Національна академія наук України

ІНСТИТУТ
ФІЗИКИ
КОНДЕНСОВАНИХ
СИСТЕМ

ICMP–03–26E

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OPTICAL IMAGES SUPERPOSITION IN THE FRACTIONAL FOURIER TRANSFORM DOMAIN

ЛЬВІВ

УДК: 532; 533; 533.9:530.182; 536.75; 536-12.01.

PACS: 03.65.Bz,42.30.K, 42.30.V, 42.79, 42.25.F, 42.50.Dv, 52.25.Fi

Суміщення оптичних зображень в області дробового фур'є-перетворення.

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Анотація. Досліджено особливості оптичного суміщення двох зміщених і промодульованих плоскою хвилею зображень в області дробового фур'є-перетворення. Теоретично обгрунтована принципова можливість суміщення двох зображень в області ДФП при довільному значенні параметра ДФП p.Наведені чисельні результати, які ілюструють формування висококонтрасної інтерференційної картини в області ДФП. Детально досліджено області реалізації ДФП в оптичних системах.

Optical images superposition in the fractional fourier transform domain.

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Abstract. The FFT images optical superposition regularities are investigated for the general case of two shifted and modulated by the plane wave optical signals. Principal possibility of the optical superposition of two images in the FFT domain at an arbitrary value of parameter p is theoretically proved. Numerical reults which demonstrate forming of the hight-contrast interference band pattern in the FFT images optical superposition point are given. Domains of the FFT realization in the optical systems are investigated in detail.

1. Introduction

One of the perspective trends of the optical information proccesing shemes investigagation is associated with using the methodology of the fractional Fourier transform (FFT), as generalization of the Fourier optics methods. Physical basis of this theory was first introduced by Namias [1] for the quantum mechanics problems.

Basis of the FFT theory was investigated in the first works [2, 3]. In particular, it was shown in [2, 4, 5] that based on Wingner distribution function, FFT can be interpretated as rotation of the optical signal distribution on the informational diagram, on an angle proportiolan to the FFT parameter p. It is known many works [6–10], in which different FFT optical interpretation is given. Different cases of the optical signals correlation analysis in the FFT domain were investigated. The purpose of such investigations was to constract principally new sheme of generalizated correlator [12–14].

In the present paper theory of the FFT conjugate images forming is given based on the signal distribution method, for the general case of two shifted and modulated by the plane wave input optical signals. Theoretical background of the principally new possibility of two optical signals superposition in the FFT domains at an arbitrary value of the FFT parameter p is given.

2. Images forming in the FFT domain

It us known that the FFT is described by the equation

$$u_p(x) = \hat{\mathcal{F}}^p[f(x)] = \int_{-\infty}^{\infty} f(y) K_p(x,y) dy. \tag{1}$$

where the FFT kernel

$$K_p(x,y) = \sqrt{\frac{k}{2\pi d_0}} \, \frac{\exp\left(i\left[\frac{\pi}{4} - \frac{\phi}{2}\right]\right)}{\sqrt{\sin\phi}} \times \exp\left(i\frac{k[x^2+y^2]}{2d_0\tan\phi}\right)\exp\left(-i\frac{kxy}{d_0\sin\phi}\right). \tag{2}$$

where $u_p(x)$ is the fractional Fourier image of the function $f(x)$, d_0 - constant factor, $\varphi = p\pi/2$, p - is the FFT parameter.

We introduse conjugate FFT

$$U_p(x) = \hat{\mathcal{F}}^p\left[F\left(\frac{k}{d_0}x\right)\right] = \sqrt{\frac{k}{2\pi d_0}} \int_{-\infty}^{\infty} F\left(\frac{k}{d_0}y\right) K_p(x,y) dy, \tag{3}$$

Let us consider the general case of two modulated by the plane wave optical signals [15]

$$g(x) = f_1(x+b)\exp(i\omega_l x) \pm f_2(x-b)\exp(-i\omega_l x), \tag{4}$$

Then, for image calculation in the FFT donain the next general formula can be obtained [16]

