{-2\alpha} \\ &\qquad {}_2F_1\left(\alpha,\frac{\alpha{+}\beta}{2};\alpha{+}\beta;\frac{4\sqrt{-z(1-z)}}{\big(\sqrt{1-z}+\sqrt{-z}\,\big)^2}\right)\end{aligned} \tag{45}$$

and yields

$${}_2F_1\Big(\frac{1}{3},\frac{2}{3};1;-\phi^{-1}\Big) = \phi^{-1}\,{}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;4\sqrt{5}-8\Big). \tag{46}$$

The argument of the last ${}_2F_1$ function $4\sqrt{5}-8 = 1-\phi^{-6}$ as can be easily checked. It has the same algebraic form as in (8) and thus identifies $b = \phi^{-1}$ by inspection.

Finally, using $b = \phi^{-1}$ in (8) and taking into account (46) and (44) we derive ${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;a(\phi^{-1})\right)$ in a closed form.

However, in doing this we follow Vidūnas [24] and express the gamma functions from (44) in terms of a few basic functions $\Gamma(\frac{1}{3})$, $\Gamma(\frac{1}{5})$, $\Gamma(\frac{2}{5})$, and $\Gamma(\frac{1}{15})$ from the set [24, (5)] by means of the elementary relation $z\Gamma(z) = \Gamma(1+z)$ and corresponding reduction formulas from [24, p. 269–270]. Thus we end up with

$${}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}-\frac{\sqrt{3}}{32}\big(7{+}\sqrt{5}\,\big)\Big) = \frac{2}{\pi}\,\frac{3^{-21/20}5^{-1/3}\sqrt{2(1+\sqrt{5})}}{5+\sqrt{5}+\sqrt{30-6\sqrt{5}}}\,\frac{\Gamma^2(\frac{1}{15})\Gamma(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma(\frac{1}{3})}. \tag{47}$$

Denoting the argument of the last Gauss function as $x_{15} = \frac{1}{2} - \frac{\sqrt{3}}{32}(7+\sqrt{5})$ and expressing the right-hand side in terms of the golden ratio ϕ we rewrite the last result in a compact and elegant form

$${}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;x_{15}\Big) = \frac{3^{-21/20}5^{-7/12}}{2\pi}\Big(\phi\sqrt{\phi\sqrt{5}}-\sqrt{3}\Big)\,\frac{\Gamma^2(\frac{1}{15})\Gamma(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma(\frac{1}{3})}\,. \tag{48}$$

It is of interest to compare our findings in this example with expressions given in [10, Appendix A.3] where the result for K[15] is given in a quite different form. From [10, Appendix A.3] (see also [25, p. 350]), the value

$$k_{15} = \frac{\sqrt{3}\,\left(\sqrt{15}-1\right)-\sqrt{15}-1}{8\sqrt{2}} = \frac{\sqrt{15}\,\left(\sqrt{3}-1\right)-\sqrt{3}-1}{8\sqrt{2}} \tag{49}$$

can be inferred, which is symmetric with respect to an interchange of $\sqrt{3}$ and $\sqrt{15}$ and appears to be simpler as

$$k_{15} = \frac{1}{4\sqrt{2}}\sqrt{16-\sqrt{3}(7+\sqrt{5})} \tag{50}$$

which follows from our last calculation.[11]

The value of the elliptic integral itself appears in [10, Appendix A.3] as

$$\boldsymbol{K}(k_{15}) = \frac{\sqrt{6\left(5+\sqrt{5}\right)}}{60}\,\frac{b(\frac{1}{15})\,b(\frac{4}{15})}{b(\frac{5}{15})} \tag{51}$$

where the function $b(p)$ is defined via [10, p. 76]

$$b(p) := \frac{\Gamma^2(p)}{\Gamma(2p)}\sqrt{\tan(p\pi)}\,. \tag{52}$$

Using this definition along with reduction formulae for Γ functions from [24] it should be possible to reproduce our result (48) starting from (51). We have checked the agreement between these two expressions numerically.

6. Special values of Gauss hypergeometric functions

It is also possible to use the MSR relation (8) in a reverse direction, and thus derive some special values of the Gauss function ${}_2F_1(\frac{1}{3},\frac{1}{2};1;z)$ in the cases when the elliptic integral on the right of (8) is known.

As an example let us consider the $n = 9$ singular value [9, p. 117], [26, p. 259], [10, Appendix A.3], [25]

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x_9\right) = \frac{1+\sqrt{3}}{(2\sqrt{3}\,\pi)^{3/2}}\,\Gamma^2\left(\frac{1}{4}\right) \tag{53}$$

[11] In [21], k_{15} has an essentially more complicated form.

where $\qquad x_9 = k_9^2 \qquad$ and $\qquad k_9 = \dfrac{\sqrt{2}-\sqrt[4]{3}}{1+\sqrt{3}}\,.$

Fortunately, Mathematica [19] is able to solve the equation $a(b) = x_9$. The root lying in the interval $(0, 1)$ is

$$b_9 = \frac{1}{2}\left(1+\sqrt{3}-\sqrt{2\sqrt{3}}\right). \tag{54}$$

Another real root is $b_9^{-1} = \frac{1}{2}\left(1+\sqrt{3}+\sqrt{2\sqrt{3}}\right) > 1$.[12] Both these roots are relevant for the MSR equation cast in the form (18).

