Петро Андрійович Глушак
Михайло Васильович Токарчук

Ланцюжок кінетичних рівнянь для функцій розподілу частинок в простій рідині з врахуванням нелінійних гідродинамічних флуктуацій

Роботу отримано 8 грудня 2014 р.

Затверджено до друку Вченою радою ІФКС НАН України

Рекомендовано до друку відділом теорії нерівноважних процесів

Виготовлено при ІФКС НАН України

Національна академія наук України

ІНСТИТУТ
ФІЗИКИ
КОНДЕНСОВАНИХ
СИСТЕМ

ICMP–14–05E

P.A. Hlushak, M.V. Tokarchuk

CHAIN OF KINETIC EQUATIONS FOR THE DISTRIBUTION FUNCTIONS OF PARTICLES IN SIMPLE LIQUID TAKING INTO ACCOUNT NONLINEAR HYDRODYNAMIC FLUCTUATIONS

ЛЬВІВ

УДК: 536.75; 538.931

PACS: 05.20.Dd, 05.40.-a, 05.60.Cd

Ланцюжок кінетичних рівнянь для функцій розподілу частинок в простій рідині з врахуванням нелінійних гідродинамічних флуктуацій

П.А. Глушак, М.В. Токарчук

Анотація. Запропоновано ланцюжок кінетичних рівнянь для нерівноважних одночастинкової, двочастинкової і s частинкових функцій розподілу частинок з урахуванням нелінійних гідродинамічних флуктуацій. Нерівноважна функція розподілу нелінійних гідродинамічних флуктуацій задовольняє узагальнене рівняння Фоккера-Планка. Використовується метод нерівноважного статистичного оператора Зубарєва. Запропоновано спосіб розрахунку структурної функції розподілу гідродинамічних колективних змінних та їх гідродинамічних швидкостей (вище гаусового наближення), що містяться в узагальненому рівнянні Фоккера-Планка для нерівноважної функції розподілу гідродинамічних колективних змінних.

Chain of kinetic equations for the distribution functions of particles in simple liquid taking into account nonlinear hydrodynamic fluctuations

P.A. Hlushak, M.V.Tokarchuk

Abstract. Chain of kinetic equations for non-equilibrium single, double and s-particle distribution functions of particles is obtained taking into account nonlinear hydrodynamic fluctuations. Non-equilibrium distribution function of non-linear hydrodynamic fluctuations satisfies a generalized Fokker-Planck equation. The method of non-equilibrium statistical operator by Zubarev is applied. A way of calculating of the structural distribution function of hydrodynamic collective variables and their hydrodynamic velocities (above Gaussian approximation) contained in the generalized Fokker-Planck equation for the non-equilibrium distribution function of hydrodynamic collective variables is proposed.

Подається в Physica A
Submitted to Physica A

1. Introduction

The study of nonlinear kinetic and hydrodynamic fluctuations in dense gases, liquids and plasma, in turbulence phenomena and dynamics of phase transitions, in chemical reactions and self-organizing processes are relevant both on kinetic and hydrodynamic levels of description in statistical theory of non-equilibrium processes [1–18]. The non-equilibrium states of such systems are far from equilibrium. Therefore the study of both the processes establishing the stationary states with characteristic times of life and the relaxation processes to the known equilibrium states, that are described by means of molecular hydrodynamics [19–23], is of great importance. An important feature of theoretical modeling of non-equilibrium phenomena in dense gases, liquids, dense plasmas (dusty plasmas) is a consistent description of kinetic and hydrodynamic processes [23–27] and taking into account the characteristic short and long-range interactions between the particles of the systems. In particular, the non-equilibrium gas-liquid phase transition is characterized by nonlinear hydrodynamic fluctuations of mass, momentum and particle energy, which describe a collective nature of the process and define the spatial and temporal behavior of the transport coefficients (viscosity, thermal conductivity), time correlation functions and dynamic structure factor. At the same time, due to heterogeneity in collective dynamics of these fluctuations, liquid drops emerge in the gas phase (in case of transition from the gas phase to the liquid phase), or the gas bubbles emerge in the liquid phase (in case of transition from the liquid phase to the gas phase), formation of which has a kinetic nature described by a redistribution of momentum and energy, i.e. when a certain group of particles in the system receives a significant decrease (in the case of drops), or increase (in the case of bubbles) of kinetic energy. The particles, that form bubbles or droplets, diffuse out of their phases in the liquid or the gas and vice versa. They have different values of momentum, energy and pressure in different phases. All these features are related to the non-equilibrium unary, binary and s-particle distribution functions (which depend on coordinate, momentum and time) that satisfy the BBGKY chain of equations. Therefore, the construction of kinetic equations that take into account nonlinear hydrodynamic fluctuations [28–32] is an important problem in the theory of transport processes in dense gases and liquids. In particular, this problem arises in the description of low-frequency anomalies in the kinetic equations and related "long tail" correlation functions [33–35].

The main difficulty of the problem is that the kinetics and hydrodynamics of these processes are strongly related and should be considered simultaneously. Zubarev, Morozov, Omelyan and Tokarchuk [23, 36, 37] proposed the consistent description of kinetic and hydrodynamic processes in dense gases and liquids on the basis of Zubarev non-equilibrium statistical operator [38, 39]. In particular, this approach was used to obtain the kinetic equation of the revised Enskog theory [37,40] for a system of hard spheres and kinetic Enskog-Landau equations for one-component system of charged hard spheres from the BBGKY chain of equations.

Zubarev *et al* [23] obtained the generalized hydrodynamic equations for the hydrodynamic variables (densities of the particle, momentum and the total energy) connected with the kinetic equation for the non-equilibrium one-particle distribution function. Later [24, 26], these equations were used to study time correlation functions and the collective excitation spectrum of the weakly non-equilibrium processes in liquids.

Obviously, the approach proposed by Zubarev *et al* [23, 36, 37] and Tokarchuk *et al* [24,26] can be used to describe both weakly and strongly non-equilibrium systems. At the same time, in order to consistently describe kinetic processes and nonlinear hydrodynamic fluctuations it is convenient to reformulate this theory so that a set of equations for non-equilibrium one-particle distribution function and the distribution functional of hydrodynamic variables, particle number densities as well as momentum and energy densities could be obtained.

In this contribution we will develop an approach for consistent description of kinetic and hydrodynamic processes that are characterized by non-linear fluctuations and are especially important for the description of non-equilibrium gas-liquid phase transition. In the second section we will obtain the non-equilibrium statistical operator for non-equilibrium state of the system when the parameters of the reduced description are a non-equilibrium one-particle distribution function and the distribution function of non-equilibrium nonlinear hydrodynamic variables. Using this operator we construct the kinetic equations for the non-equilibrium single, double, s-particle distribution functions which take into account nonlinear hydrodynamic fluctuations, for which the non-equilibrium distribution function satisfies a generalized Fokker-Planck equation. In the third section, we will consider one of the ways to calculate the structural distribution function of hydrodynamic collective variables and their hydrodynamic velocities (in higher than Gaussian approximation), which enter the generalized Fokker-Planck equation for the non-equilibrium distribution function of hydrodynamic collective variables.

2. Non-equilibrium distribution function

For a consistent description of kinetic and hydrodynamic fluctuations in a classical one-component fluid it is necessary to select the description parameters for one-particle and collective processes. As these parameters we choose the non-equilibrium one-particle distribution function $f_1(x;t) = \langle \hat{n}_1(x)\rangle^t$ and distribution function of hydrodynamic variables $f(a;t) = \langle \delta(\hat{a} - a)\rangle^t$. Here the phase function

$$\hat{n}_1(x) = \sum_{j=1}^{N} \delta(x - x_j) = \sum_{j=1}^{N} \delta(\mathbf{r} - \mathbf{r}_j)\delta(\mathbf{p} - \mathbf{p}_j) \tag{2.1}$$

is the microscopic particle number density. $x_j = (\mathbf{r}_j, \mathbf{p}_j)$ is the set of phase variables (coordinates and momentums), N is the total number of particles in a volume V. A microscopic phase distribution of hydrodynamic variables is given by

$$\hat{f}(a) = \delta(\hat{a} - a) = \prod_{m=1}^{3} \prod_{\mathbf{k}} \delta(\hat{a}_{m\mathbf{k}} - a_{m\mathbf{k}}), \tag{2.2}$$

where $\hat{a}_{1\mathbf{k}} = \hat{n}_{\mathbf{k}}$, $\hat{a}_{2\mathbf{k}} = \hat{\mathbf{J}}_{\mathbf{k}}$, $\hat{a}_{3\mathbf{k}} = \hat{\varepsilon}_{\mathbf{k}}$ are the Fourier components of the densities of particle number, momentum and energy:

$$\hat{n}_{\mathbf{k}} = \sum_{j=1}^{N} \mathrm{e}^{-i\mathbf{k}\mathbf{r}_j}, \qquad \hat{\mathbf{J}}_{\mathbf{k}} = \sum_{j=1}^{N} \mathbf{p}_j \mathrm{e}^{-i\mathbf{k}\mathbf{r}_j}, \tag{2.3}$$
$$\hat{\varepsilon}_{\mathbf{k}} = \sum_{j=1}^{N} \Big[\frac{p_j^2}{2m} + \frac{1}{2}\sum_{l\neq j=1}^{N} \Phi(|\mathbf{r}_{lj}|)\Big]\mathrm{e}^{-i\mathbf{k}\mathbf{r}_j},$$

and $a_{m\mathbf{k}} = (n_{\mathbf{k}}, \mathbf{J}_{\mathbf{k}}, \varepsilon_{\mathbf{k}})$ are the corresponding collective variables. $\Phi(|\mathbf{r}_{lj}|) = \Phi(|\mathbf{r}_l - \mathbf{r}_j|)$ is the pair interaction potential between particles. The average values $\langle \hat{n}_1(x)\rangle^t$ and $\langle \delta(\hat{a} - a)\rangle^t$ are calculated by means of the non-equilibrium N-particle distribution function $\varrho(x^N;t)$, that satisfies the Liouville equation. In line with the idea of reduced description of non-equilibrium states this function is the functional

$$\varrho(x^N;t) = \varrho(\ldots, f_1(x;t), f(a;t), \ldots). \tag{2.4}$$

In order to find a non-equilibrium distribution function $\varrho(x^N;t)$ we use Zubarev's method [38, 39, 41], in which a general solution of Liouville equation taking into account a projection procedure can be presented in the form:

$$\varrho(x^N;t) = \varrho_q(x^N;t) - \int_{-\infty}^{t} dt' \mathrm{e}^{\epsilon(t'-t)} T_q(t,t')(1-P_q(t')) iL_N \varrho_q(x^N;t'), \tag{2.5}$$

where $\epsilon \to +0$ after thermodynamic limiting transition. The source selects the retarded solutions of Liouville equation with operator iL_N. $T_q(t,t') = \exp_+(-\int_{t'}^{t} dt'(1-P_q(t'))iL_N)$ is the generalized time evolution operator taking into account Kawasaki-Gunton projection $P_q(t')$. The structure of $P_q(t')$ depends on the quasi-equilibrium distribution function $\varrho_q(x^N;t)$, which in method by Zubarev is determined from extremum of the information entropy at simultaneous conservation of normalization condition

$$\int d\Gamma_N \varrho_q(x^N;t) = 1, \ \ d\Gamma_N = \frac{(dx)^N}{N!} = \frac{(dx_1, \ \ldots, dx_N)}{N!}, \ \ dx = d\mathbf{r}d\mathbf{p}, \tag{2.6}$$

and the fact that the parameters of the reduced description, $f_1(x;t)$ and $f(a;t)$ are fixed. Then quasi-equilibrium distribution function can be written as

$$\varrho_q(x^N;t) = \exp\big\{-\Phi(t) - \int dx \gamma(x;t)\hat{n}_1(x) - \int da F(a;t)\hat{f}(a)\big\}, \tag{2.7}$$

where $da = \{dn_{\mathbf{k}}, d\mathbf{J}_{\mathbf{k}}, d\varepsilon_{\mathbf{k}}\}$. The Massieu-Planck functional $\Phi(t)$ is determined from the normalization condition for the quasi-equilibrium distribution function

$$\Phi(t) = \ln \int d\Gamma_N \exp\big\{-\int dx\gamma(x;t)\hat{n}_1(x) - \int da F(a;t)\hat{f}(a)\big\}.$$

