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ПРО УТРИМУЮЧІ ВЗАЄМОДІЇ В НЕЛІНІЙНИХ УЗАГАЛЬНЕННЯХ
МОДЕЛІ ВІКА-КУТКОСЬКОГО

Роботу отримано 30 жовтня 2009 р.

Затверджено до друку Вченою радою ІФКС НАН України

Рекомендовано до друку семінаром відділу компютерного
моделювання багато-частинкових систем

Виготовлено при ІФКС НАН України

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Національна академія наук України



ІНСТИТУТ
ФІЗИКИ
КОНДЕНСОВАНИХ
СИСТЕМ

ICMP-09-09E

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ON CONFINEMENT INTERACTIONS IN NONLINEAR
GENERALIZATIONS OF THE WICK-CUTKOSKY MODEL

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ЛЬВІВ

УДК: 530.12:530.145

PACS: 11.10.Ef, 11.10.Lm

Про утримуючі взаємодії в нелінійних узагальненнях моделі Віка-Куткоського

Ю.Даревич, А.Дувіряк

Анотація. Розглядаються узагальнення моделі Віка-Куткоського з нелінійним полем-посередником. Шляхом розкладу за параметром нелінійності та вилученням поля-посередника з допомогою коваріантної функції Гріна отримано лагранжіан із часо-нелокальними багато-точковими членами взаємодії. У нижчих наближеннях теорії φ^3 отримано звичні дво-струмові взаємодії та три-струмову взаємодію типу конфайнменту. Цей же результат отримано точно для версії дипольної моделі. При гамільтонізації та канонічному квантуванні наближено враховано часову нелокальність. Релятивістичні дво- та три-частинкові хвильові рівняння отримано варіаційним методом із застосуванням багато-частинкових пробних станів у просторі Фока. Отримано нерелятивістичні границі цих рівнянь, аналізуються та коротко обговорюються їх властивості.

On confinement interactions in nonlinear generalizations of the Wick-Cutkosky model

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Abstract. We consider nonlinear-mediating-field generalizations of the Wick-Cutkosky model. Using an expansion in the nonlinearity parameter and eliminating the mediating field by means of the covariant Green function we derive a Lagrangian with many-point time-nonlocal interaction terms. In low-order approximations of φ^3 theory we obtain the usual two-current interaction and a three-current confining one. The same result is obtained exactly for a version of the dipole model. The Hamiltonization and canonical quantization is performed with time non-locality taken into account approximately. Relativistic two- and three-particle wave equations are derived variationally by using many-particle Fock space trial states. The non-relativistic limits of these equations are obtained and their properties are analyzed and discussed briefly.

Подається в Journal of Mathematical Physics

Submitted to Journal of Mathematical Physics

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Institute for Condensed Matter Physics 2009

1. Introduction

The so-called partially reduced QFT complemented by the variational method is a promising and powerful approach to the relativistic bound state problem [1–9]. The use of many-particle Fock-space sectors in the variational trial states leads to wave equations with lower (and thus improved) energy levels of bound states. This has been shown on the example of the simple scalar Yukawa model [8, 9]. The incorporation of many-particle cluster interactions requires nonlinear terms to be added to the Lagrangian of the model. The purpose of this study is to shed light on the question: might confinement be governed by many-body interactions?

In this paper we analyse the interactions that arise from the nonlinear terms in the mediating-field sector of the QFT Lagrangian. As an example we consider the φ^3 -generalization of the Wick-Cutkosky (i.e. massless scalar Yukawa) model [10] and of a version of the dipole model.

2. Nonlocal Lagrangian from a non-linear Wick-Cutkosky model

We proceed from the classical action integral:

$$I = \int d^4x \mathcal{L}(x), \quad (2.1)$$

with the Lagrangian ($\hbar = c = 1$)

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \rho \chi + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \kappa \mathcal{V}(\chi), \quad (2.2)$$

where $\phi(x)$ is a complex scalar “matter” field with rest mass m , $\chi(x)$ is a real massless scalar field interacting with ϕ via the scalar density $\rho = -g\phi^*\phi$ and with itself via the potential $\kappa\mathcal{V}(\chi)$; here g, κ are interaction constants.

The stationary property of the action (2.1)-(2.2), i.e. $\delta I(x) = 0$, leads to the coupled set of the Euler-Lagrange equations,

$$(\square + m^2)\phi = -g\phi\chi, \quad (2.3)$$

$$(\square + m^2)\phi^* = -g\phi^*\chi, \quad (2.4)$$

$$\square\chi = \rho - \kappa\mathcal{V}'(\chi), \quad (2.5)$$

which determine the field dynamics (here $\mathcal{V}'(\chi) \equiv d\mathcal{V}(\chi)/d\chi$).

Equation (2.5) can be formally solved by means of an iterative expansion in the parameter κ (cf. ref. [11]). In 1st-order approximation we have

$$\chi = \overset{0}{\chi} + \kappa \overset{1}{\chi} + \dots = D * (\rho - \kappa \mathcal{V}'(D * \rho) + \dots), \quad (2.6)$$

where $\overset{0}{\chi} = D * \rho$ is the solution of (2.6) with $\kappa = 0$, $D(x) = \frac{1}{4\pi} \delta(x^2)$ is the symmetric Green function of the d'Alembert equation, and “ $*$ ” denotes the convolution $[D * \rho](x) \equiv \int d^4x' D(x - x') \rho(x')$. The arbitrary solution of the homogeneous d'Alembert equation is omitted because the free χ field plays no role in the investigation considered here. The use of the formal solution (2.6) in equations (2.3) and (2.4) leads to a coupled set of integro-differential equations for the fields $\phi(x)$ and $\phi^*(x)$, which we shall refer to as partially-reduced field equations. Alternatively, these equations can be derived from the partially reduced action obtained, in turn, by the use of (2.6) directly in the Lagrangian (2.2). In the 1st order this gives,

$$\begin{aligned} \mathcal{L} &\simeq \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \rho \left(\overset{0}{\chi} + \kappa \overset{1}{\chi} \right) \\ &\quad - \frac{1}{2} \left(\overset{0}{\chi} + \kappa \overset{1}{\chi} \right) \square \left(\overset{0}{\chi} + \kappa \overset{1}{\chi} \right) - \kappa \mathcal{V}(\overset{0}{\chi}) \\ &\simeq \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \overset{0}{\chi} \left(\rho - \frac{1}{2} \square \overset{0}{\chi} \right) + \kappa \overset{1}{\chi} \left(\rho - \square \overset{0}{\chi} \right) - \kappa \mathcal{V}(\overset{0}{\chi}) \\ &\simeq \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \frac{1}{2} \rho D * \rho - \kappa \mathcal{V}(D * \rho) \\ &\equiv \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}^{(2)} + \mathcal{L}_{\text{int}}^{(>2)} \end{aligned} \quad (2.7)$$

where \simeq denotes equality modulo surface terms. This Lagrangian is non-local in the space-time, and the action (2.1), (2.7) includes 1-, 2- and >2 -fold integrations over the Minkowsky space. The treatment of non-local theories of this type is a conceptually complicated, but practically realisable procedure [7].