With $b = b_9$, we obtain from (8) and (53)

$${}_2F_1\left(\frac{1}{3},\frac{1}{2};1;y_9\right) = \frac{\sqrt{1+\sqrt{2}/\sqrt[4]{3}}}{2\pi^{3/2}}\,\Gamma^2\left(\frac{1}{4}\right) \tag{55}$$

where the argument

$$y_9 = \frac{3}{\sqrt{2}}\left(13{\cdot}3^{1/4}+23{\cdot}3^{-1/4}\right)-36-21\sqrt{3} = \frac{2^{3/2}\cdot 3^{3/4}}{5-\sqrt{3}+2^{1/2}\cdot 3^{3/4}}\,. \tag{56}$$

The parameters of the last ${}_2F_1$ function are of the form $(\alpha, \beta; 2\beta)$, and we can apply to this function the quadratic transformations [15, 7.3.1.64–69]. In particular, the transformation (21) leads to the beautiful result

$${}_2F_1\left(\frac{1}{3},\frac{1}{3};1;\frac{\sqrt{3}+2}{3\sqrt{3}}\right) = \frac{3^{3/8}}{2^{1/12}(2\pi)^{3/2}}\,(\sqrt{3}-1)^{1/6}\,\Gamma^2\left(\frac{1}{4}\right) \tag{57}$$

where the argument of the Gauss function is essentially simpler than that in (55)— (56).

The Pfaff transformation (16) of the last ${}_2F_1$ function yields the following instance of the analytic continuation of the function ${}_2F_1(\frac{1}{3},\frac{2}{3};1;z)$:

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{-3\sqrt{3}-5}{4}\right) = \frac{\sqrt{\sqrt{3}-1}}{2^{1/4}\cdot 3^{1/8}\,2\pi^{3/2}}\,\Gamma^2\left(\frac{1}{4}\right). \tag{58}$$

[12] It is interesting to note that the negatives of the roots b_9 and b_9^{-1} are the real solutions of the equation $a(b) = x_1$ with $x_1 = \frac{1}{2}$ corresponding to the case $n = 1$: we have $b_1 = -b_9$ and $b_1^{-1} = -b_9^{-1}$. Though, with $b_1 < 0$, the case $n = 1$ is not covered by (8) (see Figure 2), the MSR equation involves some relation between the cases $n = 1$ and $n = 9$, which still has to be uncovered.

6.1. A digression: Berndt-Chan-Ramanujan evaluations

At this point it is interesting to recall two explicit determinations of ${}_2F_1$ appearing in the parer by Bernd and Chan [27, p.280] and in Brendt's book [5, pp. 327–8] as corollaries of an entry from Ramanujan's notebooks:

$$\textit{If} \qquad p = p_9 := \frac{\sqrt{6\sqrt{3}-9}-1}{2}\,, \tag{59}$$

then

$$_2F_1\Big(\frac{1}{2},\frac{1}{2};1;\alpha(p_9)\Big) = \frac{\sqrt{\pi}}{\sqrt{6\sqrt{3}-9}\,\Gamma^2(\frac{3}{4})} \tag{60}$$

and

$$_2F_1\Big(\frac{1}{3},\frac{2}{3};1;\beta(p_9)\Big) = \frac{\sqrt{\pi}}{2^{1/4}\cdot 3^{1/8}\sqrt{\sqrt{3}-1}\,\Gamma^2(\frac{3}{4})} \tag{61}$$

where

$$\alpha(p_9) = \left(\frac{\sqrt{2}-\sqrt[4]{3}}{1+\sqrt{3}}\right)^2 = x_9 \qquad \textit{and} \qquad \beta(p_9) = \frac{3\sqrt{3}-5}{4}\,, \tag{62}$$

while functions $\alpha(p)$ and $\beta(p)$ are defined in (9), as before.

With the indicated identification $\alpha(p_9) = x_9$, the evaluation (60) is equivalent to that in (53), the singular ${}_2F_1$ value in the case $n = 9$.

The two functions from (60) and (61) satisfy the equation (10) with the coefficient

$$\gamma(p_9) = \frac{3^{5/8}}{2^{1/4}\sqrt{\sqrt{3}+1}}\,. \tag{63}$$

Expressing the original equation (61) in the style of (53) and (58) we write

$$_2F_1\Big(\frac{1}{3},\frac{2}{3};1;\frac{3\sqrt{3}-5}{4}\Big) = \frac{\sqrt{\sqrt{3}+1}}{2^{1/4}\cdot 3^{1/8}(2\pi)^{3/2}}\,\Gamma^2\Big(\frac{1}{4}\Big) \tag{64}$$

which is very similar to (58) obtained via MSR equation.