The functions $\gamma(x;t)$ and $F(a;t)$ are the Lagrange multipliers and are determined from the consistency conditions:

$$\begin{aligned} f_1(x;t) &= \langle \hat{n}_1(x)\rangle^t = \langle \hat{n}_1(x)\rangle_q^t, \\ f(a;t) &= \langle \delta(\hat{a}-a)\rangle^t = \langle \delta(\hat{a}-a)\rangle_q^t, \end{aligned} \tag{2.8}$$

where $\langle\ldots\rangle^t = \int d\Gamma_N \ldots \varrho(x^N;t)$ and $\langle\ldots\rangle_q^t = \int d\Gamma_N \ldots \varrho_q(x^N;t)$. To find the explicit form of non-equilibrium distribution function $\varrho(x^N;t)$ we exclude the factor $F(a;t)$ in quasi-equilibrium distribution function and thereafter, by means of consistency conditions (2.8), we have

$$\varrho_q(x^N;t) = \varrho_q^{kin}(x^N;t)\frac{f(a;t)}{W(a;t)}\Big|_{a=\hat{a}}. \tag{2.9}$$

Here

$$\begin{aligned} W(a;t) &= \int d\Gamma_N \mathrm{e}^{-\Phi^{kin}(t)-\int dx\gamma(x;t)\hat{n}_1(x)}\hat{f}(a) \\ &= \int d\Gamma_N \varrho_q^{kin}(x^N;t)\hat{f}(a) \end{aligned} \tag{2.10}$$

is the structure distribution function of hydrodynamic variables, which could be also considered as a Jacobian for transition from $\hat{f}(a)$ into space of collective variables $n_{\mathbf{k}}$, $\mathbf{J}_{\mathbf{k}}$, $\varepsilon_{\mathbf{k}}$ averaged with the "kinetic" quasi-equilibrium distribution function

$$\begin{aligned} \varrho_q^{kin}(x^N;t) &= \exp\big\{-\Phi^{kin}(t) - \int dx\gamma(x;t)\hat{n}_1(x)\big\}, \\ \Phi^{kin}(t) &= \ln \int d\Gamma_N \exp\big\{-\int dx\gamma(x;t)\hat{n}_1(x)\big\}. \end{aligned} \tag{2.11}$$

Here the entropy

$$\begin{aligned} S(t) &= -\langle \ln \varrho_q(x^N;t)\rangle_q^t \\ &= \Phi(t) + \int dx\gamma(x;t)\langle\hat{n}_1(x)\rangle^t + \int da f(a;t)\ln\frac{f(a;t)}{W(a;t)}. \end{aligned} \tag{2.12}$$

corresponds to the quasi-equilibrium distribution (2.9). In combination with the self-consistency conditions (2.8), it can be considered as entropy of non-equilibrium state. In accordance with (2.5), in order to obtain the explicit form of non-equilibrium distribution function, it is necessary to disclose the action of Liouville operator on $\varrho_q(x^N;t)$ and action of the Kawasaki-Gunton projection operator, which in our case has the following structure according to (2.9):

$$\begin{aligned} P_q(t)\varrho' &= \varrho_q(x^N;t)\int d\Gamma_N\varrho' + \int dx \frac{\partial\varrho_q(x^N;t)}{\partial\langle\hat{n}_1(x)\rangle \mathrm{e}^t} \\ &\times\Big(\int d\Gamma_N\hat{n}_1(x)\varrho' - \langle\hat{n}_1(x)\rangle^t\int d\Gamma_N\varrho'\Big) \\ &+ \int da \frac{\partial\varrho_q(x^N;t)}{\partial(\frac{f(a;t)}{W(a;t)})}\frac{1}{W(a;t)}\Big(\int d\Gamma_N\hat{f}(a)\varrho' - f(a;t)\int d\Gamma_N\varrho'\Big) \\ &+ \int dx\int da\frac{\partial\varrho_q(x^N;t)}{\partial(\frac{f(a;t)}{W(a;t)})}\frac{f(a;t)}{W(a;t)}\frac{\partial \ln W(a;t)}{\partial\langle\hat{n}_1(x)\rangle^t} \\ &\times\Big(\int d\Gamma_N\hat{n}_1(x)\varrho' - \langle\hat{n}_1(x)\rangle^t\int d\Gamma_N\varrho'\Big). \end{aligned} \tag{2.13}$$

Next, we consider the action of Liouville operator on quasi-equilibrium distribution function (2.9):

$$iL_N\varrho_q(x^N;t) = -\int dx\gamma(x;t)\dot{\hat{n}}_1(x)\varrho_q(x^N;t) + \left[iL_N\frac{f(a;t)}{W(a;t)}\Big|_{a=\hat{a}}\right]\varrho_q^{kin}(x^N;t), \tag{2.14}$$

where $\dot{\hat{n}}_1(x) = iL_N\hat{n}_1(x)$. Having used thereafter the relation

$$iL_N\hat{f}(a) = iL_N\hat{f}(n_{\mathbf{k}},\mathbf{J}_{\mathbf{k}},\varepsilon_{\mathbf{k}}) = \sum_{\mathbf{k}}\Big[\frac{\partial}{\partial n_{\mathbf{k}}}\hat{f}(a)\dot{\hat{n}}_{\mathbf{k}} + \frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\hat{f}(a)\dot{\hat{\mathbf{J}}}_{\mathbf{k}} + \frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\hat{f}(a)\dot{\hat{\varepsilon}}_{\mathbf{k}}\Big],$$

where $\dot{\hat{n}}_{\mathbf{k}} = iL_N\hat{n}_{\mathbf{k}}$, $\dot{\hat{\mathbf{J}}}_{\mathbf{k}} = iL_N\hat{\mathbf{J}}_{\mathbf{k}}$, $\dot{\hat{\varepsilon}}_{\mathbf{k}} = iL_N\hat{\varepsilon}_{\mathbf{k}}$, the last expression in (2.14) can be rewritten in following form:

$$\left[iL_N\frac{f(a;t)}{W(a;t)}\Big|_{a=\hat{a}}\right]\varrho_q^{kin}(x^N;t) = \int da\sum_{\mathbf{k}}W(a;t)\Big[\dot{\hat{n}}_{\mathbf{k}}\frac{\partial}{\partial n_{\mathbf{k}}}\frac{f(a;t)}{W(a;t)} + \dot{\hat{\mathbf{J}}}_{\mathbf{k}}\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\frac{f(a;t)}{W(a;t)} + \dot{\hat{\varepsilon}}_{\mathbf{k}}\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\frac{f(a;t)}{W(a;t)}\Big]\varrho_L(x^N;t). \tag{2.15}$$

Here we introduced new quasi-equilibrium distribution function $\varrho_L(x^N,a;t)$ with the microscopic distribution of large-scale collective variables

$$\varrho_L(x^N,a;t) = \varrho_q^{kin}(x^N;t)\frac{\hat{f}(a)}{W(a;t)}. \tag{2.16}$$

This quasi-equilibrium distribution function is connected with $\varrho_q(x^N;t)$ by the relation

$$\varrho_q(x^N;t) = \int da f(a;t)\varrho_L(x^N,a;t) \tag{2.17}$$

and is obviously normalized to unity

$$\int d\Gamma_N\varrho_L(x^N,a;t) = 1. \tag{2.18}$$

Using then the relation (2.16), the average values with quasi-equilibrium distribution is convenient to represent in following form:

$$\langle\ldots\rangle_q^t = \int da\langle\ldots\rangle_L^t f(a;t), \quad \langle\ldots\rangle_L^t = \int d\Gamma_N\ldots\varrho_L(x^N,a;t). \tag{2.19}$$

Now in accordance with (2.15) and (2.16) we can rewrite the action of the Liouville operator on $\varrho_q(x^N;t)$ as follows

$$iL_N\varrho_q(x^N;t) = -\int da\int dx\gamma(x;t)\dot{\hat{n}}_1(x)\varrho_L(x^N,a;t)f(a;t) + \int da\sum_{\mathbf{k}}W(a;t)\Big[\dot{\hat{n}}_{\mathbf{k}}\frac{\partial}{\partial n_{\mathbf{k}}}\frac{f(a;t)}{W(a;t)} + \dot{\hat{\mathbf{J}}}_{\mathbf{k}}\cdot\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\frac{f(a;t)}{W(a;t)} + \dot{\hat{\varepsilon}}_{\mathbf{k}}\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\frac{f(a;t)}{W(a;t)}\Big]\varrho_L(x^N,a;t). \tag{2.20}$$