3. Nonlocal Lagrangian from nonlinear dipole model

The non-local Lagrangian (2.7) is the 1st-order approximate result of the reduction procedure applied to nonlinear generalizations of the Wick-Cutkosky model. Here we propose another local model which can be reduced to the Lagrangian (2.7) exactly. The model is built in analogy to the linear “dipole” model [12] that simulates the confinement interaction of quarks in mesons. The present model is nonlinear and gives Yukawa + cluster interactions.

Let us consider the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \rho \left(\chi + \frac{1}{2} \varphi \right) + \partial_\mu \chi \partial^\mu \varphi - \kappa \mathcal{V}(\varphi), \quad (3.1)$$

where both the $\chi(x)$ and $\varphi(x)$ are real massless scalar fields and $\rho = -g \phi^* \phi$ as in (2.2).

The variation of the action (2.1), (3.1) leads to the coupled set of the Euler-Lagrange equations,

$$(\square + m^2) \phi = -g \phi \left(\chi + \frac{1}{2} \varphi \right), \quad (3.2)$$

$$(\square + m^2) \phi^* = -g \phi^* \left(\chi + \frac{1}{2} \varphi \right), \quad (3.3)$$

$$\square \varphi = \rho, \quad (3.4)$$

$$\square \chi = \frac{1}{2} \rho - \kappa \mathcal{V}'(\varphi), \quad (3.5)$$

which determine the field dynamics.

Equations (3.4) and (3.5) possess exact formal solution:

$$\varphi = D * \rho, \quad (3.6)$$

$$\chi = D * \left\{ \frac{1}{2} \rho - \kappa \mathcal{V}'(\varphi) \right\} = D * \left\{ \frac{1}{2} \rho - \kappa \mathcal{V}'(D * \rho) \right\}, \quad (3.7)$$

which can immediately be used in the r.h.s. of eqs. (3.2), (3.3):

$$(\square + m^2) \phi = -g \phi D * \left\{ \rho - \kappa \mathcal{V}'(D * \rho) \right\}, \quad (3.8)$$

and similarly for ϕ^* . These equations can be derived from $\delta I = 0$, with a Lagrangian identical to (2.7) (but note that no iterative expansions in κ need to be made in this case).

4. φ^3 -interaction

We consider the simplest non-linear Wick-Cutkosky model, namely that where

$$\mathcal{V}(\varphi) = \frac{1}{3} \varphi^3. \quad (4.1)$$

Then, the corresponding term in the non-local action integral has the form:

$$\begin{aligned} I_{\text{int}}^{(3)} &= -\frac{1}{3} \kappa \iiint d^4x d^4x' d^4x'' d^4x''' D(x - x') D(x - x'') D(x - x''') \times \\ &\quad \times \rho(x') \rho(x'') \rho(x'''). \end{aligned} \quad (4.2)$$

In subsequent sections we will discuss the role of this term in the quantized version of this model. However, it is of interest to examine first the

physical content of (4.2) at the classical level. Had the action term (4.2) the standard form $I_{\text{int}}^{(3)} = \int dt L^{(3)}[\rho]$, where $L^{(3)}[\rho]$ is a time-independent functional of the field source ρ , the contribution of $I_{\text{int}}^{(3)}$ to the energy of the system would be $H^{(3)} = -L^{(3)}$. This is not the case unless we use the static approximation where the source is considered to be time-independent: $\rho(x) = \rho(\mathbf{x})$. In that (static) case the choice of the Green function is not important and we choose the retarded function, $D_+(x-x') = \frac{1}{4\pi} \delta(t-t' - |\mathbf{x} - \mathbf{x}'|)/|\mathbf{x} - \mathbf{x}'|$, instead of the symmetric one (though, with a little more effort, the same result is obtained if the symmetric function is used). Then, we have

$$\begin{aligned} I_{\text{int}}^{(3)} &= -\frac{1}{3}\kappa \int dt' \int d^3x' \int d^3x'' \int d^3x''' \rho(\mathbf{x}') \rho(\mathbf{x}'') \rho(\mathbf{x}''') \times \\ &\quad \times \int dt'' \int dt''' \int d^4x D_+(x-x') D_+(x-x'') D_+(x-x''') \\ &= -\frac{\kappa}{3(4\pi)^3} \int dt \int d^3x' \int d^3x'' \int d^3x''' \rho(\mathbf{x}') \rho(\mathbf{x}'') \rho(\mathbf{x}''') \times \\ &\quad \times U(\mathbf{x}', \mathbf{x}'', \mathbf{x}'''), \end{aligned} \quad (4.3)$$

where the kernel

$$U(\mathbf{x}', \mathbf{x}'', \mathbf{x}''') = \int \frac{d^3x}{|\mathbf{x} - \mathbf{x}'| |\mathbf{x} - \mathbf{x}''| |\mathbf{x} - \mathbf{x}'''|} \quad (4.4)$$

is a time-independent function which has the structure (modulo a constant factor) of a three-point interaction potential. Its properties are studied in the Appendix A. It is shown there (Prop. 2) that the integral (4.4) diverges. However, the corresponding force is well behaved, as can be seen from the potential difference

$$\begin{aligned} \Delta U(\mathbf{x}', \mathbf{x}'', \mathbf{x}''') &\equiv \Delta U(\mathbf{a} \rightarrow \mathbf{x}', \mathbf{b} \rightarrow \mathbf{x}'', \mathbf{c} \rightarrow \mathbf{x}''') \\ &= U(\mathbf{a}, \mathbf{b}, \mathbf{c}) - U(\mathbf{x}', \mathbf{x}'', \mathbf{x}'''), \end{aligned} \quad (4.5)$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are arbitrary constant vectors. This can be expressed as the sum

$$\begin{aligned} \Delta U(\mathbf{a} \rightarrow \mathbf{x}', \mathbf{b} \rightarrow \mathbf{x}'', \mathbf{c} \rightarrow \mathbf{x}''') &= \Delta U(\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{x}''') + \Delta U(\mathbf{x}''', \mathbf{a}, \mathbf{b} \rightarrow \mathbf{x}'') \\ &\quad + \Delta U(\mathbf{x}'', \mathbf{x}''', \mathbf{a} \rightarrow \mathbf{x}') \end{aligned} \quad (4.6)$$

of the partial potential differences of the form $\Delta U(\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{r}) = U(\mathbf{a}, \mathbf{b}, \mathbf{c}) - U(\mathbf{a}, \mathbf{b}, \mathbf{r})$ (see (A2)). These are well defined (see Prop. 3) and indicate logarithmic confinement (Prop. 4). The calculation of the