Applying the Pfaff transformation (16) on the left of the last equation we obtain a companion evaluation to that in (57):

$$_2F_1\Big(\frac{1}{3},\frac{1}{3};1;\frac{\sqrt{3}-2}{3\sqrt{3}}\Big) = \frac{3^{3/8}}{2^{1/12}4\pi^{3/2}}\,(\sqrt{3}+1)^{1/6}\,\Gamma^2\Big(\frac{1}{4}\Big). \tag{65}$$

We learn from the above findings that in the case $n = 9$ the relations (8) and (10) transform the elliptic integral ${}_2F_1(\frac{1}{2},\frac{1}{2};1;x_9)$ into different, albeit "similar" ${}_2F_1$ functions with arguments of the form $a \pm b$ with real a and b. These pairs of companion functions appear in (58) and (64). Their Pfaff transformations yield again the Gauss functions with arguments of the same general form — see (57) and (65).

Just for curiosity's sake we calculate the ratios of these pairs of functions. These are

$$\frac{_2F_1\big(\frac{1}{3},\frac{2}{3};1;\frac{-3\sqrt{3}-5}{4}\big)}{_2F_1\big(\frac{1}{3},\frac{2}{3};1;\frac{3\sqrt{3}-5}{4}\big)} = \sqrt{3}-1 \quad \text{and} \quad \frac{_2F_1\big(\frac{1}{3},\frac{1}{3};1;\frac{\sqrt{3}+2}{3\sqrt{3}}\big)}{_2F_1\big(\frac{1}{3},\frac{1}{3};1;\frac{\sqrt{3}-2}{3\sqrt{3}}\big)} = 2^{1/3}\big(\sqrt{3}-1\big)^{1/3}.$$

7. Some instances of complex b

7.1. $b = \mathrm{e}^{i\pi/6}$

In [28, p. 1895, Theorem 3.1], Mingari Scarpello and Ritelli derived an explicit expression for an analytic continuation of a special Gauss hypergeometric function of argument 2. We transcribe it in the form

$$_2F_1(\alpha,\beta;2\beta,2) = (-i)^{\alpha}\sqrt{\pi}\,\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\frac{\alpha+1}{2})\Gamma(\beta+\frac{1}{2}-\frac{\alpha}{2})}\,, \qquad \alpha > 0. \tag{66}$$

In the special case of $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$ this reduces to

$$_2F_1\Big(\frac{1}{3},\frac{1}{2};1;2\Big) = \frac{(-i)^{1/3}\sqrt{\pi}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})} = \mathrm{e}^{-i\frac{\pi}{6}}\,\frac{3}{2^{7/3}\pi^2}\,\Gamma^3\Big(\frac{1}{3}\Big), \quad \mathrm{e}^{-i\frac{\pi}{6}} = \frac{\sqrt{3}-i}{2}\,. \tag{67}$$

In (8), the argument of the function ${}_2F_1(\frac{1}{3},\frac{1}{2};1;1-b^6)$ becomes 2 when b satisfies the equation $b^6 = -1$ whose solutions are $b = \pm\mathrm{e}^{i\frac{\pi}{2}}$ and $b = \pm\mathrm{e}^{\pm i\frac{\pi}{6}}$. A numerical check shows that the main equation (8) is satisfied only with the solution $\tilde{b} = \mathrm{e}^{i\frac{\pi}{6}}$. For this value of b, the argument of the Gauss function on the right of (8) is

$$a(\tilde{b}) = a\big(\mathrm{e}^{i\frac{\pi}{6}}\big) = \frac{1}{16}\Big(8+3\sqrt{2}-5\sqrt{6}\Big) \lesssim 0\,. \tag{68}$$

Considering the present example we face a situation which differs from that encountered before: As we saw in Fig. 2, real non-negative values of b lead to small positive arguments $a(b)$ lying in the interval $[0,(2-\sqrt{3})/4]$. The upper bound of $a(b)$ has been attributed to the case $n = 3$ of the singular value theory. In the interval $a(b) \in [0,x_3]$ with $x_3 = (2-\sqrt{3})/4$ we found several other instances of singular moduli squared, $a(b_n) \equiv x_n$, with $n = 9$, 15, 27, 75, — all of them greater than

3. In the previous example we had a hint (see footnote 12) that to the smallest n's, $n=1$ and $n=2$, will correspond negative b's and associated singular moduli $a(b_n)$ in the interval $\big((2-\sqrt{3})/4, 1\big)$, in some way related to certain cases with $n>3$.