$$\begin{aligned} |w_p(x)|^2 &= \hat{\mathcal{F}}^{-1}\left[\mathcal{A}_{f_1 f_1^*}(a_{12}\omega_0; a_{22}\omega_0)\exp\left(i\left[a_{22}b + a_{12}\omega_1\right]\omega_0\right)\right] \\ &+ \hat{\mathcal{F}}^{-1}\left[\mathcal{A}_{f_2 f_2^*}(a_{12}\omega_0; a_{22}\omega_0)\exp(-i[a_{22}b + a_{12}\omega_1])\omega_0\right] \\ &+ \hat{\mathcal{F}}^{-1}\left[\mathcal{A}_{f_1 f_2^*}(a_{12}\omega_0 + 2b; a_{22}\omega_0 - 2\omega_1)\right] \\ &+ \hat{\mathcal{F}}^{-1}\left[\mathcal{A}_{f_2 f_1^*}(a_{12}\omega_0 - 2b; a_{22}\omega_0 + 2\omega_1)\right]. \end{aligned} \tag{5}$$

where $\mathcal{A}_{f_1 f_1^*}(x_0, \omega_0)$ - is the ambiguity function, (x_0, ω_0) - are conjugate difference coordinates, a_{ij} - coefficients of the general matrix [18]

$$\mathbf{A} = [a_{ij}] = \begin{pmatrix} a_{11} & -\frac{d_0}{k}a_{12} \\ \frac{k}{d_0}a_{21} & a_{22} \end{pmatrix} \tag{6}$$

Two first terms describe shifted FFT images from two input signals

$$|u_p(x; b, \omega_1)|^2 = |u_p(x + [a_{22}b + a_{12}\omega_1])|^2, \tag{7}$$

$$|v_p(x; b, \omega_1)|^2 = |v_p(x - [a_{22}b + a_{12}\omega_1])|^2. \tag{8}$$

Accordingly, third and fourth terms of equation (5) form the interference term of the FFT intensity distribution

$$\begin{aligned} i_p(x; b, \omega_1) &= u_p(x; b, \omega_1)v_p^*(x; b, \omega_1) \\ &+ [u_p(x; b, \omega_1)v_p^*(x; b, \omega_1)]^*. \end{aligned} \tag{9}$$

For the first term of the interference member we obtain

$$\begin{aligned} &u_p(x;b,\omega_1)v_p^*(x;b,\omega_1) = \\ &= \exp\left(2i\left[a_{12}b + a_{22}\omega_1\right]x\right) \\ &\times u_p\left(x + \left[a_{22}b + a_{12}\omega_1\right]\right)v_p^*\left(x - \left[a_{22}b + a_{12}\omega_1\right]\right). \end{aligned} \quad (10)$$

As a result we obtain the next general formula which form the FFT image of two shifted and modulated optical signals

$$\begin{aligned} &|w_p(x;B_\phi,\Omega_\phi)|^2 = \\ &= \left|u_p(x+B_\phi)\exp\left(i\Omega_\phi x\right)v_p(x-B_\phi)\exp\left(-i\Omega_\phi x\right)\right|^2, \end{aligned} \quad (11)$$

The structure of this Equation is equivalent to the form of an input signal (4). The generalized shifting parameter B_ϕ and modulation frequency Ω_ϕ of the FFT image can be determined by the matrix equation

$$\begin{pmatrix} B_\phi \\ \Omega_\phi \end{pmatrix} = \begin{pmatrix} a_{22} & -\dfrac{d_0}{k}a_{12} \\ \dfrac{k}{d_0}\dfrac{(1-a_{22}^2)}{a_{12}} & a_{22} \end{pmatrix} \begin{pmatrix} b \\ \omega_1 \end{pmatrix}, \quad (12)$$

in other words elements of general matrix **A** are factors of the linear equations system Thus, we can conclude that the FFT image forming of two optical signals may be interpreted as a parallel process of the FFT image cross shifting proportionally to the value B_ϕ and the FFT image modulation by the plane wave with frequency Ω_ϕ. In this connection, the more cross shifting of the FFT images the less its modulation and vise versa.

In the case of two isolated slits we can obtain the formula [16]

$$\begin{aligned} &|u_p(x)|^2 = \\ &= \frac{2}{\pi}\int_0^{1/\sin\phi} \frac{\sin[4\pi F_0\ \omega\cos\phi(1-\omega\sin\phi)]}{\omega\cos\phi}\cos(4\pi F_0 x\omega)d\omega, \end{aligned} \quad (13)$$

where $F_0 = a^2/\lambda d_0$ - is Frenels number. This formula is generalization of Assakura formula [17] at FFT. Note, that in the FFT case in integral formula top limit of integration is variable $(1 \div \infty)$ and depends of the distribution rotation angle ϕ.