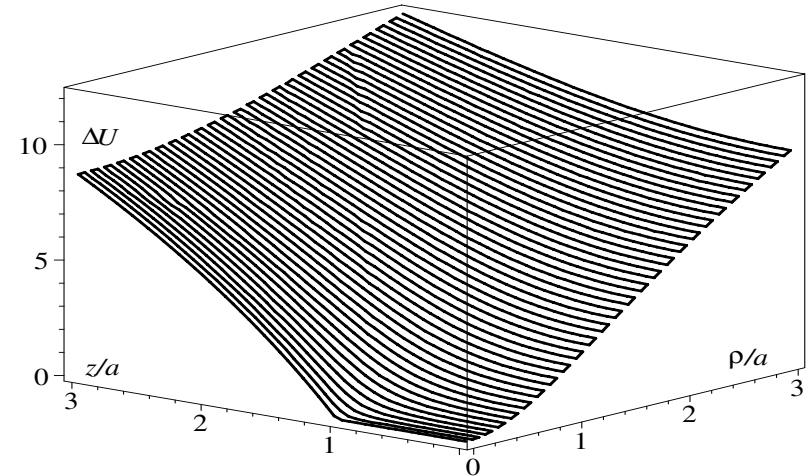


Figure 1. The potential $\Delta U(\mathbf{a}, -\mathbf{a}, \mathbf{0} \rightarrow \mathbf{r})$ as a function of $\mathbf{r} = \{x, y, z\}$; $\rho = \sqrt{x^2 + y^2}$. The function is symmetric under the inversion $z \rightarrow -z$ and rotation around $0z$. In particular, $\Delta U = 4\pi\theta(|z| - a) \log \frac{1}{2}(|z|/a + 1)$ if $\rho = 0$.

integral $\Delta U(\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{r})$ is complicated, and we illustrate the behavior of this function in Figure 1 for the particular case $\mathbf{b} = -\mathbf{a}, \mathbf{c} = \mathbf{0}$.

The subtraction procedure (4.5) can be regarded as a regularisation of the divergent integral (4.4). Another version of the subtraction procedure is presented in the Appendix B. It is possible to use other regularisation procedures, such as inserting a cut-off factor, for example $e^{-b|\mathbf{x}|}$ with $b > 0$, into the integrand of (4.4) (whereupon the integral converges) and studying the results in the limit of $b \rightarrow 0$. Evidently, the cut-off procedure is not unique.

5. Quantization: the $\mathcal{V} = \frac{1}{3} \varphi^3$ model

Following Refs. [7, 9] we proceed to the Hamiltonian formalism and canonical quantization. Formally, the Hamiltonization procedure is as follows. We construct the Hamiltonian density,

$$\mathcal{H} = \mathcal{H}_{\text{free}} + \mathcal{H}_{\text{int}}^{(2)} + \mathcal{H}_{\text{int}}^{(3)}, \quad (5.1)$$

where $\mathcal{H}_{\text{int}}^{(2)} = -\frac{1}{2} \int dx' \rho(x) D(x-x') \rho(x')$ and $\mathcal{H}_{\text{int}}^{(3)} = -\mathcal{L}_{\text{int}}^{(3)}$ is specified by $I_{\text{int}}^{(3)} = \int d^4x \mathcal{L}_{\text{int}}^{(3)}$, given in equation (4.2). The total interaction

Hamiltonian density (5.1) is then expressed in terms of the Fourier amplitudes $A_{\mathbf{k}}$, $B_{\mathbf{k}}$ and $A_{\mathbf{k}}^\dagger$, $B_{\mathbf{k}}^\dagger$, of the field $\phi(x)$ (see eq. (2.14) in [9]). Upon quantization these amplitudes satisfy the standard commutation relations and become the creation and annihilation operators. Then the canonical Hamiltonian operator is given by

$$H = \int d^3x : \mathcal{H}(t=0, \mathbf{x}) :, \quad (5.2)$$

where “: :” denotes the normal ordering of operators. Other canonical generators, such as linear and angular momentum, can be easily obtained.

The term H_{free} is the standard Hamiltonian of the free complex scalar field. The explicit form of the pair interaction term $H_{\text{int}}^{(2)}$ is known (see [3]b, [4, 7]) and so we shall concentrate on the $H_{\text{int}}^{(3)}$ term. It has the following somewhat cumbersome form:

$$\begin{aligned} H_{\text{int}}^{(3)} = & -\frac{\kappa g^3}{24(2\pi)^6} \int \frac{d^3k_1 \dots d^3k_6}{\sqrt{k_{10} \dots k_{60}}} \sum_{\eta_1=\pm \dots \eta_6=\pm} \tilde{D}(\eta_1 k_1 + \eta_2 k_2) \times \\ & \times \tilde{D}(\eta_3 k_3 + \eta_4 k_4) \tilde{D}(\eta_5 k_5 + \eta_6 k_6) \delta(\eta_1 \mathbf{k}_1 + \dots + \eta_6 \mathbf{k}_6) \times \\ & \times : \overset{\eta_1}{B}_{\mathbf{k}_1} \overset{\eta_2}{A}_{\mathbf{k}_2} \overset{\eta_3}{B}_{\mathbf{k}_3} \overset{\eta_4}{A}_{\mathbf{k}_4} \overset{\eta_5}{B}_{\mathbf{k}_5} \overset{\eta_6}{A}_{\mathbf{k}_6} :, \quad (5.3) \end{aligned}$$

where $\overset{+}{B} = B$, $\overset{-}{B} = A^\dagger$, $\overset{+}{A} = A$, $\overset{-}{A} = B^\dagger$ and the Fourier transform, $\tilde{D}(k) = -\mathcal{P}/k^2$, of the symmetric Green function of d'Alembert equation depends on the on-shell 4-momentum $k = \{k_0, \mathbf{k}\}$, where $k_0 = \sqrt{m^2 + \mathbf{k}^2}$. The expression (5.3) includes $2^6 = 64$ terms.

6. Variational three-particle wave equations

In the variational approach to QFT the trial state of the system is built of few particle channel components [8, 9] such as the two-particle state vector $|2\rangle = \frac{1}{\sqrt{2}} \int d^3p_1 d^3p_2 F_2(\mathbf{p}_1, \mathbf{p}_3) A_{\mathbf{p}_1}^\dagger A_{\mathbf{p}_2}^\dagger |0\rangle$, the particle-antiparticle one $|1 + \bar{1}\rangle = \int d^3p_1 d^3p_2 G(\mathbf{p}_1, \mathbf{p}_3) A_{\mathbf{p}_1}^\dagger B_{\mathbf{p}_2}^\dagger |0\rangle$, and so on. The three-particle component has the form

$$|3\rangle = \frac{1}{\sqrt{3!}} \int d^3p_1 d^3p_2 d^3p_3 F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) A_{\mathbf{p}_1}^\dagger A_{\mathbf{p}_2}^\dagger A_{\mathbf{p}_3}^\dagger |0\rangle, \quad (6.1)$$

where the channel wave function F , which is to be determined variationally, is completely symmetric under the permutation of particle variables: $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. In the variational method the channel components, $|\psi_i\rangle$,

are used to determine the matrix elements of the Hamiltonian, namely $\langle \psi_i | H | \psi_j \rangle$, where i, j stand for $1, \bar{1}, 2, 1+\bar{1}, \bar{2}, 3, 2+\bar{1}, 2+\bar{2}, \dots$