Now, the complex value $\tilde{b}$ of the parameter b satisfying (8) does not belong to the established region of validity of this equation, $b \geq 0$. The associated argument $a(\tilde{b})$ is negative and thus does not match the picture reproduced in Fig. 2. This argument cannot be associated to any integer n because the check (30) for n yields the complex number $\tilde{n} \simeq 11+6.93i$. With $b=\tilde{b}=\mathrm{e}^{i\frac{\pi}{6}}$, our equation (8) looks like

$$ {}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;2\Big) = \frac{3^{3/4}}{2\sqrt{2}}\,\frac{\sqrt{3}-i}{(2+\sqrt{3})^{1/4}}\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;\frac{1}{16}\big(8+3\sqrt{2}-5\sqrt{6}\big)\Big). \tag{69} $$

Remembering that the Pfaff transformation (16) converts small negative arguments of Gauss functions into small positive ones we apply this transformation to the right-hand side of the last equation. The resulting relation is

$$ {}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;2\Big) = \frac{3^{3/4}\sqrt{2}}{\sqrt{8-3\sqrt{2}+5\sqrt{6}}}\,\frac{\sqrt{3}-i}{(2+\sqrt{3})^{1/4}}\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;x_{12}\Big), \tag{70} $$
$$ x_{12} = \frac{5\sqrt{6}-3\sqrt{2}-8}{5\sqrt{6}-3\sqrt{2}+8}, $$

where x_{12}, the square of the singular modulus k_{12} associated to $n=12$ in (30) appears. In the present form, $k_{12}=\sqrt{x_{12}}$ appears in the handbook by Brychkov, [29, 7.2.16.21]. The case $n=12$ is given also in [29, 7.2.16.18] with $k_{12}=(5-2\sqrt{6})(3-2\sqrt{2})$ with reference to [30] where we find $k_{12}=(\sqrt{3}-\sqrt{2})^2(\sqrt{2}-1)^2$ in Table 1, and in [10, Appendix A.3]. For the corresponding Gauss function that appears in (70) we have

$$ {}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;x_{12}\Big) = \frac{3^{1/4}}{2^{23/6}\pi^2}\,(\sqrt{3}+2\sqrt{2}+1)\Gamma^3\Big(\frac{1}{3}\Big). \tag{71} $$

Substituting the explicit expressions (67) and (71) for involved Gauss functions into (70) we obtain

$$ \mathrm{e}^{-i\frac{\pi}{6}}\,\frac{3}{2^{7/3}\pi^2}\,\Gamma^3\Big(\frac{1}{3}\Big) = \frac{3\sqrt{2}}{\sqrt{8-3\sqrt{2}+5\sqrt{6}}}\,\frac{\sqrt{3}-i}{(2+\sqrt{3})^{1/4}}\,\frac{1}{2^{23/6}\pi^2}\,(\sqrt{3}+2\sqrt{2}+1)\Gamma^3\Big(\frac{1}{3}\Big) \tag{72} $$

which reduces after several evident cancellations to

$$ \mathrm{e}^{-i\frac{\pi}{6}} = \frac{\sqrt{3}-i}{2}\,\frac{\sqrt{3}+2\sqrt{2}+1}{(2+\sqrt{3})^{1/4}\sqrt{8-3\sqrt{2}+5\sqrt{6}}} \tag{73} $$

and we finish with

$$ 1 = \frac{\sqrt{3}+2\sqrt{2}+1}{(2+\sqrt{3})^{1/4}\sqrt{8-3\sqrt{2}+5\sqrt{6}}}. \tag{74} $$

MATHEMATICA tells us that the last expression is indeed an ingenious representation of unity, and thus we confirm the numerically suggested validity of equation (8) with $b=\mathrm{e}^{i\frac{\pi}{6}}$ along with its implications discussed in the present section.

Further numerical calculations indicate that it is possible to find other complex values of $b=\mathrm{e}^{i\frac{\pi}{m}}$ with $m>6$ for which the equation (8) is satisfied. On the other hand, there were no such matches when m was chosen to be less than 6.

7.2. $b=\mathrm{e}^{i\frac{\pi}{12}}$

There is a numerical evidence that the equation (8) is satisfied with $b=\mathrm{e}^{i\frac{\pi}{12}}$, and in this special case we are able to perform analytical calculations. A direct substitution yields

$$ {}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;1-i\Big) = c\big(\mathrm{e}^{i\frac{\pi}{12}}\big)\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;a\big(\mathrm{e}^{i\frac{\pi}{12}}\big)\Big) \tag{75} $$

with

$$ c\big(\mathrm{e}^{i\frac{\pi}{12}}\big) = \frac{(-1)^{23/12}\,3^{3/4}}{(1+\sqrt{3})^{1/4}\sqrt{1+\sqrt{2+\sqrt{3}}}} \tag{76} $$

and

$$ a\big(\mathrm{e}^{i\frac{\pi}{12}}\big) = \frac{1}{8}\Big[4-\sqrt{6\big(9\sqrt{6}+14\sqrt{3}-16\sqrt{2}-21\big)}\Big] \lesssim 0. $$