Substituting this expression into (2.5), one obtains for non-equilibrium distribution function the following result:

$$\begin{aligned}\varrho(x^N;t) &= \int da f(a;t)\varrho_L(x^N,a;t) \\ &+ \int da\int dx\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}T_q(t,t')\Big(1-P_q(t')\Big) \\ &\times\dot{\hat{n}}_1(x)\varrho_L(x^N,a;t')f(a;t')\gamma(x;t') \\ &-\int da\sum_{\mathbf{k}}\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}T_q(t,t')\Big(1-P_q(t')\Big)W(a;t')\Big[\dot{\hat{n}}_{\mathbf{k}}\frac{\partial}{\partial n_{\mathbf{k}}}\frac{f(a;t')}{W(a;t')} \\ &+\dot{\hat{\mathbf{J}}}_{\mathbf{k}}\cdot\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\frac{f(a;t')}{W(a;t')} + \dot{\hat{\varepsilon}}_{\mathbf{k}}\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\frac{f(a;t')}{W(a;t')}\Big]\varrho_L(x^N,a;t').\end{aligned} \tag{2.21}$$

and the corresponding generalized transport equations:

$$\begin{aligned}&\Big[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\cdot\frac{\partial}{\partial \mathbf{r}}\Big]f_1(x;t) - \int dx'\frac{\partial}{\partial \mathbf{r}}\Phi(|\mathbf{r}-\mathbf{r}'|)\cdot\Big[\frac{\partial}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}'}\Big]g_2(x,x';t) \\ &= \int dx'\int da\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}\phi_{nn}(x,x',a;t,t')f(a;t')\gamma(x';t') \\ &-\sum_{\mathbf{k}}\int da\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}\Big\{\phi_{nj}(x,\mathbf{k},a;t,t')\cdot\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}} \\ &+\phi_{n\varepsilon}(x,\mathbf{k},a;t,t')\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\Big\}\frac{f(a;t')}{W(a;t')},\end{aligned} \tag{2.22}$$

$$\frac{\partial}{\partial t}f(a;t)=\sum_{\mathbf{k}}\Big\{\frac{\partial}{\partial n_{\mathbf{k}}}v_{n,\mathbf{k}}(a;t) \tag{2.23}$$
$$+\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\cdot\mathbf{v}_{j,\mathbf{k}}(a;t)+\frac{\partial}{\partial\varepsilon_{\mathbf{k}}}v_{\varepsilon,\mathbf{k}}(a;t)\Big\}f(a;t)$$
$$=\sum_{\mathbf{k}}\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\cdot\int dx'\int da'\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}$$
$$\times\phi_{jn}(x',\mathbf{k},a,a';t,t')f(a;t')\gamma(x';t')$$
$$-\sum_{\mathbf{k}}\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\int dx'\int da'\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}$$
$$\times\phi_{\varepsilon n}(x',\mathbf{k},a,a';t,t')f(a;t')\gamma(x';t')$$
$$+\sum_{\mathbf{k},\mathbf{q}}\int da'\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\cdot\phi_{jj}(\mathbf{k},\mathbf{q},a,a';t,t')\cdot\frac{\partial}{\partial \mathbf{J}_{\mathbf{q}}}\frac{f(a;t')}{W(a;t')}$$
$$+\sum_{\mathbf{k},\mathbf{q}}\int da'\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\phi_{\varepsilon\varepsilon}(\mathbf{k},\mathbf{q},a,a';t,t')\frac{\partial}{\partial \varepsilon_{\mathbf{q}}}\frac{f(a;t')}{W(a;t')}$$
$$+\sum_{\mathbf{k},\mathbf{q}}\int da'\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}\Big\{\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\cdot\phi_{j\varepsilon}(\mathbf{k},\mathbf{q},a,a';t,t')\frac{\partial}{\partial \varepsilon_{\mathbf{q}}}$$
$$+\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\phi_{\varepsilon j}(\mathbf{k},\mathbf{q},a,a';t,t')\cdot\frac{\partial}{\partial \mathbf{J}_{\mathbf{q}}}\Big\}\frac{f(a;t')}{W(a;t')}.$$

The generalized transport equations (2.22), (2.23) include the quasi-equilibrium binary distribution function of particles $g_2(x,x';t)$:

$$g_2(x,x';t)=\langle G_2(x,x')\rangle_q^t=\langle\hat{n}_1(x)\hat{n}_1(x')\rangle_q^t \tag{2.24}$$
$$=\int d\Gamma_{N-2}\varrho_q(x^N;t)=\int da g_2^L(x,x';a;t)f(a;t),$$

where

$$g_2^L(x,x';a;t)=\int d\Gamma_{N-2}\varrho_L(x^N;a;t)$$

is the binary quasi-equilibrium distribution function of large-scale collective variables. The generalized transport kernels $\phi_{\alpha\beta}$ $(\alpha,\beta=\{n,\mathbf{j},\varepsilon\})$, that describe non-Markovian kinetic and hydrodynamic processes, are non-equilibrium correlation functions of generalized fluxes I_α, I_β:

$$\phi_{\alpha\beta}(t,t')=\langle I_\alpha(t)T_q(t,t')I_\beta(t')\rangle_L^{t'}, \tag{2.25}$$

$$\hat{I}_n(x;t)=\big(1-P(t)\big)\dot{\hat{n}}_1(x), \tag{2.26}$$

$$\hat{I}_{\mathbf{j}}(\mathbf{k};t)=\big(1-P(t)\big)\dot{\hat{\mathbf{J}}}_{\mathbf{k}}, \tag{2.27}$$
$$\hat{I}_{\varepsilon}(\mathbf{k};t)=\big(1-P(t)\big)\dot{\hat{\varepsilon}}_{\mathbf{k}}. \tag{2.28}$$

Here $P(t)$ is the generalized Mori operator related to Kawasaki-Gunton projection operator $P_q(t)$ by following relation

$$P_q(t)a(x)\varrho_q(x^N;t)=\varrho_q(x^N;t)P(t)a(x).$$

It should be emphasized that in (2.25) the averages are calculated with distribution function $\varrho_L(x^N,a;t)$ (2.19), so that the transport kernels are some functions of collective variables $a_{\mathbf{k}}$. In equation (2.23) the functions (called hydrodynamic velocities) $v_{n,\mathbf{k}}(a;t)$, $\mathbf{v}_{j,\mathbf{k}}(a;t)$, $v_{\varepsilon,\mathbf{k}}(a;t)$ represent the fluxes in the space of collective variables and are defined as:

$$v_{n,\mathbf{k}}(a;t)=\int d\Gamma_N\dot{\hat{n}}_{\mathbf{k}}\varrho_L(x^N,a;t)=\langle\dot{\hat{n}}_{\mathbf{k}}\rangle_L^t,$$
$$\mathbf{v}_{j,\mathbf{k}}(a;t)=\int d\Gamma_N\dot{\hat{\mathbf{J}}}_{\mathbf{k}}\varrho_L(x^N,a;t)=\langle\dot{\hat{\mathbf{J}}}_{\mathbf{k}}\rangle_L^t, \tag{2.29}$$
$$v_{\varepsilon,\mathbf{k}}(a;t)=\int d\Gamma_N\dot{\hat{\varepsilon}}_{\mathbf{k}}\varrho_L(x^N,a;t)=\langle\dot{\hat{\varepsilon}}_{\mathbf{k}}\rangle_L^t.$$

The presented system of transport equations gives consistent description of kinetic and hydrodynamic processes of classical fluids which take into account long-living fluctuations.

The system of transport equations (2.22), (2.23) is a not closed due to Lagrange parameter $\gamma(x;t)$, which is determined from the corresponding self-consistent conditions. From the kinetic processes standpoint, we must supplement this system of transport equations with the kinetic equation $f_2(x,x';t)$, and hence for $f_s(x_1\ldots x_s;t)$, $s<N$:

$$\frac{\partial}{\partial t}f_2(x,x';t)+iL_2f_2(x,x';t) \tag{2.30}$$
$$-\int dx''\{iL(x,x'')+iL(x',x'')\}f_3(x,x',x'';t)$$
$$=iL_2\Delta f_2(x,x';t)-\int dx''\{iL(x,x'')+iL(x',x'')\}\Delta f_3(x,x',x'';t)$$
$$+\int dx''\int da\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}\phi_{Gn}(x,x',x'',a;t,t')f(a;t')\gamma(x'';t')$$
$$-\sum_{\mathbf{k}}\int da\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}\Big\{\phi_{Gj}(x,x',\mathbf{k},a;t,t')\cdot\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}$$
$$+\phi_{G\varepsilon}(x,x',\mathbf{k},a;t,t')\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\Big\}\frac{f(a;t')}{W(a;t')},$$

$$\frac{\partial}{\partial t} f_s(x_1 \dots x_s;t) + iL_s f_s(x_1 \dots x_s;t) \tag{2.31}$$
$$- \sum_j \int dx_{s+1} iL(x_j, x_{s+1}) f_{s+1}(x_1 \dots x_s, x_{s+1};t)$$
$$= iL_s \Delta f_s(x_1 \dots x_s;t) - \sum_j \int dx_{s+1} iL(x_j, x_{s+1}) \Delta f_{s+1}(x_1 \dots x_s, x_{s+1};t)$$
$$+ \int dx'' \int da \int_{-\infty}^{t} dt' \mathrm{e}^{\epsilon(t'-t)} \phi_{G_s n}(x, x', x'', a; t, t') f(a;t') \gamma(x'';t')$$
$$- \sum_{\mathbf{k}} \int da \int_{-\infty}^{t} dt' \mathrm{e}^{\epsilon(t'-t)} \Big\{ \phi_{G_s j}(x, x', \mathbf{k}, a; t, t') \cdot \frac{\partial}{\partial \mathbf{J_k}}$$
$$+ \phi_{G_s \varepsilon}(x, x', \mathbf{k}, a; t, t') \frac{\partial}{\partial \varepsilon_{\mathbf{k}}} \Big\} \frac{f(a;t')}{W(a;t')},$$

where

$$\Delta f_2(x, x';t) = f_2(x, x';t) - g_2(x, x';t),$$
$$\Delta f_s(x_1 \dots x_s;t) = f_s(x_1 \dots x_s;t) - g_s(x_1 \dots x_s;t).$$

In Eq.(2.30) the two-particle Liouville operator

$$iL_2 = iL_0(x) + iL_0(x') + iL(x, x')$$

was introduced. It contains one-particle operator

$$iL_0(x) = \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}}, \quad x = \{\mathbf{r}, \mathbf{p}\},$$

and also a potential part

$$iL(x, x') = \frac{\partial}{\partial \mathbf{r}} \Phi(|\mathbf{r} - \mathbf{r}'|) \cdot \Big[\frac{\partial}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}'} \Big].$$

Accordingly, in Eq.(2.31), iL_s is the s-particle Liouville operator, and

$$g_s(x_1 \dots x_s;t) = \langle \hat{G}_s(x_1 \dots x_s) \rangle^t = \int da g_s^L(x_1 \dots x_s;a;t) f(a;t),$$

where

$$g_s^L(x_1 \dots x_s;a;t) = \int d\Gamma_N \hat{G}_s(x_1 \dots x_s) \varrho_L(x^N;a;t)$$

is the s-particle quasi-equilibrium distribution function of large-scale variables and $\hat{G}_s(x^s) = \hat{n}_1(x_1)....\hat{n}_1(x_s)$.

Thus we obtained a system of equations for non-equilibrium single, double, s-particle distribution functions which take into account nonlinear hydrodynamic fluctuations.