In the same way formula for the FFT conjugate imagea calculation on isolated slit can be obtained

For the interference term (9) we obtain

$$\begin{aligned} i(x;B_\phi) &= \frac{2}{\pi}\int_0^{1/\sin\phi} S(\omega;+B_\phi)\cos\left(4\pi F_0\left[\omega + \frac{\beta}{\sin\phi}\right]x\right)d\omega \\ &+ \frac{2}{\pi}\int_0^{1/\sin\phi} S(\omega;-B_\phi)\cos\left(4\pi F_0\left[\omega - \frac{\beta}{\sin\phi}\right]x\right)d\omega, \end{aligned} \quad (14)$$

where $\beta = b/a$ and

$$\begin{aligned} &S(\omega;\pm B_\phi) = 4\pi F_0(1-\omega\sin\phi) \\ &\times \operatorname{sinc}\left(4F_0\cos\phi\left[\omega \pm \frac{B_\phi}{\sin\phi\cos\phi}\right][1-\omega\sin\phi]\right). \end{aligned} \quad (15)$$

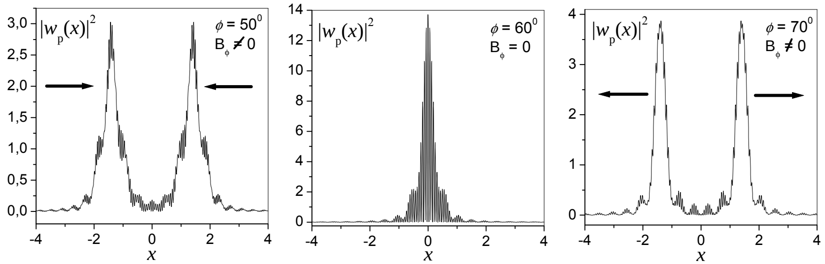

Figure 1. Superposition of two shifted and modulated by the plane wave slits in the FFT domain.

Using solution (11) and matrix equation (12), we obtain the next condition of images optical superposition in the FFT domain

$$B_\phi = b\cos\phi - \omega_1\frac{d_0}{k}\sin\phi = 0. \quad (16)$$

It is important to note that at condition of normal incidence of the plane wave ($\omega_1 = 0$) images optical superposition can be realized only in Fourier plane ($\phi = \pi/2$). It is classical sheme of join Fourier transform correlator. In the general case of oblique icidence of the plane wave ($\omega_1 \neq 0$) optical superposition condition can be realized in the FFT donain at an arbitrary values of parameter p ($p \neq 1$). Then, superposition forming describes by the next formula

$$|w_p(x;0,\Omega_\phi^{\max})|^2 = \\ = |u_p(x)|^2 + |v_p(x)|^2 + u_p(x)v_p^*(x)\exp\left(2i\Omega_\phi^{\max}x\right) \\ + u_p^*(x)v_p(x)\exp\left(-2i\Omega_\phi^{\max}x\right), \quad (17)$$

where $\Omega_\phi^{\max}$ - is maximum modulation frequency. It is obviously that at condition of the FFT images optical superposition maximum cross shifting $B_\phi^{\max}$ of the FFT conjugate images is realized. Values $B_\phi^{\max}$ and $\Omega_\phi^{\max}$ can be find using the matrix Eq.(12).

On figure 1 typical dependences of the two FFT images from two cross shifted infinity slits full circle at optical superposition in the point $\phi = \pi/3$ are given. In the superposition points of the FFT images high-contrast pattern of the interference band is forming.

In this connection, the more cross shifting of the FFT images, the less influence of the interference pattern. In the case of the FFT conjugate images in that point maximum cross shifting $B_\phi^{\max}$ is realized but interference influence is missing ($\Omega_\phi = 0$). Such pattern is periodically repeated accordingly to parameter p.

At the condition $B_\phi = 0$ under integral function $S(\omega;0)$ is equal to Eq.(13). Thus, we can conclude that in the arbitrary FFT domain optically superimposed images from two slits are described by the next general formula

$$|w_p(x;0,\Omega_\phi^{\max})|^2 = 4\cos^2\left(2\pi\frac{\beta F_0}{\sin\phi}x\right)|u_p(x)|^2, \quad (18)$$

In the close FFT domains before and after the optical superposition point interference band contrast abruptly drop. Interference influence is stronger in the central part of the shifted FFT images, because diffractive intensity distribution is non-central. In is clear that the more B_ϕ, the less influence of the interference effects on images forming. In the boundary point $B_\phi^{\max}$ above mentioned influence is wanting.