Similarly as we did in the previous case, we apply the Pfaff transformation (16) to the Gauss function on the right. This leads to an expression involving a tabulated elliptic integral with $n=24$,

$$ {}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;1-i\Big) = \frac{c\big(\mathrm{e}^{i\frac{\pi}{12}}\big)}{\sqrt{1-a\big(\mathrm{e}^{i\frac{\pi}{12}}\big)}}\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;x_{24}\Big), \tag{77} $$

where

$$x_{24} = \frac{\sqrt{6\big(9\sqrt{6}+14\sqrt{3}-16\sqrt{2}-21\big)}-4}{\sqrt{6\big(9\sqrt{6}+14\sqrt{3}-16\sqrt{2}-21\big)}+4}\,. \tag{78}$$

Petrović [21] quotes an alternative representation of k_{24} whose square is

$$x_{24} = \left(\sqrt{6+3\sqrt{3}}-\sqrt{5+3\sqrt{3}}\right)^4\left(\sqrt{2+\sqrt{3}}-\sqrt{1+\sqrt{3}}\right)^4. \tag{79}$$

In turn, in [10, Appendix A.3] we find[13]

$$x_{24} = \left(\frac{1-u}{1+u}\right)^2 \qquad \text{where} \qquad u = (\sqrt{2}+1)(\sqrt{3}-1)\sqrt{\sqrt{3}-\sqrt{2}} \lesssim 1 \tag{80}$$

along with

$$\boldsymbol{K}(k_{24}) = \frac{1+u}{2}\,\boldsymbol{K}(k_6) \quad \text{where} \quad k_{24} = \frac{1-u}{1+u} \gtrsim 0\,. \tag{81}$$

The elliptic integral associated with $n=6$ is [26, p. 259, Table 1], [29, 7.2.16.13], [10, Appendix A.3, p. 331]

$$\boldsymbol{K}(k_6) = \frac{\sqrt{v}}{48\sqrt{\pi}}\,\Gamma\Big(\frac{1}{24}\Big)\Gamma\Big(\frac{11}{24}\Big), \qquad \text{where} \qquad k_6 = (2-\sqrt{3}\,)(\sqrt{3}-\sqrt{2}\,)\,, \tag{82}$$

and we introduced the notation

$$v := \sqrt{6}\,(\sqrt{2}-1)(\sqrt{3}+1)(\sqrt{3}+\sqrt{2}\,)\,. \tag{83}$$

Thus we have on the right-hand side of (77)

$${}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;x_{24}\Big) = \frac{(1+u)\sqrt{v}}{48\pi^{3/2}}\,\Gamma\Big(\frac{1}{24}\Big)\Gamma\Big(\frac{11}{24}\Big)\,, \tag{84}$$

explicit expressions for u and v are given in (80) and (83).

Finally, with some simplifications provided by Mathematica [19] we obtain the evaluation

$${}_2F_1\Big(\frac{1}{3},\frac{1}{2};1;1-i\Big) = \frac{(7-4\sqrt{3})^{1/8}}{16\pi^{3/2}}\,\big[1+\sqrt{3}+i(1-\sqrt{3})\big]\,\Gamma\Big(\frac{1}{24}\Big)\Gamma\Big(\frac{11}{24}\Big). \tag{85}$$

[13] There is a misprint in the expression for u in [10, Appendix A.3]: There, the middle factor is given with a plus sign.

The linear transformation (16) of the last ${}_2F_1$ function produces its counterpart with the complex conjugate argument, ${}_2F_1(\frac{1}{3},\frac{1}{2};1;1+i)$. The both Gauss functions ${}_2F_1(\frac{1}{3},\frac{1}{2};1;z)$ with $z=1\pm i$ and $|z|=\sqrt{2}>1$ are given by (85) via analytic continuation, similarly as in the case (66).

However, quadratic transformations [15, 7.3.1.64–68] for functions ${}_2F_1(\alpha,\beta;2\beta,z)$ produce the results for convergent ${}_2F_1$ series of real arguments. Thus, an application of the transformation [15, 7.3.1.64]

$${}_2F_1(\alpha,\beta;2\beta,z) = (1-z)^{-\alpha/2}\,{}_2F_1\Big(\frac{\alpha}{2},2\beta-\frac{\alpha}{2};\beta+\frac{1}{2};\frac{z^2}{4(z-1)}\Big) \tag{86}$$

in (85) yields a Gauss function with the argument $z=-1$:

$${}_2F_1\Big(\frac{1}{6},\frac{2}{3};1;-1\Big) = 2^{1/3}\,\frac{(7-4\sqrt{3})^{1/8}}{8\pi^{3/2}}\,\Gamma\Big(\frac{1}{24}\Big)\Gamma\Big(\frac{11}{24}\Big). \tag{87}$$

In turn, its Pfaff transformation (16) leads us to the formula

$${}_2F_1\Big(\frac{1}{6},\frac{1}{3};1;\frac{1}{2}\Big) = \frac{\sqrt{2}}{8\pi^{3/2}}\,(7-4\sqrt{3})^{1/8}\,\Gamma\Big(\frac{1}{24}\Big)\Gamma\Big(\frac{11}{24}\Big). \tag{88}$$