We now discuss the equation (2.23) that is of Fokker-Planck type one for non-equilibrium distribution function of collective variables which take into account the kinetic processes. The transport kernel in this equation $\phi_{nn}(x, x';t, t')$ describes a dissipation of kinetic processes, while the kernels $\phi_{nj}(x, \mathbf{k}, a;t, t')$, $\phi_{n\varepsilon}(x, \mathbf{k}, a;t, t')$, $\phi_{jn}(x, \mathbf{k}, a;t, t')$, $\phi_{\varepsilon n}(x, \mathbf{k}, a;t, t')$ describe a dissipation of correlations between kinetic and hydrodynamic processes. To uncover more detailed a structure of transport kernels $\phi_{nn}(x;x', a;t, t')$, $\phi_{Gn}(x;x', x'', a;t, t')$ we consider action of Liouville operator on $\hat{n}_1(x)$ and $\hat{G}(x, x')$:

$$iL_N \hat{n}_1(x) = -\frac{\partial}{\partial \mathbf{r}} \cdot \frac{1}{m} \hat{\mathbf{j}}(\mathbf{r}, \mathbf{p}) + \frac{\partial}{\partial \mathbf{p}} \cdot \hat{\mathbf{F}}(\mathbf{r}, \mathbf{p}), \tag{2.32}$$

$$iL_N \hat{G}(x, x') = -\frac{\partial}{\partial \mathbf{r}} \cdot \frac{1}{m} \hat{\mathbf{j}}(\mathbf{r}, \mathbf{p}) \hat{n}_1(x') - \hat{n}_1(x) \frac{\partial}{\partial \mathbf{r}'} \cdot \frac{1}{m} \hat{\mathbf{j}}(\mathbf{r}', \mathbf{p}')$$
$$+ \frac{\partial}{\partial \mathbf{p}} \cdot \hat{\mathbf{F}}(\mathbf{r}, \mathbf{p}) \hat{n}_1(x') + \hat{n}_1(x) \frac{\partial}{\partial \mathbf{p}'} \cdot \hat{\mathbf{F}}(\mathbf{r}', \mathbf{p}'), \tag{2.33}$$

where

$$\hat{\mathbf{j}}(\mathbf{r}, \mathbf{p}) = \sum_{j=1}^{N} \mathbf{p}_j \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{p} - \mathbf{p}_j) \tag{2.34}$$

is the microscopic density of momentum vector in coordinate-momentum space,

$$\hat{\mathbf{F}}(\mathbf{r}, \mathbf{p}) = \sum_{l \neq j} \frac{\partial}{\partial \mathbf{r}_j} \Phi(|\mathbf{r}_j - \mathbf{r}_l|) \delta(\mathbf{r} - \mathbf{r}_j) \delta(\mathbf{p} - \mathbf{p}_j) \tag{2.35}$$

is the microscopic density of force vector in coordinate-momentum space.

Taking into account equations (2.32)-(2.35), for the kinetic transport kernels, we obtain:

$$\phi_{nn}(x;x', a;t, t') = -\Big[\frac{\partial}{\partial \mathbf{r}} \cdot D_{jj}(x, x', a;t, t') \cdot \frac{\partial}{\partial \mathbf{r}'} \tag{2.36}$$
$$- \frac{\partial}{\partial \mathbf{p}} \cdot D_{Fj}(x, x', a;t, t') \cdot \frac{\partial}{\partial \mathbf{r}'}$$
$$- \frac{\partial}{\partial \mathbf{r}} \cdot D_{jF}(x, x', a;t, t') \cdot \frac{\partial}{\partial \mathbf{p}'} + \frac{\partial}{\partial \mathbf{p}} \cdot D_{FF}(x, x', a;t, t') \cdot \frac{\partial}{\partial \mathbf{p}'} \Big],$$

where

$$D_{jj}(x,x',a;t,t') = \int d\Gamma_N \hat{\mathbf{j}}(x) T_q(t,t')(1-P(t'))\hat{\mathbf{j}}(x')\rho_L(x^N;t'),$$

$$D_{FF}(x,x',a;t,t') = \int d\Gamma_N \hat{\mathbf{F}}(x) T_q(t,t')(1-P(t'))\hat{\mathbf{F}}(x')\rho_L(x^N;t')$$

are the generalized diffusion and the particle friction coefficients in the coordinate-momentum space. Moreover,

$$\int d\mathbf{p} \int d\mathbf{p}' D_{jj}(x,x';t,t') = D_{jj}(\mathbf{r},\mathbf{r}';t,t'),$$

$$\int d\mathbf{p} \int d\mathbf{p}' D_{FF}(x,x';t,t') = D_{FF}(\mathbf{r},\mathbf{r}';t,t')$$

are the generalized coefficients of diffusion and friction. Similarly, we obtain the expression for the transport kernel $\phi_{Gn}(x;x',x'';t,t')$:

$$\begin{aligned}\phi_{Gn}(x;x',x'',a;t,t') = -\Big[&\frac{\partial}{\partial \mathbf{r}} \cdot D_{jjn}(x,x',x'',a;t,t') \cdot \frac{\partial}{\partial \mathbf{r}'} \\ &+\frac{\partial}{\partial \mathbf{r}} \cdot D_{jnj}(x,x',x'',a;t,t') \cdot \frac{\partial}{\partial \mathbf{r}''} \\ &-\frac{\partial}{\partial \mathbf{p}} \cdot D_{Fjn}(x,x',x'',a;t,t') \cdot \frac{\partial}{\partial \mathbf{r}'} - \frac{\partial}{\partial \mathbf{p}} \cdot D_{Fnj}(x,x',x'',a;t,t') \cdot \frac{\partial}{\partial \mathbf{r}''} \\ &-\frac{\partial}{\partial \mathbf{r}} \cdot D_{jFn}(x,x',x'',a;t,t') \cdot \frac{\partial}{\partial \mathbf{p}'} - \frac{\partial}{\partial \mathbf{r}} \cdot D_{jnF}(x,x',x'',a;t,t') \cdot \frac{\partial}{\partial \mathbf{p}''} \\ &+\frac{\partial}{\partial \mathbf{p}} \cdot D_{FFn}(x,x',x'',a;t,t') \cdot \frac{\partial}{\partial \mathbf{p}'} + \frac{\partial}{\partial \mathbf{p}} \cdot D_{FnF}(x,x',x'',a;t,t') \cdot \frac{\partial}{\partial \mathbf{p}''}\Big],\end{aligned} \tag{2.37}$$

It is remarkable that expression

$$\int dx' \int da \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \phi_{nn}(x,x',a;t,t') f(a;t')\gamma(x';t')$$

in equation (2.23) with (2.36) is the generalized collision integral of Fokker-Planck type in the coordinate-momentum space. That is, taking into account (2.23) and (2.36), the kinetic equation (2.22) can be

written as follows:

$$\begin{aligned}\Big[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{r}}\Big] f_1(x;t) - \int dx' \int da \frac{\partial}{\partial \mathbf{r}}\Phi(|\mathbf{r}-\mathbf{r}'|)& \\ \times\Big[\frac{\partial}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}'}\Big] g_2^l(x,x',a;t) f(a;t) &= \\ -\int dx' \int da \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)}& \\ \times \frac{\partial}{\partial \mathbf{r}} \cdot D_{jj}(x,x',a;t,t') \cdot \frac{\partial}{\partial \mathbf{r}'}\gamma(x';t') f(a;t')& \\ +\int dx' \int da \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \Big[\frac{\partial}{\partial \mathbf{p}} \cdot D_{Fj}(x,x',a;t,t') \cdot \frac{\partial}{\partial \mathbf{r}'}& \\ +\frac{\partial}{\partial \mathbf{r}} \cdot D_{jF}(x,x',a;t,t') \cdot \frac{\partial}{\partial \mathbf{p}'} - \frac{\partial}{\partial \mathbf{p}} \cdot D_{FF}(x,x',a;t,t') \cdot \frac{\partial}{\partial \mathbf{p}'}\Big]& \\ \times\gamma(x';t') f(a;t') - \sum_{\mathbf{k}} \int da \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \Big\{\phi_{nj}(x,\mathbf{k},a;t,t') \cdot \frac{\partial}{\partial \mathbf{J_k}}& \\ + \phi_{n\varepsilon}(x,\mathbf{k},a;t,t')\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\Big\}\frac{f(a;t')}{W(a;t')}&.\end{aligned} \tag{2.38}$$

In the equation (2.23) the quantities $\phi_{jj}(\mathbf{k},\mathbf{q},a,a';t,t')$, $\phi_{j\varepsilon}(\mathbf{k},\mathbf{q},a,a';t,t')$, $\phi_{\varepsilon j}(\mathbf{k},\mathbf{q},a,a';t,t')$, $\phi_{\varepsilon\varepsilon}(\mathbf{k},\mathbf{q},a,a';t,t')$ correspond to the dissipative processes connected with the correlations between viscous and heat hydrodynamic processes. The set of equations (2.22), (2.23), (2.30), (2.31) allows two limiting cases. First, if the description of non-equilibrium processes does not take into account nonlinear hydrodynamic fluctuations, we will obtain generalized kinetic equation for the non-equilibrium distribution function of the particles [43]:

$$\begin{aligned}\Big[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}}\Big] f_1(x;t) - \int dx' \frac{\partial}{\partial \mathbf{r}}\Phi(|\mathbf{r}-\mathbf{r}'|) \cdot \Big[\frac{\partial}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}'}\Big] g_2(x,x';t)& \\ = \int dx' \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \phi_{nn}(x,x';t,t')\gamma(x';t')&.\end{aligned} \tag{2.39}$$

Second, if we do not take into account kinetic processes then we will obtain generalized (non-Markov) Fokker-Planck equation for non-equilibrium distribution function of collective variables, which can be obtained by the method of Zwanzig projection operators or by the method of

Zubarev non-equilibrium statistical operator [43]:

$$\begin{aligned}\frac{\partial}{\partial t}f(a;t) &= \sum_{\mathbf{k}}\Big\{\frac{\partial}{\partial n_{\mathbf{k}}}v_{n,\mathbf{k}}(a;t) \\ &+\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\cdot \mathbf{v}_{j,\mathbf{k}}(a;t)+\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}v_{\varepsilon,\mathbf{k}}(a;t)\Big\}f(a;t) \\ &= \sum_{\mathbf{k},\mathbf{q}}\int da'\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\cdot\phi_{jj}(\mathbf{k},\mathbf{q},a,a';t,t')\cdot\frac{\partial}{\partial \mathbf{J}_{\mathbf{q}}}\frac{f(a;t')}{W(a;t')} \\ &+\sum_{\mathbf{k},\mathbf{q}}\int da'\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\phi_{\varepsilon\varepsilon}(\mathbf{k},\mathbf{q},a,a';t,t')\frac{\partial}{\partial \varepsilon_{\mathbf{q}}}\frac{f(a;t')}{W(a;t')} \\ &+\sum_{\mathbf{k},\mathbf{q}}\int da'\int_{-\infty}^{t}dt' e^{\epsilon(t'-t)}\Big\{\frac{\partial}{\partial \mathbf{J}_{\mathbf{k}}}\cdot\phi_{j\varepsilon}(\mathbf{k},\mathbf{q},a,a';t,t')\frac{\partial}{\partial \varepsilon_{\mathbf{q}}} \\ &+\frac{\partial}{\partial \varepsilon_{\mathbf{k}}}\phi_{\varepsilon j}(\mathbf{k},\mathbf{q},a,a';t,t')\cdot\frac{\partial}{\partial \mathbf{J}_{\mathbf{q}}}\Big\}\frac{f(a;t')}{W(a;t')}.\end{aligned} \tag{2.40}$$

The hard problem for analyzing of the transport equations (2.22), (2.23) and transport kernels (2.25) is calculation of structure function $W(a;t)$ of collective variables and of the hydrodynamic velocities $v_{n,\mathbf{k}}(a;t)$, $\mathbf{v}_{j,\mathbf{k}}(a;t)$, $v_{\varepsilon,\mathbf{k}}(a;t)$.