3. Domains of the FTT realization in optical systems

The FFT parameter p can changes from 0 to 4π and than it is possible to investigate four domains of the FFT realization. Such domains can be realized in one or double optical stages. For more detailed investigation of such approach let us consider different domain of the generalized FFT. For this purpose we elucidate the physical meaning of the invariant parameters a_{ij}.

Rotational matrix $\mathbf{T}_\varphi$ gives the possibility of the symetrical case ($d_1 = d_2$) realization only, and with the restricted by $2F$ distances. Due to the form of generalized matrix (5), for the generalized FFT the nonsymetric case ($d_1 \neq d_2$) can be considered. Superposition of the input signals at arbitrary distances as before and behind the lens is realized.

Let us describe the methodology of the stage parameters calculation at the optical superposition of the input images. For the optical stage the elements a_{12} and a_{22} (the invariant parameters of such stage) of matrix (5) are calculated where the "effective" rotation angle φ_{EF} at the generalized FFT is defined from (9). The expediency of the "effective" rotation angle using is caused by the fact that in the optical stage the rotation matrix $\mathcal{T}_\varphi$ can not be realized. The angle φ_{EF} lets calculate the value of the relative distance d_1 from the input plane and the corresponding value of the relative distance d_2 to the output plane of the optical stage. The signes of the invariant coefficients determine the domain of the FFT realization. Let us consider the "motion" of images on the informational diagram. Each domain has the two boundary points. The FFT domain is placed between these points. One can see that to any point on the informational diagram correponds the certain construction of the optical stage where the generalized FFT is realized.

3.1. Fractional Fourier transform domain

The first domain Fig. 2(a) of the gereralized FFT image formation corresponds to the values of the FFT parameter ($0 < p < 1$). The extreme cases $p = 0$ (the coordinate plane) and $p = 1$ (the frequency plane) correspond to the usual FT. Between these planes the FFT domain is placed and the FFT parameter in this domain changes continually from 0 to 1. Such domain is discribed by the matrix (5) at the following conditions of invariant parameters ($a_{12} > 0; a_{22} > 0$). This domain may be realized in two types of optical stages.

First of them optical stage Fig. 2(b). In the FFT domain point ($a_{22} = 0$) correponds the next construction of the optical stage. Input image is placed at the Focal distance before the lens and output image is placed at the Focal distance after the lens. This case corresponds to the common Fourier transform. The fist domain on the informational diagram Fig. 2 corresponds to the crosshatched region on Fig. 2(b).

The FFT domain may be also realized in the doble stage Fig. 2(c) and accordingly this domain corresponds to the crosshatched region on Fig. 2(c).

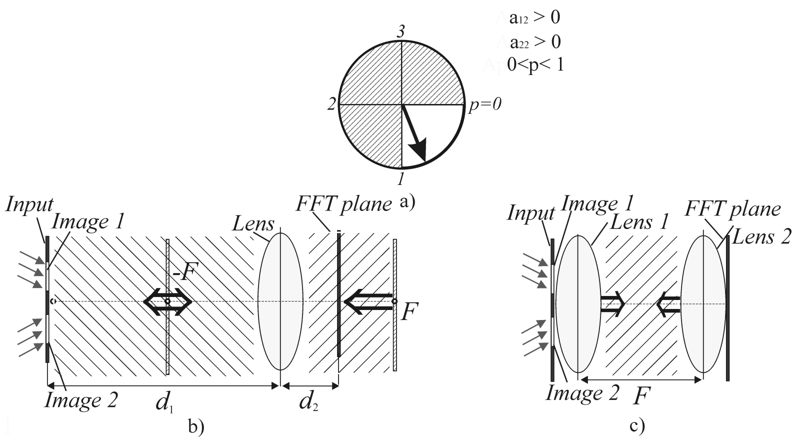

Figure 2. The FFT domain.

3.2. Conjugate fractional Fourier transform domain

In the conjugate FFT domain Fig. 3(a) invariant parameters have the next values ($a_{12} > 0; a_{22} < 0$). The boundary points of this domain are ($1 < p < 2$). The point ($a_{12} = 0$) is peculiar, i.e. this point corresponds to the identical operation.

Conjugate FFT domain also has two variants of realization in the optical systems. In the optical stage this domain corresponds to the crosshatched region on Fig. 3(b) and in the double optical stage the conjugate FFT domain corresponds to the crosshatched region on Fig. 3(c).