The parameters of the Gauss functions in the last two summation formulas (87) and (88) do not fit the classical Kummer theorems and their generalizations discussed in [31]. An identical result follows on direct application of the quadratic transformation (19) to equation (85), so that we have an intersting chain of equalities

$$\begin{aligned}{}_2F_1\Big(\frac{1}{6},\frac{1}{3};1;\frac{1}{2}\Big) &= {}_2F_1\Big(\frac{1}{3},\frac{2}{3};1;\frac{1}{4}(2-\sqrt{2})\Big) = \\ &\frac{\sqrt{2}}{8\pi^{3/2}}\,(7-4\sqrt{3})^{1/8}\,\Gamma\Big(\frac{1}{24}\Big)\Gamma\Big(\frac{11}{24}\Big).\end{aligned} \tag{89}$$

Finally, we notice the result

$${}_2F_1\Big(\frac{1}{3},\frac{1}{3};1;2\sqrt{2}-3\Big) = \frac{\sqrt{2}}{8\pi^{3/2}}\,\Big(\frac{2+\sqrt{2}}{4}\Big)^{1/3}(7-4\sqrt{3})^{1/8}\,\Gamma\Big(\frac{1}{24}\Big)\Gamma\Big(\frac{11}{24}\Big) \tag{90}$$

which follows from the second equality in (89) by Pfaff transformation (16).

8. Summary and outlook

In the present communication we we have presented a study of the hypergeometric transformation (8) recorded by Mingari Scarpello and Ritelli in 2018 [1] and having its roots in the work of Legendre. We essentially extended its validity range for real non-negative parameters b as compared to the one declared in the original paper [1]. We showed that the MSR relation (8) holds also for certain complex values of b.

An interesting question have been raised in this connection: how should we modify the left-hand side of the equation (8) to express the function $c(b)\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;a(b))$ with *negative* b in terms of ${}_2F_1$ functions depending on some other sets of parameters and arguments that differ from those in ${}_2F_1(\frac{1}{3},\frac{1}{2};1;1-b^6)$? The resulting function has to be no more even in b!

Moreover, it would be highly desirable to provide an analytic continuation of (appropriately modified) equation (8) to the complex plane.

We have shown how the equation (8) gives access to singular values of elliptic integrals of the first kind and their singular moduli k_n for $n \geq 3$. Sometimes, besides reproducing the known results, we obtained certain new alternative representations of $\boldsymbol{K}(k_n)$ itself and their arguments k_n.

It was also shown how the MSR relation works in the reverse direction, which opens a possibility to discover some new explicit determinations of Gauss ${}_2F_1$ functions from the information available for singular values of elliptic integrals.

Interesting relationships arise in comparison with a similar hypergeometric transformation due to Ramanujan.

Another interesting question is raised:

If we start with some known explicit values of the function ${}_2F_1(\frac{1}{3},\frac{1}{2};1;z)$*, do they necessarily couple to certain singular values of elliptic integrals* $\boldsymbol{K}(k_n)$*?*

Or, could it be possible that the knowledge the Gauss function ${}_2F_1(\frac{1}{3},\frac{1}{2};1;z)$ *at certain argument* z *would yield some unknown value of the elliptic integral* $\boldsymbol{K}$ *not necessarily belonging to the established set of singular values* $\boldsymbol{K}(k_n)$*?*

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Appendix

LEGENDRE [2, Chap. XXVII, § III]:

On the integral $$Z = \int \frac{\mathrm{d}\varphi}{\sqrt[3]{1-c^2\sin^2\varphi}}\,.$$

Note: *Z is an indefinite integral. The definite integral over the interval* $(0,\pi/2)$

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt[3]{1-c^2\sin^2\varphi}}$$

is termed "complete" and is denoted by $Z^{\rceil}$ *by Legendre.*

147. With $\sqrt[3]{1-c^2\sin^2\varphi} = x$, we have[14] $c^2\sin^2\varphi = 1-x^3$ and $c^2\cos^2\varphi = x^3-b^2$ (*where* $b^2 := 1-c^2$); thus[1]

$$Z = -\frac{3}{2}\int \frac{x\,\mathrm{d}x}{\sqrt{1-x^3}\cdot\sqrt{x^3-b^2}}\,.$$

The product $(1-x^3)(x^3-b^2) = (1+b^2)x^3 - x^6 - b^2$. With $b := n^3$ and $x^2+n^2 = xz$, we shall have first

$$Z = -\frac{3}{2}\int \frac{x^{-\frac{1}{2}}\,\mathrm{d}x}{\sqrt{1+n^6+3n^2z-z^3}}\,;$$

and, from the accepted change of variables we draw

$$\begin{aligned}
x+n &= x^{\frac{1}{2}}\sqrt{z+2n},\\
x-n &= x^{\frac{1}{2}}\sqrt{z-2n},\\
2x^{\frac{1}{2}} &= \sqrt{z+2n}+\sqrt{z-2n},\\
2x^{-\frac{1}{2}}\mathrm{d}x &= \frac{dz}{\sqrt{z+2n}} + \frac{dz}{\sqrt{z-2n}}\,.
\end{aligned}$$

Hence, the transformed integral over z is

$$\begin{aligned}
Z =& \frac{3}{4}\int \frac{-\mathrm{d}z}{\sqrt{z+2n}\cdot\sqrt{1+n^6+3n^2z-z^3}} + \\
& \frac{3}{4}\int \frac{-\mathrm{d}z}{\sqrt{z-2n}\cdot\sqrt{1+n^6+3n^2z-z^3}}\,,
\end{aligned}$$

and it is seen that these integrals depend only on elliptic functions.