3. Calculation of structure function $W(a;t)$ and hydrodynamical velocities $v_{l,\mathbf{k}}(a;t)$

In the Kawasaki theory [44] of non-linear fluctuations, the structure function is approximated by a gaussian dependence on collective variables. In this case, as it can be seen, the hydrodynamic velocities $v_{l,\mathbf{k}}(a;t)$, $l=n,j,\varepsilon$ are the linear function of a. Other approach for calculation of hydrodynamical velocities $v_{l,\mathbf{k}}(a;t)$ was proposed on the basis of local thermodynamics [43]. The resulting expressions are valid obviously at low frequencies and for small values of the wave vector, when the ratios of the local thermodynamics are valid. Structure function $W(a;t)$ and hydrodynamical velocities $v_{l,\mathbf{k}}(a;t)$ in a case of study of hydrodynamic fluctuations were calculated in [45, 47] using the method of collective variables [46]. The basic idea of this approach is that the structure function $W(a;t)$ and hydrodynamic velocities $v_{l,\mathbf{k}}(a;t)$ can be calculated in approximations higher than Gaussian. Next, we apply the method of collective variables [45–47] for calculating the structure function $W(a;t)$ and hydrodynamic velocities $v_{l,\mathbf{k}}(a;t)$.

First, we calculate the structure function $W(a;t)$ for collective variables. To do this, we use the integral representation for δ-functions:

$$\hat{f}(a)=\int d\omega\exp\Big\{-i\pi\sum_{l,\mathbf{k}}\omega_{l,\mathbf{k}}(\hat{a}_{l,\mathbf{k}}-a_{l,\mathbf{k}})\Big\},\quad l=n,\mathbf{j},\varepsilon. \tag{3.1}$$

Next, using a cumulant expansion [47] for $W(a;t)$ one obtains:

$$\begin{aligned}W(a;t) &= \int d\Gamma_N\varrho_q^{kin}(x^N;t)\hat{f}(a) \\ &= \int d\omega\exp\Big\{-i\pi\sum_{l,\mathbf{k}}\omega_{l,\mathbf{k}}\bar{a}_{l,\mathbf{k}} \\ &-\frac{\pi^2}{2}\sum_{l_1,l_2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\mathfrak{M}_2^{l_1,l_2}(\mathbf{k}_1,\mathbf{k}_2;t)\omega_{l_1,\mathbf{k}_1}\omega_{l_2,\mathbf{k}_2}\Big\}\exp\Big\{\sum_{n\geq 3}D_n(\omega;t)\Big\},\end{aligned} \tag{3.2}$$

where

$$\bar{a}_{l,\mathbf{k}}=a_{l,\mathbf{k}}-\langle\hat{a}_{l,\mathbf{k}}\rangle^t_{kin},\ \ d\omega=\prod_{l,\mathbf{k}}d\omega^r_{l,\mathbf{k}}d\omega^s_{l,\mathbf{k}},$$
$$\omega_{l,\mathbf{k}}=\omega^r_{l,\mathbf{k}}-i\omega^s_{l,\mathbf{k}},\ \ \omega_{l,-\mathbf{k}}=\omega^*_{l,\mathbf{k}},$$

$$D_n(\omega;t)=\frac{(-i\pi)^n}{n!}\sum_{l_1,\dots,l_n}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_n}\mathfrak{M}_n^{l_1,\dots,l_n}(\mathbf{k}_1,\dots,\mathbf{k}_n;t)\times\omega_{l_1,\mathbf{k}_1}\dots\omega_{l_n,\mathbf{k}_n}, \tag{3.3}$$

$$\mathfrak{M}_n^{l_1,\dots,l_n}(\mathbf{k}_1,\dots,\mathbf{k}_n;t)=\langle\hat{a}_{l_1,\mathbf{k}_1}\dots\hat{a}_{l_n,\mathbf{k}_n}\rangle^{t,c}_{kin} \tag{3.4}$$

are the non-equilibrium cumulant averages in approximations of the n-order, which are calculated using distribution $\varrho_q^{kin}(x^N;t)$ (2.11). We present the structure function $W(a;t)$ for further calculations in following form:

$$\begin{aligned}W(a;t) &= \int d\omega\exp\{-i\pi\sum_{l,\mathbf{k}}\omega_{l,\mathbf{k}}\bar{a}_{l,\mathbf{k}} \\ &-\frac{\pi^2}{2}\sum_{l_1,l_2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\mathfrak{M}_2^{l_1,l_2}(\mathbf{k}_1,\mathbf{k}_2;t)\omega_{l_1,\mathbf{k}_1}\omega_{l_2,\mathbf{k}_2}\} \\ &\times\Big(1+B+\frac{1}{2!}B^2+\frac{1}{3!}B^3+\dots+\frac{1}{n!}B^n+\dots\Big),\end{aligned} \tag{3.5}$$

where $B = \sum_{n\geq 3} D_n(\omega;t)$. If in series of exponent (3.5), namely, $e^{\sum_{n\geq 3} D_n(\omega;t)}$, one retains only the first term equal to unity, one will obtain the Gaussian approximation for $W(a;t)$:

$$W^G(a;t) = \int d\omega \exp\{i\pi \sum_{l,\mathbf{k}} \omega_{l,\mathbf{k}} \bar{a}_{l,\mathbf{k}} \tag{3.6}$$

$$-\frac{\pi^2}{2} \sum_{l_1,l_2} \sum_{\mathbf{k}_1,\mathbf{k}_2} \mathfrak{M}_2^{l_1,l_2}(\mathbf{k}_1,\mathbf{k}_2;t)\omega_{l_1,\mathbf{k}_1}\omega_{l_2,\mathbf{k}_2}\},$$

where $\mathfrak{M}_2^{l_1,l_2}(\mathbf{k}_1,\mathbf{k}_2;t)$ are the matrix elements of non-equilibrium correlation functions:

$$\mathfrak{M}_2(\mathbf{k}_1,\mathbf{k}_2;t) = \begin{vmatrix} \langle\hat{n}\hat{n}\rangle^c_{kin} & \langle\hat{n}\hat{\mathbf{J}}\rangle^c_{kin} & \langle\hat{n}\hat{\varepsilon}\rangle^c_{kin} \\ \langle\hat{\mathbf{J}}\hat{n}\rangle^c_{kin} & \langle\hat{\mathbf{J}}\hat{\mathbf{J}}\rangle^c_{kin} & \langle\hat{\mathbf{J}}\hat{\varepsilon}\rangle^c_{kin} \\ \langle\hat{\varepsilon}\hat{n}\rangle^c_{kin} & \langle\hat{\varepsilon}\hat{\mathbf{J}}\rangle^c_{kin} & \langle\hat{\varepsilon}\hat{\varepsilon}\rangle^c_{kin} \end{vmatrix}_{(\mathbf{k}_1,\mathbf{k}_2)}, \tag{3.7}$$

and the non-equilibrium cumulant average

$$\langle\hat{n}_{\mathbf{k}}\hat{n}_{-\mathbf{k}}\rangle^{t,c}_{kin} = \langle\hat{n}_{\mathbf{k}}\hat{n}_{-\mathbf{k}}\rangle^{t}_{kin} - \langle\hat{n}_{\mathbf{k}}\rangle^{t}_{kin}\langle\hat{n}_{-\mathbf{k}}\rangle^{t}_{kin}. \tag{3.8}$$

For integrating over $d\omega$ in (3.5) we should transform the quadratic form in exponential expression into a diagonal form with respect to $\omega_{l,\mathbf{k}}$. To this end it is necessary to find the eigenvalues of the matrix (3.7) by solving the equation

$$\det\left|\tilde{\mathfrak{M}}_2(\mathbf{k}_1,\mathbf{k}_2;t) - \tilde{E}(\mathbf{k}_1,\mathbf{k}_2;t)\right| = 0,$$

$\tilde{E}(\mathbf{k}_1,\mathbf{k}_2;t)$ is the diagonal matrix. Further, obtained eigenvalues $E_l(\mathbf{k};t)$, $l=1,\ldots,5$ of the expression (3.6) are as follows:

$$W^G(a;t) = \int d\tilde{\omega}\, \det\tilde{W}\, \exp\Big\{-i\pi \sum_{l,\mathbf{k}} \tilde{a}_{l\mathbf{k}}\tilde{\omega}_{l\mathbf{k}} \tag{3.9}$$

$$-\frac{\pi^2}{2}\sum_{l}\sum_{\mathbf{k}} E_l(\mathbf{k};t)\tilde{\omega}_{l\mathbf{k}}\tilde{\omega}_{l,-\mathbf{k}}\Big\},$$

where new variables $\tilde{a}_{l\mathbf{k}}$, $\tilde{\omega}_{l\mathbf{k}}$ are connected with the old variables by ratio:

$$\tilde{a}_{n\mathbf{k}} = \sum_{l} \bar{a}_{l\mathbf{k}}\omega_{ln}, \qquad \omega_{l\mathbf{k}} = \sum_{m=1}^{3}\omega_{lm}\tilde{\omega}_{m\mathbf{k}},$$

and ω_{lm} are matrix elements $\tilde{W} = \begin{vmatrix} \omega_{11} & \ldots & \omega_{15} \\ \vdots & \ddots & \vdots \\ \omega_{51} & \ldots & \omega_{55} \end{vmatrix}_{(\mathbf{k};t)}$. Integrand in (3.9) is a quadratic function $\tilde{\omega}_{n\mathbf{k}}$ and after integrating over $d\omega_{n\mathbf{k}}$ we will obtain following structural function in Gaussian approximation $W^G(a;t)$:

$$W^G(a;t) = \exp\Big\{-\frac{1}{2}\sum_{l,\mathbf{k}} E_l^{-1}(\mathbf{k};t)\tilde{a}_{l\mathbf{k}}\tilde{a}_{l,-\mathbf{k}}\Big\} \tag{3.10}$$

$$\times \exp\Big\{-\frac{1}{2}\sum_{\mathbf{k}} \ln \pi \det \tilde{E}(\mathbf{k};t)\Big\} \exp\Big\{\sum_{\mathbf{k}} \ln \det \tilde{W}(\mathbf{k};t)\Big\},$$

or through variables $\bar{a}_{l\mathbf{k}}$:

$$W^G(a;t) = Z(t)\, \exp\Big\{-\frac{1}{2}\sum_{l,\mathbf{k}} \bar{E}_l(\mathbf{k};t)\bar{a}_{l\mathbf{k}}\bar{a}_{l,-\mathbf{k}}\Big\}, \tag{3.11}$$

where

$$\bar{E}_l(\mathbf{k};t) = \sum_{l'} \omega_{ll'}E_{l'}^{-1}(\mathbf{k};t)\omega_{l'l},$$

$$Z(t) = \exp\Big\{-\frac{1}{2}\sum_{\mathbf{k}} \ln \pi \det \tilde{E}(\mathbf{k};t)\Big\} \exp\Big\{\sum_{\mathbf{k}} \ln \det \tilde{W}(\mathbf{k};t)\Big\}.$$