3.3. Inverse fractional Fourier transform domain

At the point ($a_{12} = 0$) the inverse image of the input signal is formed, whereas in the domain Fig. 4(a) between the extreme points ($p = 2$) and ($p = 3$) the conjugate FFT is formed. Whereas the third domain of the generalized FFT exists and describes the inverse image of the input signal.

Variants of realization in the following. In the optical stage this domain corresponds to the crosshatched region on Fig. 4(b) and in the double optical stage the inverse FFT domain corresponds to the crosshatched region on Fig. 4(c).

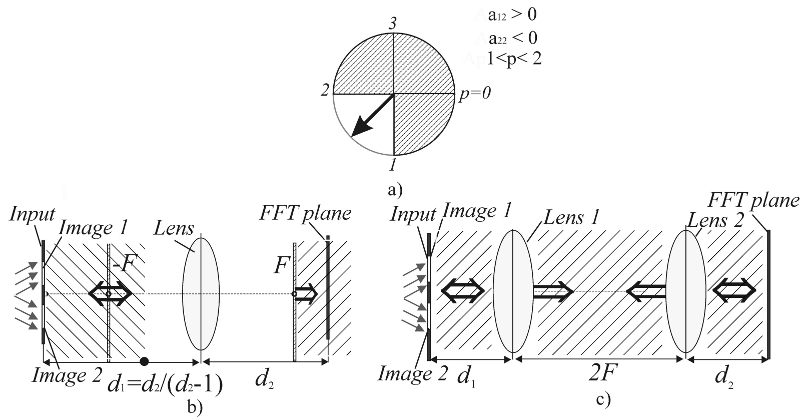

Figure 3. Conjugate FFT domain.

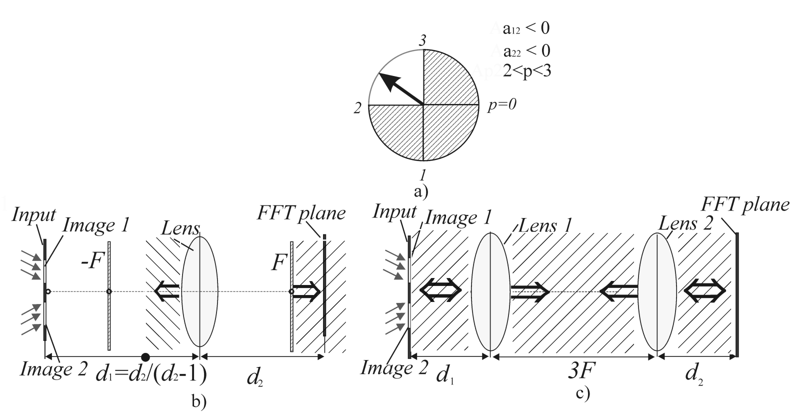

Figure 4. Inverse FFT domain.

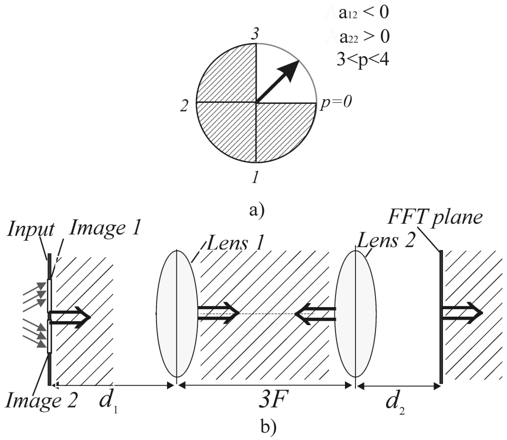

Figure 5. Conjugate inverse FFT domain.

3.4. Conjugate inverse fractional Fourier transform domain

Conjugate inverse fractional Fourier transform domain Fig. 5(a) has its perticularties becouse it is not realized in the optical stage. Realization in the double stage is shown on Fig. 5(b) (crosshatched region on this Figure corresponds to the conjugate inverse FFT domain).

4. Conclusion

Conjugate images forming is theoretically described based on ambiguity function. Obtained results of investigations give theoretical background of the two images optical superposition in the FFT domain, which are modulated by hight-constrast interference band pattern. Four domains of the FFT realization are investigated. It is shown that each domain may be realized as in optical and in double optical stage. These results may be used for generalizated fractional optical correlator construction.

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Роботу отримано 10 грудня 2003 р.

Затверджено до друку Вченою радою ІФКС НАН України

Рекомендовано до друку семінаром відділу теорії конденсованого стану

Виготовлено при ІФКС НАН України