To calculate the first one,

$$P = \frac{3}{4}\int \frac{-\mathrm{d}z}{\sqrt{z+2n}\cdot\sqrt{1+n^6+3n^2z-z^3}}\,,$$

[14] Misprint in [2].

I observe that $1+n^6+3n^2z-z^3=(1+n^2-z)(1-n^2+n^4+(1+n^2)z+z^2)$. Thus, with

$$z=\frac{1+n^2-2ny^2}{1+y^2}\,,$$

we obtain

$$P=\frac{3}{2}\int\frac{\mathrm{d}y}{\sqrt{\lambda^2+2\lambda\mu\cos\theta y^2+\mu^2y^4}}\,,$$

$$\mu=1-n+n^2,\quad \lambda^2=3(1+n^2+n^4)=3\mu(\mu+2n)$$

$$\cos\theta=\frac{\sqrt{3}}{2}\,\frac{(1-n^2)^2-2n(1-n+n^2)}{(1-n+n^2)\sqrt{1+n^2+n^4}}=\frac{\sqrt{3}}{2}\,\frac{\mu^2-3n^2}{\mu\sqrt{\mu^2+2n\mu}}\,;$$

If we additionally define $\gamma=\sqrt{\frac{\lambda}{\mu}}\cdot\tan\frac{\omega}{2}$ and $k^2=\sin^2\frac{\theta}{2}=\frac{1}{2}(1-\cos\theta)$, we obtain

$$P=\frac{3}{4\sqrt{\lambda\mu}}\,F(k,\omega).$$

The relation between φ and ω is such that ω is null at two integration limits, when $\varphi=0$ and $\varphi=\frac{\pi}{2}$. Thus, the quantity P does not enter into the complete integral

$$Z^{\rceil}\equiv\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt[3]{1-c^2\sin^2\varphi}}\,.$$

For the next integral

$$Q=\frac{3}{4}\int\frac{-\mathrm{d}z}{\sqrt{z-2n}\cdot\sqrt{1+n^6+3n^2z-z^3}}\,,$$

we do the change of variables

$$z=\frac{1+n^2+2ny'^2}{1+y'^2}\,,$$

which leads us to the transformation

$$Q=\frac{3}{2}\int\frac{\mathrm{d}y'}{\sqrt{\lambda'^2+2\lambda'\mu'\cos\theta' y'^2+\mu'^2y'^4}}\,,$$

in which

$$\lambda'^2=3(1+n^2+n^4)=3(1+n+n^2)(1-n+n^2)=3\mu'(\mu'-2n),$$

$$\mu'=1+n+n^2,$$

$$\cos\theta'=\frac{3(\mu'^2-3n^2)}{2\lambda\mu'}=\frac{\sqrt{3}}{2}\cdot\frac{\mu'^2-3n^2}{\mu'\sqrt{\mu'^2-2n\mu'}}$$

Defining, as usual, $\gamma'=\sqrt{\frac{\lambda'}{\mu'}}\cdot\tan\frac{\psi}{2}$ and $k'^2=\sin^2\frac{\theta'}{2}=\frac{1}{2}(1-\cos\theta')$

we obtain

$$Q=\frac{3}{4\sqrt{\lambda\mu'}}\,F(k',\theta)\,;$$

thus, the integral of interest is given by

$$Z=\frac{3}{4\sqrt{\lambda\mu}}\,F(k,\omega)+\frac{3}{4\sqrt{\lambda\mu'}}\,F(k',\theta)\,.$$

Moreover, the modules k and k' are not complements of each other, and have no other relations between them than those which result from the equations

$$k^2=\frac{1}{2}-\frac{\sqrt{3}}{4}\cdot\frac{\mu^2-3n^2}{\mu\sqrt{\mu\mu'}}\,,\ k'^2=\frac{1}{2}-\frac{\sqrt{3}}{4}\cdot\frac{\mu'^2-3n^2}{\mu'\sqrt{\mu\mu'}}\,;$$

$\mu=1-n+n^2,\ \mu'=1+n+n^2,\ n=\sqrt[3]{b}\,.$

When $\varphi=\frac{\pi}{2}$, we have $\omega=0$ and $\psi=2\pi$; thus the complete integral

$$Z^{\rceil}=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt[3]{1-c^2\sin^2\varphi}}=\frac{3}{\sqrt{\mu'\lambda}}\,\boldsymbol{K}(k')\,.$$