The structure function $W^G(a;t)$ gives a possibility to calculate (3.5) in higher approximations over Gaussian moments [47]:

$$W(a;t) = W^G(a;t) \exp\Big\{\sum_{n\geq 3}\langle\tilde{D}_n(a;t)\rangle_G\Big\}, \tag{3.12}$$

where one presents $\langle\tilde{D}_n(a;t)\rangle_G$ approximately as:

$$\langle\tilde{D}_3(a;t)\rangle_G = \langle\bar{D}_3(a;t)\rangle_G,$$

$$\langle\tilde{D}_4(a;t)\rangle_G = \langle\bar{D}_4(a;t)\rangle_G,$$

$$\langle\tilde{D}_6(a;t)\rangle_G = \langle\bar{D}_6(a;t)\rangle_G - \frac{1}{2}\langle\bar{D}_3(a;t)\rangle_G^2,$$

$$\langle\tilde{D}_8(a;t)\rangle_G = \langle\bar{D}_8(a;t)\rangle_G - \langle\bar{D}_3(a;t)\rangle_G\langle\bar{D}_5(a;t)\rangle_G - \frac{1}{2}\langle\bar{D}_4(a;t)\rangle_G^2,$$

$$\langle\tilde{D}_n(a;t)\rangle_G = \frac{1}{W^G(a;t)}\sum_{l_1,\ldots,l_n}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n}\tilde{\mathfrak{M}}_n^{l_1,\ldots,l_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n;t)$$

$$\times \frac{1}{(i\pi)^n} \frac{\delta^n}{\delta\bar{a}_{l_1,\mathbf{k}_1}...\delta\bar{a}_{l_n,\mathbf{k}_n}} W^G(a;t).$$

$\bar{\mathfrak{M}}_n^{l_1,\dots,l_n}(\mathbf{k}_1,\dots,\mathbf{k}_n;t)$ are the renormalized non-equilibrium cumulant averages of order n for the variables $\bar{a}_{l\mathbf{k}}$. In expression (3.12) the summands are with only even degrees over a since all odd Gaussian moments vanish.

The method of calculation of the structure function $W(a;t)$ can be applied for approximate calculations of hydrodynamic velocities $v_{l,\mathbf{k}}(a;t)$. We present general formula of velocities consistent with (2.29) in following form:

$$v_{l,\mathbf{k}}(a;t) = \int d\Gamma_N \dot{\hat{a}}_{l,\mathbf{k}} \varrho_q^{kin}(x^N;t)\hat{f}(a)$$

and introduce function $W(a,\lambda;t)$:

$$W(a,\lambda;t) = \int d\Gamma_N \; \mathrm{e}^{-i\pi\sum_{l,\mathbf{k}}\lambda_{l,\mathbf{k}}\dot{\hat{a}}_{l,\mathbf{k}}} \; \varrho_q^{kin}(x^N;t)\hat{f}(a),$$

so that

$$v_{l,\mathbf{k}}(a;t) = \frac{\partial}{\partial(-i\pi\lambda_{l,\mathbf{k}})} \ln W(a,\lambda;t)\Big|_{\lambda_{l,\mathbf{k}}=0}. \tag{3.13}$$

We calculate the function $W(a,\lambda;t)$ using the preliminary results of the calculation of the structural function $W(a;t)$, and rewrite $W(a,\lambda;t)$ as:

$$W(a,\lambda;t) = \int d\Gamma_N \int d\omega \exp\Big\{ -i\pi\sum_{l,\mathbf{k}} \lambda_{l,\mathbf{k}}\dot{\hat{a}}_{l,\mathbf{k}}\Big\} \tag{3.14}$$
$$\times \exp\Big\{ -i\pi\sum_{l,\mathbf{k}} \omega_{l,\mathbf{k}}(\hat{a}_{l,\mathbf{k}} - a_{l,\mathbf{k}})\Big\}\varrho_q^{kin}(x^N;t).$$

Now we carry out an averaging in (3.14) with $\varrho_q^{kin}(x^N;t)$ using following cumulant expansion:

$$W(a,\lambda;t) = \int d\omega \exp\Big\{ -i\pi\sum_{l,\mathbf{k}} \omega_{l,\mathbf{k}}\bar{a}_{l,\mathbf{k}} \tag{3.15}$$
$$+ \sum_{n\ge 1}\left[D_n(\omega;t) + D_n(\lambda;t) + D_n(\omega,\lambda;t)\right]\Big\},$$

where

$$D_n(\omega;t) = \frac{(-i\pi)^n}{n!} \sum_{l_1,\dots,l_n}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_n} \mathfrak{M}_n^{l_1,\dots,l_n}(\mathbf{k}_1,\dots,\mathbf{k}_n;t)\omega_{l_1,\mathbf{k}_1}\dots\omega_{l_n,\mathbf{k}_n},$$

$$D_n(\lambda;t) = \frac{(-i\pi)^n}{n!} \sum_{l_1,\dots,l_n}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_n} \mathfrak{M}_n^{(1)l_1,\dots,l_n}(\mathbf{k}_1,\dots,\mathbf{k}_n;t)\lambda_{l_1,\mathbf{k}_1}\dots\lambda_{l_n,\mathbf{k}_n},$$

$$D_n(\omega,\lambda;t) = \frac{(-i\pi)^n}{n!} \sum_{l_1,\dots,l_n}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_n} \mathfrak{M}_n^{(2)l_1,\dots,l_n}(\mathbf{k}_1,\dots,\mathbf{k}_n;t)$$
$$\times\, \omega_{l_1,\mathbf{k}_1}\dots\omega_{l_{n-1},\mathbf{k}_{n-1}}\dots\lambda_{l_n,\mathbf{k}_n},$$

with the cumulants of following structure:

$$\mathfrak{M}_n^{l_1,\dots,l_n}(\mathbf{k}_1,\dots,\mathbf{k}_n;t) = \langle \hat{a}_{l_1,\mathbf{k}_1},\dots\hat{a}_{l_n,\mathbf{k}_n}\rangle_{kin}^{t,c},$$
$$\mathfrak{M}_n^{(1)l_1,\dots,l_n}(\mathbf{k}_1,\dots,\mathbf{k}_n;t) = \langle \dot{\hat{a}}_{l_1,\mathbf{k}_1},\dots\dot{\hat{a}}_{l_n,\mathbf{k}_n}\rangle_{kin}^{t,c},$$
$$\mathfrak{M}_n^{(2)l_1,\dots,l_n}(\mathbf{k}_1,\dots,\mathbf{k}_n;t) = n[(n-j)+(j-n+1)\delta_{l_1,\dots,l_{n-1}}]$$
$$\times\,\langle \hat{a}_{l_1,\mathbf{k}_1},\dots,\hat{a}_{l_{n-j},\mathbf{k}_{n-j}},\dots,\dot{\hat{a}}_{l_{n-j+1},\mathbf{k}_{n-j+1}},\dots,\dot{\hat{a}}_{l_n,\mathbf{k}_n}\rangle_{kin}^{t,c}.$$

First, we consider a Gaussian approximation for $W(a,\lambda;t)$, namely we leave in the exponent of an integrand only the summands with $n=2$ and linear over $\lambda_{l,\mathbf{k}}$:

$$W^G(a,\lambda;t) = \int d\omega \exp\Big\{ i\pi\sum_{l,\mathbf{k}}\omega_{l,\mathbf{k}}\bar{a}_{l,\mathbf{k}} - i\pi\sum_{l,\mathbf{k}}\langle\dot{\hat{a}}_{l,\mathbf{k}}\rangle_{kin}^{t,c}\lambda_{l,\mathbf{k}} \tag{3.16}$$
$$-\frac{\pi^2}{2}\sum_{l_1,l_2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\mathfrak{M}_2^{l_1,l_2}(\mathbf{k}_1,\mathbf{k}_2;t)\omega_{l_1,\mathbf{k}_1}\omega_{l_2,\mathbf{k}_2}$$
$$-\frac{\pi^2}{2}\sum_{l_1,l_2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\mathfrak{M}_2^{(2)l_1,l_2}(\mathbf{k}_1,\mathbf{k}_2;t)\omega_{l_1,\mathbf{k}_1}\lambda_{l_2,\mathbf{k}_2}\Big\}.$$

Then, transforming this expression in the exponent to diagonal quadratic form over variables $\omega_{l,\mathbf{k}}$, similarly as for $W(a;t)$, after integrating with respect to the new variables $\bar{\omega}_{l,\mathbf{k}}$, one obtains:

$$W^G(a,\lambda;t) = \int d\omega \exp\Big\{ -i\pi\sum_{l,\mathbf{k}}\langle\dot{\hat{a}}_{l,\mathbf{k}}\rangle_{kin}^{t}\lambda_{l,\mathbf{k}} \tag{3.17}$$
$$-\frac{\pi^2}{2}\sum_{l,\mathbf{k}}E_l^{-1}(\mathbf{k};t)b_{l,\mathbf{k}}b_{l,-\mathbf{k}} - \frac{1}{2}\sum_{\mathbf{k}}\ln\pi det\tilde{E}(\mathbf{k};t)$$
$$+\sum_{\mathbf{k}}\ln det\tilde{W}(\mathbf{k};t)\Big\},$$

where

$$b_{l,\mathbf{k}} = \sum_j \omega_{lj}\Big[\bar{a}_{j,\mathbf{k}} + \frac{i\pi}{2}\sum_{j'}\mathfrak{M}_2^{(2)j,j'}(\mathbf{k};t)\lambda_{j',\mathbf{k}}\Big],$$

and ω_{lj}, $\mathfrak{M}_2^{(2)j,j'}(\mathbf{k};t)$ and $E_l(\mathbf{k};t)$ do not dependent on $\lambda_{l,\mathbf{k}}$. Here the cumulant $\mathfrak{M}_2^{(2)j,j'}(\mathbf{k};t)$ has the following structure:

$$\mathfrak{M}_2^{(2)j,j'}(\mathbf{k};t) = \langle\dot{\hat{a}}_{j,\mathbf{k}}\hat{a}_{j',-\mathbf{k}}\rangle^t_{kin} - \langle\dot{\hat{a}}_{j,\mathbf{k}}\rangle^t_{kin}\langle\hat{a}_{j',-\mathbf{k}}\rangle^t_{kin}. \tag{3.18}$$