Here, we are interested in the matrix element of the interaction $H_{\text{int}} = H_{\text{int}}^{(2)} + H_{\text{int}}^{(3)}$ of the Hamiltonian. We note that $\langle 1+\bar{1} | H_{\text{int}}^{(3)} | 1+\bar{1} \rangle = 0$, $\langle 2 | H_{\text{int}}^{(3)} | 2 \rangle = 0$. In other words, purely two-particle trial states, and so the resulting variational wave equations, do not sample the term $H_{\text{int}}^{(3)}$. Thus we first consider the three-particle case and calculate the matrix element

$$\begin{aligned} \langle 3 | H_{\text{int}} | 3 \rangle = & \int d^3p'_1 \dots d^3p'_3 d^3p_1 \dots d^3p_3 F^*(\mathbf{p}'_1 \dots \mathbf{p}'_3) F(\mathbf{p}_1 \dots \mathbf{p}_3) \times \\ & \times \mathcal{K}_{33}(\mathbf{p}'_1 \dots \mathbf{p}'_3, \mathbf{p}_1 \dots \mathbf{p}_3), \quad (6.2) \end{aligned}$$

where the kernel $\mathcal{K}_{33} = \mathcal{K}_{33}^{(2)} + \mathcal{K}_{33}^{(3)}$ consists of the following components:

$$\begin{aligned} \mathcal{K}_{33}^{(2)}(\mathbf{p}'_1 \dots \mathbf{p}'_3, \mathbf{p}_1 \dots \mathbf{p}_3) = & -\frac{3g^2}{4(2\pi)^3} \delta(\mathbf{p}'_1 + \mathbf{p}'_2 + \mathbf{p}'_3 - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \times \\ & \times \frac{\tilde{D}(p'_2 - p_2) \delta(\mathbf{p}'_3 - \mathbf{p}_3)}{\sqrt{p'_{10} p'_{20} p_{10} p_{20}}}, \quad (6.3) \end{aligned}$$

$$\begin{aligned} \mathcal{K}_{33}^{(3)}(\mathbf{p}'_1 \dots \mathbf{p}'_3, \mathbf{p}_1 \dots \mathbf{p}_3) = & -\frac{\kappa g^3}{4(2\pi)^6} \delta(\mathbf{p}'_1 + \mathbf{p}'_2 + \mathbf{p}'_3 - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \times \\ & \times \frac{\tilde{D}(p'_1 - p_1) \tilde{D}(p'_2 - p_2) \tilde{D}(p'_3 - p_3)}{\sqrt{p'_{10} \dots p'_{30} p_{10} \dots p_{30}}}, \quad (6.4) \end{aligned}$$

and $p_{i0} = \sqrt{m^2 + \mathbf{p}_i^2}$ and similarly for p'_{i0} .

This kernel determines the interaction in the relativistic three-particle wave equation that follows from the variational principle $\delta \langle 3 | H - E | 3 \rangle = 0$, namely

$$\begin{aligned} \{p_{10} + p_{20} + p_{30} - E\} F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \\ + \int d^3p'_1 d^3p'_2 d^3p'_3 \mathcal{K}_{33}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3) F(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3) = 0, \quad (6.5) \end{aligned}$$

where the kernel is understood to be the completely symmetrized expression (with respect to the variables $\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3$ and $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$) of (6.3) and (6.4).

The term $\mathcal{K}_{33}^{(2)}$ of the kernel corresponds to the attractive interaction via massless boson exchange between each pair of particles while $\mathcal{K}_{33}^{(3)}$ describes a cluster three-particle interaction.

In order to have some understanding of the properties of the cluster interaction we consider the non-relativistic limit of the equation (6.5).

In coordinate space, it is the Schrödinger equation for the three-particle eigenfunction $\Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ (see [8]) and eigenenergy $\epsilon = E - 3m$:

$$\left\{ \frac{1}{2m}(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{p}_3^2) + V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) - \epsilon \right\} \Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = 0, \quad (6.6)$$

where $\mathbf{p}_a = -i\nabla_a$ ($a = 1, 2, 3$), and the potential $V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, like the relativistic kernel, consists of two parts, $V = V_{33}^{(2)} + V_{33}^{(3)}$:

$$V_{33}^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = -\frac{g^2}{16\pi m^2} \left\{ \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} + \frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|} + \frac{1}{|\mathbf{x}_3 - \mathbf{x}_1|} \right\}, \quad (6.7)$$

$$V_{33}^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = -\frac{2\kappa g^3}{(8\pi m)^3} \int \frac{d^3 z}{|z - \mathbf{x}_1||z - \mathbf{x}_2||z - \mathbf{x}_3|} \equiv -\frac{2\kappa g^3}{(8\pi m)^3} U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3). \quad (6.8)$$

The function $U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ on the r.h.s. of (6.8) is discussed in Sec. 4 and Appendix A. It is a divergent and thus equation (6.6) may seem to be meaningless. However, one can resort to regularisation, as already noted in section 4. One way would be to subtract an infinite constant from the potential $V_{33}^{(3)}$ and add it to the eigenenergy ϵ as follows:

$$\begin{aligned} V_{33}^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &\rightarrow \tilde{V}_{33}^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \\ &= -\frac{2\kappa g^3}{(8\pi m)^3} \{U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) - U(\mathbf{a}, \mathbf{b}, \mathbf{c})\} \\ &\equiv \frac{2\kappa g^3}{(8\pi m)^3} \Delta U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \end{aligned} \quad (6.9)$$

$$\epsilon \rightarrow \tilde{\epsilon} = E - 3m - \frac{2\kappa g^3}{(8\pi m)^3} U(\mathbf{a}, \mathbf{b}, \mathbf{c}), \quad (6.10)$$

where \mathbf{a} , \mathbf{b} and \mathbf{c} are arbitrary constant vectors. The potential $\tilde{V}_{33}^{(3)}$ is well defined and possesses the confining property, provided that $\kappa > 0$. Thus, equation (6.6) with $V_{33}^{(3)}$ replaced by the finite quantity $\tilde{V}_{33}^{(3)}$ and ϵ replaced by $\tilde{\epsilon}$, makes sense and presumably possesses bound states solutions only. Of course, the results would be meaningful to the extent that they were independent of the choice of the regularisation procedure (choice of \mathbf{a} , \mathbf{b} and \mathbf{c} in the subtraction procedure).

The problem of divergences is expected in the relativistic case too. But the analysis of the integral equation (6.5) is a more subtle problem which shall not be undertaken in this work.