References

1. G. Mingari Scarpello and D. Ritelli, "On a class of new hypergeometric transformations," *Open J. Math. Sci.*, vol. 2, no. 1, pp. 84–92, 2018.
2. A. M. Legendre, *Traité des fonctions elliptiques et des intégrales Euleriennes. Tome premier.* Paris: Imprimerie de Huzard-Courcier, 1825.
3. T. Amdeberhan, M. L. Glasser, M. C. Jones, V. Moll, R. Posey, and D. Varela, "The Cauchy-Schlömilch transformation," *Integral Transforms Spec. Funct.*, vol. 30, pp. 940–961, 2019.
4. B. C. Berndt, S. Bhargava, and F. Garvan, "RamanujanвЂ™s theories of elliptic functions to alternative bases," *Trans. Amer. Math. Soc.*, vol. 347, no. 11, pp. 4163–4244, 1995.
5. B. Berndt, *Ramanujan's Notebooks. Part V.* Springer, 1998.
6. F. G. Garvan, "Ramanujan's theories of elliptic functions to alternative bases — a symbolic excursion," *J. Symbolic Computation*, vol. 20, no. 5, pp. 517–536, 1995.

7. A. Selberg and S. Chowla, "On Epstein's zeta function," *J. Reine. Angew. Math.*, vol. 227, pp. 88–110, 1967.
8. G. N. Watson, "Some singular moduli (I)," *Q. J. Math.*, vol. os-3, no. 1, pp. 81–98, 1932.
9. I. J. Zucker, "The evaluation in terms of Γ-functions of the periods of elliptic curves admitting complex multiplication," *Math. Proc. Camb. Phil. Soc.*, vol. 82, pp. 111–118, 1977.
10. J. M. Borwein, M. L. Glasser, R. C. McPhedran, J. G. Wan, and I. J. Zucker, *Lattice sums then and now*, vol. 150 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2013.
11. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*. Cambridge: Cambridge University Press, edited by V. H. Moll, Fifth ed., 2021.
12. F. Cooper, A. Khare, and A. Saxena, "Exact elliptic compactons in generalized Korteweg-De Vries equations," *Complexity*, vol. 11, no. 6, pp. 30–34, 2006.
13. E. D. Rainville, *Special Functions*. New York: MacMillan, 1960.
14. G. E. Andrews, R. Askey, and R. Roy, "Special functions," in *Encyclopedia of Mathematics and its Applications*, vol. 71, Cambridge: Cambridge University Press, 1999.
15. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series. More Special Functions*, vol. 3. New York: Gordon and Breach, 1990.
16. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions*. Cambridge: Cambridge University Press, 2010.
17. É. Goursat, "Sur l'équation différentielle linéaire, qui admet pour intégrale la série hypergéométrique," in *Annales scientifiques de l'École Normale Supérieure*, vol. 10, pp. 3–142, 1881.
18. M. Spiegel, "Some interesting special cases of the hypergeometric series," *Am. Math. Mon.*, vol. 69, no. 9, pp. 894–896, 1962.
19. 2021. Mathematica 12.3, a product of Wolfram Research.
20. A. Ebisu, "Special values of Gauss's hypergeometric series derived from Appell's series F_1 with closed forms." arXiv:math.CA/1607.04742, 2016.
21. V. V. Petrović, "Comprehensive table of singular moduli."
22. Wikipedia, "Golden ratio."
23. A. Ebisu, *Special values of the hypergeometric series*, vol. 248 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2017.

24. R. Vidūnas, "Expressions for values of the gamma function," *Kyushu J. Math.*, vol. 59, no. 2, pp. 267–283, 2005.
25. G. Mingari Scarpello and D. Ritelli, "Legendre hyperelliptic integrals, π new formulae and Lauricella functions through the elliptic singular moduli," *J. Number Theory*, vol. 135, pp. 334–352, 2014.
26. G. S. Joyce and I. J. Zucker, "Special values of the hypergeometric series," *Math. Proc. Camb. Phil. Soc*, vol. 109, pp. 257–261, 1991.
27. B. C. Berndt and H. Chan, "Ramanujan's explicit values for the classical theta-function," *Mathematika*, vol. 42, no. 2, pp. 278–294, 1995.
28. G. Mingari Scarpello and D. Ritelli, "π and the hypergeometric functions of complex argument," *Journal of Number Theory*, vol. 131, no. 10, pp. 1887 – 1900, 2011.
29. Yu. A. Brychkov, *Handbook of special functions derivatives, integrals, series and other formulas*. Boca Raton: Chapman and Hall/CRC, 2008.
30. I. J. Zucker and G. S. Joyce, "Special values of the hypergeometric series II," *Math. Proc. Camb. Phil. Soc*, vol. 131, pp. 309–319, 2001.
31. J. Choi, A. K. Rathie, and H. M. Srivastava, "A generalization of a formula due to Kummer," *Integral Transforms Spec. Funct.*, vol. 22, no. 11, pp. 851–859, 2011.