Now we calculate the hydrodynamic velocities $v_{l,\mathbf{k}}(a;t)$ in Gaussian approximation according to the formula (3.13) :

$$v_{l,\mathbf{k}}(a;t) = \frac{\partial}{\partial(-i\pi\lambda_{l,\mathbf{k}})}\ln W^G(a,\lambda;t)\Big|_{\lambda_{l,\mathbf{k}}=0} \tag{3.19}$$

$$= \langle\dot{\hat{a}}_{j,\mathbf{k}}\rangle^t_{kin} - \frac{1}{2}\sum_{j,j'}E_l^{-1}(\mathbf{k};t)\omega_{jl}\omega_{j'l}\mathfrak{M}_2^{(2)j',l}(\mathbf{k};t)\bar{a}_{l,\mathbf{k}}.$$

Specifically, we consider the particular case when one can divide the longitudinal and transverse fluctuations for collective variables. That is, we choose the direction of the wave vector $\mathbf{k}$ along the axis of oz. Thus, one obtains:

$$\begin{aligned} W^G(a;t) = \int d\omega \exp\{ & i\pi\sum_{l,\mathbf{k}}\omega_{l,\mathbf{k}}\bar{a}_{l,\mathbf{k}} \\ & -\frac{\pi^2}{2}\sum_{l_1,l_2=1}^{3}\sum_{\mathbf{k}_1,\mathbf{k}_2}\mathfrak{M}_2^{\parallel,l_1,l_2}(\mathbf{k}_1,\mathbf{k}_2;t)\omega_{l_1,\mathbf{k}_1}\omega_{l_2,\mathbf{k}_2} \\ & -\frac{\pi^2}{2}\sum_{l_1,l_2=1}^{4}\sum_{\mathbf{k}_1,\mathbf{k}_2}\mathfrak{M}_2^{\parallel,\perp,l_1,l_2}(\mathbf{k}_1,\mathbf{k}_2;t)\omega_{l_1,\mathbf{k}_1}\omega_{l_2,\mathbf{k}_2}\}, \end{aligned} \tag{3.20}$$

where $\mathfrak{M}_2^{\parallel,l_1,l_2}(\mathbf{k}_1,\mathbf{k}_2;t)$ are the matrix elements of the non-equilibrium correlation functions of longitudinal fluctuations

$$\mathfrak{M}_2^{\parallel}(\mathbf{k}_1,\mathbf{k}_2;t) = \begin{vmatrix} \langle\hat{n}\hat{n}\rangle^c_{kin} & \langle\hat{n}\hat{\mathbf{J}}^{\parallel}\rangle^c_{kin} & \langle\hat{n}\hat{\varepsilon}\rangle^c_{kin} \\ \langle\hat{\mathbf{J}}^{\parallel}\hat{n}\rangle^c_{kin} & \langle\hat{\mathbf{J}}^{\parallel}\hat{\mathbf{J}}^{\parallel}\rangle^c_{kin} & \langle\hat{\mathbf{J}}^{\parallel}\hat{\varepsilon}\rangle^c_{kin} \\ \langle\hat{\varepsilon}\hat{n}\rangle^c_{kin} & \langle\hat{\varepsilon}\hat{\mathbf{J}}^{\parallel}\rangle^c_{kin} & \langle\hat{\varepsilon}\hat{\varepsilon}\rangle^c_{kin} \end{vmatrix}_{(\mathbf{k_1},\mathbf{k_2})}, \tag{3.21}$$

and $\mathfrak{M}_2^{\perp l_1,l_2}(\mathbf{k}_1,\mathbf{k}_2;t)$ are the matrix elements of the non-equilibrium correlation functions of transverse and transverse-longitudinal fluctuations

$$\mathfrak{M}_2^{\parallel,\perp}(\mathbf{k}_1,\mathbf{k}_2;t) = \begin{vmatrix} 0 & \langle\hat{n}\hat{\mathbf{J}}^{\perp}_x\rangle^c_{kin} & \langle\hat{n}\hat{\mathbf{J}}^{\perp}_y\rangle^c_{kin} & 0 \\ \langle\hat{\mathbf{J}}^{\perp}_x\hat{n}\rangle^c_{kin} & \langle\hat{\mathbf{J}}^{\perp}_x\hat{\mathbf{J}}^{\perp}_x\rangle^c_{kin} & \langle\hat{\mathbf{J}}^{\perp}_x\hat{\mathbf{J}}^{\perp}_y\rangle^c_{kin} & \langle\hat{\mathbf{J}}^{\perp}_x\hat{\varepsilon}\rangle^c_{kin} \\ \langle\hat{\mathbf{J}}^{\perp}_y\hat{n}\rangle^c_{kin} & \langle\hat{\mathbf{J}}^{\perp}_y\hat{\mathbf{J}}^{\perp}_x\rangle^c_{kin} & \langle\hat{\mathbf{J}}^{\perp}_y\hat{\mathbf{J}}^{\perp}_y\rangle^c_{kin} & \langle\hat{\mathbf{J}}^{\perp}_y\hat{\varepsilon}\rangle^c_{kin} \\ 0 & \langle\hat{\varepsilon}\hat{\mathbf{J}}^{\perp}_x\rangle^c_{kin} & \langle\hat{\varepsilon}\hat{\mathbf{J}}^{\perp}_y\rangle^c_{kin} & 0 \end{vmatrix}_{(\mathbf{k_1},\mathbf{k_2})}. \tag{3.22}$$

In this case, the hydrodynamic velocities in the Gaussian approximation are as follows:

$$\begin{aligned} v^{\parallel G}_{n\mathbf{k}}(a;t) &= \langle\dot{\hat{n}}_{\mathbf{k}}\rangle^t_{kin} + E_1^{-1}(\mathbf{k};t)(\omega_{11}\bar{n}_{\mathbf{k}} + \omega_{21}\bar{\mathbf{J}}^{\parallel}_{\mathbf{k}} + \omega_{31}\bar{\varepsilon}_{\mathbf{k}})\Omega_n(\mathbf{k};t), \\ v^{\parallel G}_{J\mathbf{k}}(a;t) &= \langle\dot{\hat{\mathbf{J}}}^{\parallel}_{\mathbf{k}}\rangle^t_{kin} + E_2^{-1}(\mathbf{k};t)(\omega_{12}\bar{n}_{\mathbf{k}} + \omega_{22}\bar{\mathbf{J}}^{\parallel}_{\mathbf{k}} + \omega_{32}\bar{\varepsilon}_{\mathbf{k}})\Omega_J(\mathbf{k};t), \\ v^{\parallel G}_{\varepsilon\mathbf{k}}(a;t) &= \langle\dot{\hat{\varepsilon}}_{\mathbf{k}}\rangle^t_{kin} + E_3^{-1}(\mathbf{k};t)(\omega_{13}\bar{n}_{\mathbf{k}} + \omega_{23}\bar{\mathbf{J}}^{\parallel}_{\mathbf{k}} + \omega_{33}\bar{\varepsilon}_{\mathbf{k}})\Omega_\varepsilon(\mathbf{k};t), \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} \Omega_n(\mathbf{k};t) &= \omega_{11}\langle\hat{n}_{\mathbf{k}}\dot{\hat{n}}_{-\mathbf{k}}\rangle^{t,c}_{kin} + \omega_{21}\langle\hat{\mathbf{J}}^{\parallel}_{\mathbf{k}}\dot{\hat{n}}_{-\mathbf{k}}\rangle^{t,c}_{kin} + \omega_{31}\langle\hat{\varepsilon}_{\mathbf{k}}\dot{\hat{n}}_{-\mathbf{k}}\rangle^{t,c}_{kin}, \\ \Omega_J(\mathbf{k};t) &= \omega_{12}\langle\hat{n}_{\mathbf{k}}\dot{\hat{\mathbf{J}}}^{\parallel}_{-\mathbf{k}}\rangle^{t,c}_{kin} + \omega_{22}\langle\hat{\mathbf{J}}^{\parallel}_{\mathbf{k}}\dot{\hat{\mathbf{J}}}^{\parallel}_{-\mathbf{k}}\rangle^{t,c}_{kin} + \omega_{32}\langle\hat{\varepsilon}_{\mathbf{k}}\dot{\hat{\mathbf{J}}}^{\parallel}_{-\mathbf{k}}\rangle^{t,c}_{kin}, \\ \Omega_\varepsilon(\mathbf{k};t) &= \omega_{13}\langle\hat{n}_{\mathbf{k}}\dot{\hat{\varepsilon}}_{-\mathbf{k}}\rangle^{t,c}_{kin} + \omega_{23}\langle\hat{\mathbf{J}}^{\parallel}_{\mathbf{k}}\dot{\hat{\varepsilon}}_{-\mathbf{k}}\rangle^{t,c}_{kin} + \omega_{33}\langle\hat{\varepsilon}_{\mathbf{k}}\dot{\hat{\varepsilon}}_{-\mathbf{k}}\rangle^{t,c}_{kin}, \end{aligned} \tag{3.24}$$

and ω_{lj} are the elements of matrix $\tilde{W}(\mathbf{k};t)$. As one can be see, the hydrodynamic velocities (3.23) in the Gaussian approximation for $W^G(a,\lambda;t)$ are the linear functions of collective variables $n_{\mathbf{k}}$, $\mathbf{J}_{\mathbf{k}}$ and $\varepsilon_{\mathbf{k}}$. It is remarkable that if the kinetic processes are not taken into account, then $\varrho_q^{kin}(x^N;t) = 1$ $\langle\ldots\rangle^t_{kin} \to \langle\ldots\rangle_0$ is an averaging over a microscopic ensemble $W(a)$; in this case the expressions (3.23) for hydrodynamic velocities transform into the results of previous work [47], in which the nonlinear hydrodynamic fluctuations in simple fluids were investigated. The collective variable method [45–47] give a possibility to calculate the hydrodynamic velocities in approximations higher than the Gaussian one. In particular, the approximation for the Gaussian, based on (3.15) and hydrodynamic velocities (3.23) will be proportional to $\bar{a}_{l,\mathbf{k}}\bar{a}_{l',\mathbf{k}}$, and transport kernels in the Fokker-Planck equation will be the fourth-order correlation functions over the variables $\hat{a}_{l,\mathbf{k}}$.