7. An improved treatment of the particle-antiparticle system

It was pointed out in the previous section that the simple variational particle-antiparticle trial state $|1+\bar{1}\rangle = \int d^3 p_1 d^3 p_2 F(\mathbf{p}_1, \mathbf{p}_3) A_{\mathbf{p}_1}^\dagger B_{\mathbf{p}_2}^\dagger |0\rangle$ does not sample the $H_{\text{int}}^{(3)}$ term of the Hamiltonian. Thus, this term does not influence the variational wave equation derived by using only $|1+\bar{1}\rangle$. But the inclusion of both the $|1+\bar{1}\rangle$ and $|2+\bar{2}\rangle$ sectors [9] does show the effect of the $H_{\text{int}}^{(3)}$ term. Indeed,

$$\langle 1+\bar{1} | H_{\text{int}}^{(3)} | 2+\bar{2} \rangle = \int d^3 p'_1 d^3 p'_2 d^3 p_1 \dots d^3 p_4 F^*(\mathbf{p}'_1, \mathbf{p}'_2) G(\mathbf{p}_1 \dots \mathbf{p}_4) \times \times \mathcal{K}_{24}^{(3)}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}_1 \dots \mathbf{p}_4), \quad (7.1)$$

$$\langle 2+\bar{2} | H_{\text{int}}^{(3)} | 2+\bar{2} \rangle = \int d^3 p'_1 \dots d^3 p'_4 d^3 p_1 \dots d^3 p_4 G^*(\mathbf{p}'_1 \dots \mathbf{p}'_4) G(\mathbf{p}_1 \dots \mathbf{p}_4) \times \times \mathcal{K}_{44}^{(3)}(\mathbf{p}'_1 \dots \mathbf{p}'_4, \mathbf{p}_1 \dots \mathbf{p}_4), \quad (7.2)$$

where

$$\begin{aligned} \mathcal{K}_{24}^{(3)}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}_1 \dots \mathbf{p}_4) &= -\frac{\kappa g^3}{4(2\pi)^6} \delta(\mathbf{p}'_1 + \mathbf{p}'_2 - \mathbf{p}_1 - \dots - \mathbf{p}_4) \times \\ &\times \tilde{D}(p_3 + p_4) \frac{2\tilde{D}(p'_1 - p_1)\tilde{D}(p'_2 - p_2) + \tilde{D}(p'_1 + p'_2)\tilde{D}(p_1 + p_2)}{\sqrt{p'_{10}p'_{20}p_{10} \dots p_{40}}}, \end{aligned} \quad (7.3)$$

$$\begin{aligned} \mathcal{K}_{44}^{(3)}(\mathbf{p}'_1 \dots \mathbf{p}'_4, \mathbf{p}_1 \dots \mathbf{p}_4) &= -\frac{\kappa g^3}{2(2\pi)^6} \delta(\mathbf{p}'_1 + \dots + \mathbf{p}'_4 - \mathbf{p}_1 - \dots - \mathbf{p}_4) \times \\ &\times \left\{ 2\tilde{D}(p_3 + p_4) \left[\frac{\tilde{D}(p'_1 + p'_4)\tilde{D}(p'_3 - p_1)\delta(\mathbf{p}'_2 - \mathbf{p}_2)}{\sqrt{p'_{10}p'_{30}p'_{40}p_{10}p_{30}p_{40}}} \right. \right. \\ &\quad \left. \left. + \frac{\tilde{D}(p'_2 + p'_3)\tilde{D}(p'_4 - p_2)\delta(\mathbf{p}'_1 - \mathbf{p}_1)}{\sqrt{p'_{20}p'_{30}p'_{40}p_{20}p_{30}p_{40}}} \right] \right. \\ &\quad \left. + \tilde{D}(p'_3 - p_3)D(p'_4 - p_4) \left[\frac{\tilde{D}(p'_1 - p_1)\delta(\mathbf{p}'_2 - \mathbf{p}_2)}{\sqrt{p'_{10}p'_{30}p'_{40}p_{10}p_{30}p_{40}}} \right. \right. \\ &\quad \left. \left. + \frac{\tilde{D}(p'_2 - p_2)\delta(\mathbf{p}'_1 - \mathbf{p}_1)}{\sqrt{p'_{20}p'_{30}p'_{40}p_{20}p_{30}p_{40}}} \right] \right\}. \end{aligned} \quad (7.4)$$

Therefore, if we use the trial state $|\Psi\rangle = |1+\bar{1}\rangle + |2+\bar{2}\rangle$ and vary the energy expectation value $\bar{E} = \langle \Psi | H | \Psi \rangle / \langle \Psi | \Psi \rangle$ with respect to the channel amplitudes F and G , we obtain the coupled pair of relativistic

wave equations:

$$\{p_{10} + p_{20} - E\}F(\mathbf{p}_1, \mathbf{p}_2) + \int d^3p'_1 d^3p'_2 \mathcal{K}_{22}^{(2)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1, \mathbf{p}'_2)F(\mathbf{p}'_1, \mathbf{p}'_2) \\ + \int d^3p'_1 \dots d^3p'_4 \mathcal{K}_{24}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1 \dots \mathbf{p}'_4)G(\mathbf{p}'_1 \dots \mathbf{p}'_4) = 0, \quad (7.5)$$

$$\{p_{10} + \dots + p_{40} - E\}G(\mathbf{p}_1 \dots \mathbf{p}_4) \\ + \int d^3p'_1 \dots d^3p'_4 \mathcal{K}_{44}(\mathbf{p}_1 \dots \mathbf{p}_4, \mathbf{p}'_1 \dots \mathbf{p}'_4)G(\mathbf{p}'_1 \dots \mathbf{p}'_4) \\ + \int d^3p'_1 d^3p'_2 \mathcal{K}_{24}(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}_1 \dots \mathbf{p}_4)F(\mathbf{p}'_1, \mathbf{p}'_2) = 0 \quad (7.6)$$

with $\mathcal{K}_{24} = \mathcal{K}_{24}^{(2)} + \mathcal{K}_{24}^{(3)}$, $\mathcal{K}_{44} = \mathcal{K}_{44}^{(2)} + \mathcal{K}_{44}^{(3)}$, where $\mathcal{K}_{22}^{(2)}$, $\mathcal{K}_{24}^{(2)}$ and $\mathcal{K}_{44}^{(2)}$ are contributions of the pair interaction defined in [8, 9] (with the mass of mediating field $\mu = 0$), and $\mathcal{K}_{24}^{(3)}$, $\mathcal{K}_{44}^{(3)}$ are defined in (7.3), (7.4). We note that the kernels \mathcal{K}_{24} and \mathcal{K}_{44} in (7.6) are the expressions given in (7.3) and (7.4), symmetrized with respect to variables $\mathbf{p}_1, \mathbf{p}_3$ and $\mathbf{p}_2, \mathbf{p}_4$.

In the domain $E \sim 2m$, equations (7.5), (7.6) can be regarded as describing a two-body particle-antiparticle system in which account is taken of a virtual pair. Similarly, in the domain $E \sim 4m$ they can be regarded as describing a four-body (two-pair) system in which account is taken of the virtual annihilation of a pair. Of course, if $F = 0$, equation (7.5) does not arise, and (7.6) becomes a relativistic equation for the four-body, two-pair system ("quadronium"), analogous to (6.2) for the three-body system.