It is important that in Gaussian approximation for $\tilde{W}^G(\mathbf{k};t)$ and $v^G_{l,\mathbf{k}}(a;t)$, the Fokker-Planck equation leads to the transport equations for $\langle\hat{a}_{l,\mathbf{k}}\rangle^t$, which are similar in structure to the case of the molecular hydrodynamics, averaged only over $\varrho_L(x^N,a;t) = \varrho_q^{kin}(x^N;t)\frac{\hat{f}(a)}{W^G(a;t)}$. The proposed approach makes possible to go beyond the Gaussian approximation for $\tilde{W}(\mathbf{k};t)$ and $v_{l,\mathbf{k}}(a;t)$, and hence to do the same in the transport kernels in Foker-Planck equation. This allows us to obtain a nonlinear equation system for $\langle \hat{a}_{l,\mathbf{k}} \rangle^t$.

It is noteworthy that kinetic equation (2.23) contains a generalized integral of Fokker-Planck type with generalized coefficients of diffusion and particle friction in the phase space $(\mathbf{r}, \mathbf{p}, t)$. This region of changes $|\mathbf{r}|$ is limited by values $|\mathbf{k}|^{-1}_{hydr}$, that correspond to collective nonlinear hydrodynamic processes. This means that in regions of limited $|\mathbf{k}|^{-1}_{hydr}$ the processes are described by the generalized coefficients of diffusion and friction, and at small $|\mathbf{k}|^{-1}_{hydr}$ they are described by generalized viscosity, thermal conductivity and by cross coefficients $\phi_{jj}(\mathbf{k}, \mathbf{q}, a, a'; t, t')$, $\phi_{j\varepsilon}(\mathbf{k}, \mathbf{q}, a, a'; t, t')$, $\phi_{\varepsilon j}(\mathbf{k}, \mathbf{q}, a, a'; t, t')$, $\phi_{\varepsilon\varepsilon}(\mathbf{k}, \mathbf{q}, a, a'; t, t')$. Correlations between these regions are described by cross kernels $\phi_{nj}(x, \mathbf{q}, a, a'; t, t')$, $\phi_{n\varepsilon}(x, \mathbf{q}, a, a'; t, t')$, $\phi_{\varepsilon n}(\mathbf{k}, x', a, a'; t, t')$, $\phi_{jn}(\mathbf{k}, x', a, a'; t, t')$, that are present both in the kinetic equation and in the Fokker-Planck equation. The calculations of these kernels is very important because they describe the cross-correlations between kinetic and hydrodynamic processes.

4. Conclusions

Using the method of Zubarev non-equilibrium statistical operator, we have developed an approach [42, 43] for consistent description of kinetic and hydrodynamic processes, that are characterized by non-linear fluctuations. We have obtained the non-equilibrium statistical operator of non-equilibrium state of the system when the parameters of the reduced description are a non-equilibrium one-particle distribution function and the non-equilibrium distribution function of the non-linear hydrodynamic variables (densities of mass, momentum and energy). By using this operator we constructed a chain of kinetic equations (of BBGKY type) for non-equilibrium single, double, s-particle distribution functions of particles that take into account the nonlinear hydrodynamic fluctuations. At the same time the non-equilibrium distribution function of hydrodynamic fluctuations satisfy a generalized Fokker-Planck equation.

We proposed a method to calculate the structural distribution function of hydrodynamic collective variables and their hydrodynamic velocities (above Gaussian approximation) contained in a generalized Fokker-Planck equation for the non-equilibrium distribution function of hydrodynamic collective variables. In the future studies, we will go beyond the Gaussian approximation and carry out approximate calculations of kinetic transport coefficients for a specific system of interacting particles.

References

1. M. Tokuyama, Physica A 395 (2014) 31.
2. Guo Ran, Du Jiulin, Physica A 406 (2014) 281.
3. P. Mendoza-Mendez, L. Lopez-Flores, A. Vizcarra-Redon, L.P. Sanchez-Diaz, M. Medina-Noyola, Physica A 394 (2014) 1.
4. Du Jiulin, Physica A 391 (2012) 1718.
5. S. Lang, R. Schilling, V. Krakoviack, T. Franosch, Phys. Rev. E 86 (2012) 021502.
6. K. Yoshida, T.Arimitsu, J. Phys. A: Math. Theor. 46 (2013) 335501.
7. J.P. Boon, J.F. Lutsko, C. Lutsko, Phys.Rev. E 85 (2012) 0211126.
8. G.F. Mazenko, Phys. Rev. E 81 (2010) 061102.
9. G.F. Mazenko, Phys. Rev. E 83 (2011) 041125.
10. S.P. Das, G.F. Mazenko, J. Stat. Phys. 152 (2013) 159.
11. P. Kostrobij, R. Tokarchuk, M. Tokarchuk, B. Markiv, Conden. Matter Phys. 17 (2014) 33005.
12. P.A. Hlushak, M.V. Tokarchuk, Cond. Matt. Phys. 17 (2014) 23606.
13. C.A.B. Silva, A.R. Vasconcellos, J.G. Ramos, R. Luzzi, J. Stat. Phys. 143 (2011) 1020.
14. C.A.B. Silva, J.G. Ramos, A.R. Vasconcellos, R. Luzzi, arXiv: 1210.7280 [physics. flu-dyn], 2012.
15. V.N. Tsytovich, U. de Andelis, Phys. Plasmas 11 (2004) 496.
16. A.I. Olemskoi, Theory of structure transformations in non-equilibrium condensed matter, Horizons in World Physics Series, Vol. 231, NOVA Science Publishers, New York, 1999,
17. T.D. Frank, Nonlinear Fokker-Planck Equations. Fundamentals and Applications, Springer, 2004.
18. B. Markiv, R. Tokarchuk, P. Kostrobij, M. Tokarchuk, Physica A 390 (2011) 785.
19. J. Boon, S. Yip, Molecular Hydrodynamics, New-York:McGraw-Hill Inc, 1980.
20. I.M. Mryglod , M.V. Tokarchuk , Prob. At. Sci. Technol. 3 (1992) 134 (in Russian)
21. I.M. Mryglod, I.P. Omelyan, M.V. Tokarchuk, Mol. Phys. 84 (1995) 235.
22. B.B. Markiv, I.P. Omelyan, M.V. Tokarchuk, Phys. Rev. E. 82 (2010) 041202.
23. D. N. Zubarev, V. G. Morozov, I. P. Omelyan, M. V. Tokarchuk, Theor. Math. Phys. 96 (1993) 997.
24. M.V. Tokarchuk, I.P. Omelyan, A.E. Kobryn A.E., Condens. Matter

Phys. 1 (1998) 687.
25. A.E. Kobryn, I.P. Omelyan, M.V. Tokarchuk, J. Stat. Phys. 92 (1998) 973.
26. B. Markiv, I. Omelyan, M. Tokarchuk, J. Stat. Phys. 155 (2014) 843.
27. B. Markiv, M. Tokarchuk, Phys. Plasmas 21 (2014) 023707.
28. N.N. Bogolubov, Fizika Elementarnyh Chastic i Atomnogo Yadra (Particles Nuclei), 9 (1978) 501(in Russian).
29. J.R. Dorfman, Physica A, 106 (1981) 77.
30. Yu.L. Klimontovich, Theor. Math. Phys. 92 (1992) 909.
31. Yu.L. Klimontovich, Phys. Lett. A 170 (1992) 434.
32. E.G.D. Cohen, Physica A 194 (1993) 229.
33. P. Résibois, M. de Leener, Classical Kinetic Theory of Fluids, New York, John Willey and Sons, 1977.
34. S.K. Schnyder, F. Hofling, T. Franosch, Th. Voigtmann, J. Phys.: Conden. Matter 47 (2011) 234121.
35. T. Franosch, J. Phys. A: Math. Theor. 47 (2014) 325004.
36. D.N. Zubarev, V.G. Morozov, Theor. Math. Phys. 60 (1984) 814.
37. D.N. Zubarev, V.G. Morozov, I.P. Omelyan, M.V.Tokarchuk, Theor. Math. Phys. 87 (1991) 412.
38. D. Zubarev, V. Morozov, G. Röpke, Statistical mechanics of nonequilibrium processes. Vol. 1, Basic concepts, kinetic theory. Berlin, Academie Verlag, 1996.
39. D. Zubarev, V. Morozov, G. Röpke, Statistical mechanics of nonequilibrium processes. Vol. 2, Relaxation and hydrodynamical processes. Berlin, Academie Verlag, 1997.
40. A.E. Kobryn, V.G. Morozov V.G., I.P. Omelyan, M.V. Tokarchuk, Physica A 230 (1996) 189.
41. D.N. Zubarev, V.G. Morozov, Nonequlibrium ctatistical ansambles in kinetic theory and hydrodynamics. In: Collection of scientific works of Mathematical Institute of USSR Academy of Sciences. Moscow, Nauka, 191 (1989), 140 (in Russian).
42. V.G. Morozov, A.E. Kobryn, M.V. Tokarchuk, Cond. Matt. Phys. 4 (1994) 117.
43. D.N. Zubarev, V.G. Morozov, Physia A 120 (1983) 411.
44. K. Kawasaki, Phase transition and ritial phenomena, 1976, vol. 5A, p. 165-411.
45. D.N. Zubarev, Theor. Math. Phys. 53 (1982) 1004.
46. I.R. Yukhnovskii, M.F. Holovko, Statistical Theory of Classical Equilibrium Systems, Naukova Dumka, Kiev, 1980 (in Russian).
47. I.M. Idzyk, V.V. Ighatyuk, M.V. Tokarchuk, Ukr. Phys. Journal, 41, No. 10, (1996) 1017(in Ukrainian).

CONDENSED MATTER PHYSICS

The journal **Condensed Matter Physics** is founded in 1993 and published by Institute for Condensed Matter Physics of the National Academy of Sciences of Ukraine.

AIMS AND SCOPE: The journal **Condensed Matter Physics** contains research and review articles in the field of statistical mechanics and condensed matter theory. The main attention is paid to physics of solid, liquid and amorphous systems, phase equilibria and phase transitions, thermal, structural, electric, magnetic and optical properties of condensed matter. Condensed Matter Physics is published quarterly.

ABSTRACTED/INDEXED IN: Chemical Abstract Service, Current Contents/Physical, Chemical&Earth Sciences; ISI Science Citation Index-Expanded, ISI Alerting Services; INSPEC; "Referatyvnyj Zhurnal"; "Dzherelo".

CONTACT INFORMATION:
Institute for Condensed Matter Physics
of the National Academy of Sciences of Ukraine
1 Svientsitskii Str., 79011 Lviv, Ukraine
Tel: +38(032)2761978; Fax: +38(032)2761158
E-mail: cmp@icmp.lviv.ua http://www.icmp.lviv.ua