As before, it is of interest to consider the non-relativistic limit of the wave equations (7.5), (7.6). In this approximation the equations (7.5), (7.6) reduce to a coupled set of Schrödinger-like equations (see (4.8), (4.9) in [9]). The three-particle interaction does not change the non-relativistic potential V_{22} ((4.10) in [9]) and contributes in V_{24} and V_{44} ((4.11), (4.12) of [9]) as

$$V_{24}^{(3)} = \frac{g^3 \kappa}{(2\pi)^2 (2m)^5} \frac{\delta(\mathbf{x}_3 - \mathbf{x}_4)}{|\mathbf{x}_1 - \mathbf{x}_3| |\mathbf{x}_2 - \mathbf{x}_4|} \\ + \frac{2g^3 \kappa}{(2\pi)^3 (2m)^9} \delta(\mathbf{x}_1 - \mathbf{x}_3) \delta(\mathbf{x}_2 - \mathbf{x}_4) \delta(\mathbf{x}_3 - \mathbf{x}_4) \quad (7.7)$$

$$V_{44}^{(3)} = -\frac{g^3 \kappa}{4(4\pi m)^3} \int d^3z \left\{ \frac{1}{|z - \mathbf{x}_1| |z - \mathbf{x}_3|} \left[\frac{1}{|z - \mathbf{x}_2|} + \frac{1}{|z - \mathbf{x}_4|} \right] \right. \\ \left. + \left[\frac{1}{|z - \mathbf{x}_1|} + \frac{1}{|z - \mathbf{x}_3|} \right] \frac{1}{|z - \mathbf{x}_2| |z - \mathbf{x}_4|} \right\}$$

$$-\frac{g^3 \kappa}{2\pi (2m)^7} \left\{ [\delta(\mathbf{x}_1 - \mathbf{x}_2) + \delta(\mathbf{x}_3 - \mathbf{x}_4)] \left[\frac{1}{|\mathbf{x}_1 - \mathbf{x}_4|} + \frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|} \right] \right. \\ \left. + [\delta(\mathbf{x}_1 - \mathbf{x}_4) + \delta(\mathbf{x}_2 - \mathbf{x}_3)] \left[\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} + \frac{1}{|\mathbf{x}_3 - \mathbf{x}_4|} \right] \right\} \quad (7.8)$$

Divergent integrals in the potential $V_{44}^{(3)}$ should be regularized giving confining terms in the non-relativistic version of the equation (7.6), similarly to the potential $V_{33}^{(3)}$ (6.8) in the three-particle case. There are no divergent terms in $V_{24}^{(3)}$ (and all the more in V_{22}) so confining potentials are absent in the non-relativistic limit of (7.5). This disparity of equations (7.5) and (7.6) makes the simple subtractive regularization scheme, used in the three-particle case (by re-definition of the energy), not applicable (unless $F = 0$, that is, a pure four-body problem), so that another regularisation procedure must be used. In any case, the role of the three-point interaction in the particle-antiparticle problem needs to be investigated further.

8. Concluding remarks

We have considered generalizations of the Wick-Cutkosky (massless scalar Yukawa) model to include nonlinear mediating fields. Covariant Green functions were used to eliminate the mediating field, thus arriving at a Lagrangian that contains nonlocal interaction terms. In the case of a massless mediating field with a $\frac{1}{3}\kappa\varphi^3$ nonlinearity, we evaluate the corresponding interaction term explicitly and show that, in the non-relativistic limit, the kernel has the form of a non-local three-point potential that exhibits a logarithmic-confinement form.

We consider the quantized version of this model in the Hamiltonian formalism, and use the variational method, with trial states built from Fock-space components, to derive relativistic integral wave equations for three-particle and particle-antiparticle systems. The kernels (relativistic potentials) are shown to contain local (one-quantum exchange) and three-point non-local terms. In the non-relativistic limit we evaluate the explicit coordinate-space form of the interaction potentials and show that they consist of local Coulombic potentials and nonlocal three-point confining potentials. The nonlocal potentials, which arise from the $\frac{1}{3}\kappa\varphi^3$ term in the Hamiltonian, are divergent (and so need regularisation), but the potential differences are finite.

The many-body wave-equations derived in this paper are quite complicated and must be solved using approximation methods. This will be the subject of forthcoming work.

Appendix A. Properties of the three-point potential

First of all we list some obvious properties of the potential $U(\mathbf{a}, \mathbf{b}, \mathbf{c})$, (4.4):

1. translational invariance: $U(\mathbf{a} + \boldsymbol{\lambda}, \mathbf{b} + \boldsymbol{\lambda}, \mathbf{c} + \boldsymbol{\lambda}) = U(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where $\boldsymbol{\lambda} \in \mathbb{R}^3$;
2. rotational invariance: $U(R\mathbf{a}, R\mathbf{b}, R\mathbf{c}) = U(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where $R \in \text{SO}(3)$;
3. scaling invariance: $U(\lambda\mathbf{a}, \lambda\mathbf{b}, \lambda\mathbf{c}) = U(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where $\lambda \in \mathbb{R}$;
4. permutational invariance: $U(\mathbf{b}, \mathbf{a}, \mathbf{c}) = U(\mathbf{a}, \mathbf{c}, \mathbf{b}) = U(\mathbf{a}, \mathbf{b}, \mathbf{c})$.

These formal properties apply provided the potential $U(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is well defined. Actually, the integral (4.4) does not exist. To this show we first introduce the convenient notation:

$$U(A) \equiv U(\mathbf{a}, \mathbf{b}, \mathbf{c}; A) = \int_A \frac{d^3x}{|\mathbf{x} - \mathbf{a}||\mathbf{x} - \mathbf{b}||\mathbf{x} - \mathbf{c}|} \quad (\text{A.1})$$

where $A \subset \mathbb{R}^3$ is an integration volume. Let $d(\mathbf{a}, \epsilon)$ be a sphere of radius ϵ with center at \mathbf{a} . We consider the space \mathbb{R}^3 to be affine and use the same notation for vectors $\mathbf{a}, \mathbf{b}, \dots$ and their end points (if the starting point is $\mathbf{0}$).

Proposition 1. If $\mathbf{a} \neq \mathbf{b} \neq \mathbf{c}$ and $R > |\mathbf{a}|, |\mathbf{b}|, |\mathbf{c}|$ then $U(\mathbf{a}, \mathbf{b}, \mathbf{c}; d(\mathbf{0}, R)) < \infty$.

Proof. It is evident that $U(\mathbf{a}, \mathbf{b}, \mathbf{c}; A) < \infty$ if A is compact and does not include the singular points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of the integrand of (A.1).

Consider a neighbourhood of, say, the point \mathbf{c} , namely the sphere $d(\mathbf{c}, \epsilon)$, where $\epsilon \ll |\mathbf{a} - \mathbf{c}|, |\mathbf{b} - \mathbf{c}|$. Shifting the integration variable $\mathbf{x} \rightarrow \mathbf{x} - \mathbf{c}$ in (A.1) we have

$$\begin{aligned} U(\mathbf{a}, \mathbf{b}, \mathbf{c}; d(\mathbf{c}, \epsilon)) &= \int_{d(\mathbf{c}, \epsilon)} \frac{d^3x}{|\mathbf{x}||\mathbf{x} - \mathbf{a} + \mathbf{c}||\mathbf{x} - \mathbf{b} + \mathbf{c}|} \\ &\approx \int_{d(\mathbf{c}, \epsilon)} \frac{d^3x}{|\mathbf{x}||\mathbf{a} - \mathbf{c}||\mathbf{b} - \mathbf{c}|} = \frac{4\pi}{|\mathbf{a} - \mathbf{c}||\mathbf{b} - \mathbf{c}|} \int_0^\epsilon r dr = \frac{2\pi\epsilon^2}{|\mathbf{a} - \mathbf{c}||\mathbf{b} - \mathbf{c}|} \xrightarrow{\epsilon \rightarrow 0} 0. \end{aligned}$$

The same result holds for neighbourhoods of the other singular points \mathbf{a} and \mathbf{b} . ■

Proposition 2. Let $\bar{A} = \mathbb{R}^3 \setminus d(\mathbf{0}, R)$ where $R \gg |\mathbf{a}|, |\mathbf{b}|, |\mathbf{c}|$. Then $U(\mathbf{a}, \mathbf{b}, \mathbf{c}; \bar{A}) = \infty$.

Proof.

$$U(\mathbf{a}, \mathbf{b}, \mathbf{c}; \bar{A}) \approx \int_{\bar{A}} \frac{d^3x}{|\mathbf{x}|^3} = 4\pi \int_R^\infty \frac{dr}{r} = 4\pi \lim_{r \rightarrow \infty} \log(r/R) = \infty. \quad \blacksquare$$

Thus, the integral (4.4) diverges logarithmically at ∞ . Nevertheless, one can define the force generated by the potential, $\mathbf{F}_\mathbf{c} = -\partial U(\mathbf{a}, \mathbf{b}, \mathbf{c})/\partial \mathbf{c}$, and prove it is a well defined quantity. Alternatively, we consider the potential difference when a test particle is moved from, say, the point \mathbf{c} to \mathbf{r} :

$$\begin{aligned} \Delta U(\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{r}) &\equiv U(\mathbf{a}, \mathbf{b}, \mathbf{c}) - U(\mathbf{a}, \mathbf{b}, \mathbf{r}) \\ &= \int \frac{d^3x}{|\mathbf{x} - \mathbf{a}||\mathbf{x} - \mathbf{b}|} \left\{ \frac{1}{|\mathbf{x} - \mathbf{c}|} - \frac{1}{|\mathbf{x} - \mathbf{r}|} \right\} \quad (\text{A.2a}) \end{aligned}$$

$$= \int \frac{d^3x}{|\mathbf{x} - \mathbf{a}||\mathbf{x} - \mathbf{b}||\mathbf{x} - \mathbf{c}||\mathbf{x} - \mathbf{r}|} \{ |\mathbf{x} - \mathbf{r}| - |\mathbf{x} - \mathbf{c}| \} \quad (\text{A.2b})$$

$$= \int \frac{d^3x}{|\mathbf{x} - \mathbf{a}||\mathbf{x} - \mathbf{b}||\mathbf{x} - \mathbf{c}||\mathbf{x} - \mathbf{r}|} \frac{(\mathbf{r} + \mathbf{c} - 2\mathbf{x}) \cdot (\mathbf{r} - \mathbf{c})}{\{ |\mathbf{x} - \mathbf{c}| + |\mathbf{x} - \mathbf{r}| \}} \quad (\text{A.2c})$$

We show next that $\Delta U(\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{r})$ is a well defined finite function: It follows from (A.2a) and Prop. 1 that the integral $\Delta U(\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{r}; A)$ is well defined in a neighbourhood of every singular point $\mathbf{a}, \dots, \mathbf{r}$ of integrand except, perhaps, ∞ . For the last case we verify the following:

Proposition 3. $|\Delta U(\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{r}; \bar{A})| < \infty$ where $\bar{A} = \mathbb{R}^3 \setminus d(\mathbf{0}, R)$, $R \gg |\mathbf{a}|, |\mathbf{b}|, |\mathbf{c}|, |\mathbf{r}|$.

Proof. Taking the expression (A.2b) and using the inequality $||\mathbf{x} - \mathbf{r}| - |\mathbf{x} - \mathbf{c}|| \leq |\mathbf{r} - \mathbf{c}|$ we have

$$\begin{aligned} |\Delta U(\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{r}; \bar{A})| &\leq \int_{\bar{A}} \frac{d^3x}{|\mathbf{x} - \mathbf{a}||\mathbf{x} - \mathbf{b}||\mathbf{x} - \mathbf{c}||\mathbf{x} - \mathbf{r}|} \frac{|\mathbf{r} - \mathbf{c}|}{|\mathbf{x} - \mathbf{c}|} \\ &\approx 4\pi |\mathbf{r} - \mathbf{c}| \int_R^\infty \frac{dr}{r^2} = 4\pi |\mathbf{r} - \mathbf{c}|/R < \infty \quad \blacksquare \end{aligned}$$

We next consider some properties of the function $\Delta U(\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{r})$.

Proposition 4. Let $r \equiv |\mathbf{r}| \gg q \equiv |\max(|\mathbf{a}|, |\mathbf{b}|, |\mathbf{c}|)|$. Then $|\Delta U(\mathbf{a}, \mathbf{b}, \mathbf{c} \rightarrow \mathbf{r})| \sim 4\pi \log(r/q)$.

Proof. We divide \mathbb{R}^3 into three domains: $\mathbb{R}^3 = A_1 \cup A_2 \cup A_3$, where $A_1 = d(\mathbf{0}, 2q)$, $A_2 = d(\mathbf{0}, r/k) \setminus d(\mathbf{0}, 2q)$, $A_3 = \mathbb{R}^3 \setminus d(\mathbf{0}, r/k)$ and $k > 2$. From (A.2b) it follows that

$$|\Delta U(A_1)| \leq \int_{A_1} \frac{d^3x}{|\mathbf{x} - \mathbf{a}||\mathbf{x} - \mathbf{b}||\mathbf{x} - \mathbf{c}||\mathbf{x} - \mathbf{r}|} \approx \int_{A_1} \frac{d^3x}{|\mathbf{x} - \mathbf{a}||\mathbf{x} - \mathbf{b}||\mathbf{x} - \mathbf{c}|}.$$

Note that the function $U(\mathbf{a}, \mathbf{b}, \mathbf{c}; A_1)$, which is finite (by Prop. 1) and independent of \mathbf{r} , appears on the r.h.s. of this inequality.

Performing the change of variable $\boldsymbol{\xi} = \mathbf{x}/r$ in the integral $\Delta U(A_3)$ and using the notation $\boldsymbol{\alpha} = \mathbf{a}/r$, $\boldsymbol{\beta} = \mathbf{b}/r$, $\boldsymbol{\gamma} = \mathbf{c}/r$, $\mathbf{n} = \mathbf{r}/r$, we have:

$$\begin{aligned} |\Delta U(A_3)| &\leq \int_{A_3} \frac{d^3x}{|\mathbf{x} - \mathbf{a}||\mathbf{x} - \mathbf{b}||\mathbf{x} - \mathbf{c}||\mathbf{x} - \mathbf{r}|} \\ &= \int_{\tilde{A}_3} \frac{d^3\xi}{|\boldsymbol{\xi} - \boldsymbol{\alpha}||\boldsymbol{\xi} - \boldsymbol{\beta}||\boldsymbol{\xi} - \boldsymbol{\gamma}||\boldsymbol{\xi} - \mathbf{n}|} \approx \int_{\tilde{A}_3} \frac{d^3\xi}{|\boldsymbol{\xi}|^3 |\boldsymbol{\xi} - \mathbf{n}|}, \end{aligned}$$

where $\tilde{A}_3 = A_3/r = d(\mathbf{0}, 1/k)$. The integral on r.h.s. is finite and r -independent.

For the integral $\Delta U(A_2)$ we use the expression (A.2c) and the same change of variables:

$$\Delta U(A_2) = \int_{A_2} \frac{d^3\xi}{|\boldsymbol{\xi} - \boldsymbol{\alpha}||\boldsymbol{\xi} - \boldsymbol{\beta}||\boldsymbol{\xi} - \boldsymbol{\gamma}||\boldsymbol{\xi} - \mathbf{n}|} \frac{(\mathbf{n} + \boldsymbol{\gamma} - 2\boldsymbol{\xi}) \cdot (\mathbf{n} - \boldsymbol{\gamma})}{\{|\boldsymbol{\xi} - \mathbf{n}| + |\boldsymbol{\xi} - \boldsymbol{\gamma}|\}}, \quad (\text{A.3})$$

where $\tilde{A}_2 = A_2/r = d(\mathbf{0}, 1/k) \setminus d(\mathbf{0}, 2\delta)$ and $\delta = q/r$.

We note the following inequalities for the integrand of (A.3):

$$\begin{aligned} (\mathbf{n} + \boldsymbol{\gamma} - 2\boldsymbol{\xi}) \cdot (\mathbf{n} - \boldsymbol{\gamma}) &\geq 1 - \gamma^2 - 2\xi(1 + \gamma) > (1 + \gamma)(1 - \gamma - 2/k); \\ 1 - 1/k &\leq |\mathbf{n}| - |\boldsymbol{\xi}| \leq |\boldsymbol{\xi} - \mathbf{n}| \leq |\boldsymbol{\xi}| + |\mathbf{n}| \leq 1 + 1/k; \\ |\boldsymbol{\xi}| - \delta &\leq |\boldsymbol{\xi} - \boldsymbol{\alpha}|, |\boldsymbol{\xi} - \boldsymbol{\beta}|, |\boldsymbol{\xi} - \boldsymbol{\gamma}| \leq |\boldsymbol{\xi}| + \delta. \end{aligned}$$

Thus,

$$\Delta U(A_2) > K_- \int_{\tilde{A}_2} \frac{d^3\xi}{(|\boldsymbol{\xi}| + \delta)^3} = 4\pi K_- \int_{2\delta}^{1/k} \frac{\xi^2 d^3\xi}{(\xi + \delta)^3},$$

$$\Delta U(A_2) < K_+ \int_{\tilde{A}_2} \frac{d^3\xi}{(|\boldsymbol{\xi}| - \delta)^3} = 4\pi K_+ \int_{2\delta}^{1/k} \frac{\xi^2 d^3\xi}{(\xi - \delta)^3},$$

where

$$\begin{aligned} K_- &= \frac{(1 + \delta)(1 - \delta - 2/k)}{(1 + 1/k)(1 + \delta + 2/k)} = 1 + O(\delta, 1/k), \\ K_+ &= \frac{(1 + \delta)}{(1 - 1/k)} = 1 + O(\delta, 1/k). \end{aligned}$$

Calculating the integrals

$$\int_{2\delta}^{1/k} \frac{\xi^2 d^3\xi}{(\xi \pm \delta)^3} = -4\pi \log \delta + O(1)$$

and choosing k sufficiently large, establishes the desired result. ■

Appendix B. One-parametric subtraction procedure

The subtraction procedure (4.5) depends on the choice of three arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (i.e., nine arbitrary parameters) which is not satisfactory. It is tempting to remove this arbitrariness or, at least, to reduce a number of parameters. At first sight one can put $\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{0}$ in (4.5). In this case the term $U(\mathbf{0}, \mathbf{0}, \mathbf{0})$ compensates an infinite contribution of the integral (4.4) over a far integration volume (where the integration variable $|\mathbf{x}| \rightarrow \infty$), but brings divergency from a close volume $|\mathbf{x}| \rightarrow 0$.

Instead, we propose another subtraction procedure. We proceed from the formal expression $\tilde{\Delta}U(\mathbf{x}', \mathbf{x}'', \mathbf{x}''') \equiv \frac{1}{2}U(\mathbf{0}, \mathbf{0}, \mathbf{0}) - U(\mathbf{x}', \mathbf{x}'', \mathbf{x}''')$ which is free of any arbitrary parameter. Actually, this expression is not defined unambiguously. We specify it as follows:

$$\begin{aligned} \tilde{\Delta}U(\mathbf{x}', \mathbf{x}'', \mathbf{x}''') &\equiv \frac{1}{2}U(\mathbf{0}, \mathbf{0}, \mathbf{0}) - U(\mathbf{x}', \mathbf{x}'', \mathbf{x}''') \\ &= \lim_{R \rightarrow \infty} \int_{\varepsilon=R_0^2/R}^R r^2 dr \int d^2\Omega \left\{ \frac{1}{2|\mathbf{x}|^3} - \frac{1}{|\mathbf{x} - \mathbf{x}'||\mathbf{x} - \mathbf{x}''||\mathbf{x} - \mathbf{x}'''} \right\}, \quad (\text{B.1}) \end{aligned}$$

where Ω is a solid angle, $r = |\mathbf{x}|$, and R_0 is an arbitrary length parameter. Dividing the integration segment (ε, R) into two parts (ε, R_0)

and (R_0, R) , and using in the first one the change of integration variable $\tilde{r} = R_0^2/r$, we present the first term of (B.1) in the form:

$$\begin{aligned} \frac{1}{2} \int_{R_0^2/R}^R r^2 dr \int \frac{d^2\Omega}{|\mathbf{x}|^3} &= 2\pi \left[\int_{R_0^2/R}^{R_0} + \int_{R_0}^R \right] \frac{dr}{r} \\ &= 4\pi \int_{R_0}^R \frac{dr}{r} = 4\pi \ln(R/R_0) = \int_{R_0}^R r^2 dr \int \frac{d^2\Omega}{|\mathbf{x}|^3}. \end{aligned} \quad (\text{B.2})$$

Then

$$\begin{aligned} \tilde{\Delta}U(\mathbf{x}', \mathbf{x}'', \mathbf{x}''') &= \int_0^{R_0} r^2 dr \int \frac{d^2\Omega}{|\mathbf{x} - \mathbf{x}'||\mathbf{x} - \mathbf{x}''||\mathbf{x} - \mathbf{x}'''} \\ &+ \lim_{R \rightarrow \infty} \left[\int_{R_0}^R r^2 dr \int d^2\Omega \left\{ \frac{1}{|\mathbf{x} - \mathbf{x}'||\mathbf{x} - \mathbf{x}''||\mathbf{x} - \mathbf{x}'''} - \frac{1}{|\mathbf{x}|^3} \right\} \right]. \end{aligned} \quad (\text{B.3})$$

It follows evidently from Appendix A that both terms of this expression are finite. Thus the subtraction procedure (B.1) represent the one-parametric (i.e., R_0 -depending) regularization of the three-point potential (4.4).

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The journal **Condensed Matter Physics** is founded in 1993 and published by Institute for Condensed Matter Physics of the National Academy of Sciences of Ukraine.

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