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THERMODYNAMIC FUNCTIONS OF THE SYSTEM OF THE CHARGED PARTICLES WITH RELATIVISTIC INTERACTION IN THE RING-DIAGRAM APPROXIMATION

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Термодинамічні функції системи заряджених частинок з релятивістичною взаємодією у наближенні кільцевих діяграм

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Анотація. Досліджено статистичну суму релятивістичної системи заряджених частинок у наближенні кільцевих діяграм на базі гамільтоніяну, одержаного у лінійному наближенні за константою взаємодії. Знайдено релятивістичну поправку до теорії Дебая-Хюкеля. Записано рівняння стану.

Thermodynamic functions of the system of the charged particles with relativistic interaction in the ring-diagram approximation

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Abstract. The partition function of relativistic system of charged particles is studied in the ring-diagram approximation on the base of the Hamiltonian which is obtained in the linear approximation in the coupling constant. The relativistic correction to the Debye-Hukkel theory is found. The state equation is written.

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1. Introduction

The traditional formulation of statistical mechanics based on the Gibbs distribution demands Hamilton picture to describe the system. The Hamiltonian description of relativistic system of charged particles can be obtained within the field theory framework. However, it is known that the free energy of the classical field subsystem diverges in accordance with Relay-Jeans formula [1]. So, the correct value of the partition function may be found when the field degrees of freedom are excluded. Elimination of the field can be performed in an action integral of the system by substitution of the formal solution with the corresponding Green function. Wheeler-Feynman electrodynamics is an instanse of such a theory [2]. Nonlocality of the action in the terms of particle variables leads to seriuos difficulties in transition to the Hamiltonian description. Different ways of Hamiltanization of such a system by means of approximation approachs have been studied in literature [3–5]. More famous result was obtained by Darwin [6]. Simplicity of the weak-relativistic Darwin Lagrangian and the corresponding Hamiltonian allowed in [7,8] to examine the partition function in the ring-diagram approximation. The obtained expressions for free energy are generalized the result of the Debye-Hukkel theory.

We studied an alternative way which consists in cancelation of the field degrees of freedom after transition to the Hamiltonian description [9,10]. Using the first order approximation in the coupling constant e^2 and symmetric Green function, we found solution of the field equations in the terms of the canonical particle variables (x_a^i, k_a^i). By this way, we derived the following Hamiltonian of the system of identical charged particles [11]:

$$H = \sum_{a=1}^{N} \sqrt{m^2 + \mathbf{k}_a^2} + \frac{1}{2V} \sum_{a,b=1}^{N}{}' \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}_{ab}} \nu(\mathbf{k}) \left[1 - \frac{\mathbf{v}_a \hat{P}_\perp(\mathbf{n}) \mathbf{u}_b}{1 - (\mathbf{n}\mathbf{u}_b)^2} \right]. \quad (1)$$

Here prime over sum symbol means that $a \neq b$, V is volume of the system. Velocities and the wave-vector dependent expressions are defined as

$$\mathbf{v}_a = \frac{\mathbf{k}_a}{\sqrt{m^2 + \mathbf{k}_a^2}}, \qquad \mathbf{u}_a = \frac{\mathbf{p}_a}{\sqrt{m^2 + \mathbf{p}_a^2}},$$

$$\nu(\mathbf{k}) = \frac{4\pi e^2}{\mathbf{k}^2}, \qquad \mathbf{n} = \frac{\mathbf{k}}{k}, \qquad \hat{P}_\perp(\mathbf{n}) = \| \delta^{ij} - n^i n^j \|.$$

The relation between momenta $\mathbf{k}_a$ and $\mathbf{p}_a$ is given as

$$\mathbf{p}_a = \mathbf{k}_a - \frac{1}{2V} \sum_{b=1}^{N}{}' \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}_{ab}} \nu(\mathbf{k}) \frac{\hat{P}_\perp(\mathbf{n}) \mathbf{u}_b}{1 - (\mathbf{n}\mathbf{u}_b)^2}. \quad (2)$$

In the next section we shall calculate the partition function with the Hamiltonian (1) in the ring-diagram approximation.

2. Partition function

The partition function of this system is written by standard manner

$$Z_N = \frac{1}{N!} \int e^{-\beta H} \prod_{a=1}^{N} \frac{d^3 x_a d^3 k_a}{(2\pi)^3}, \quad (3)$$

where β is inverse temperature.

Into dynamics, where the number N of the particles is finite, equation (2) can be solved immediately under $\mathbf{p}_a$ in the linear approximation in the coupling constant. One has

$$\mathbf{p}_a = \mathbf{k}_a - \frac{1}{2V} \sum_{b=1}^{N}{}' \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}_{ab}} \nu(\mathbf{k}) \frac{\hat{P}_\perp(\mathbf{n}) \mathbf{v}_b}{1 - (\mathbf{n}\mathbf{v}_b)^2}. \quad (4)$$

This relation leads to the result in [11].

However, into statistical mechanics, when $N, V \to \infty$ and $n = N/V = \text{const}$, approximation has been done by powers of $e^2 n$ (see [8]). In order to do not solve (2), we rewrite the Hamiltonian and the partition function like to

$$H = H_0 + U_0, \quad (5)$$

$$H_0 = \sum_{a=1}^{N} \sqrt{m^2 + \mathbf{p}_a^2}, \quad U_0 = \frac{1}{2V} \sum_{a,b=1}^{N}{}' \sum_{\mathbf{k}} \nu(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}_{ab}},$$

$$Z_N = \frac{1}{N!} \int e^{-\beta(H_0 + U_0)} J \prod_{a=1}^{N} \frac{d^3 x_a d^3 p_a}{(2\pi)^3}. \quad (6)$$

Here the Jacobian is

$$J = \det \left\| \frac{\partial k_a^i}{\partial p_b^j} \right\| = \det \left\| \delta_{ab} \hat{I} + \frac{1 - \delta_{ab}}{2V} \sum_{\mathbf{k}} \nu(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}_{ab}} \hat{P}_\perp(\mathbf{n}) \hat{\mathbf{\Phi}}_b(\mathbf{n}) \right\|, \quad (7)$$

$$\hat{\Phi}_b(\mathbf{n}) = \left\| \frac{\partial}{\partial p_b^j} \frac{u_b^i - n^i(\mathbf{n}\mathbf{u}_b)}{1-(\mathbf{n}\mathbf{u}_b)^2} \right\|. \tag{8}$$

Structure of the Jacobian permits us to make exactly an integration over momenta:

$$\int e^{-\beta H_0} J \prod_{a=1}^{N} \frac{d^3p_a}{(2\pi)^3} = \xi^N \det \|\delta_{ab}\hat{I} + \hat{C}_{ab}\|, \tag{9}$$

where the Juttner result is

$$\xi = \int e^{-\beta\sqrt{m^2+\mathbf{p}^2}} \frac{d^3p}{(2\pi)^3} = \frac{2m^2K_2(\beta m)}{(2\pi)^2\beta}. \tag{10}$$

Then we have

$$\hat{C}_{ab} = \frac{1-\delta_{ab}}{2V} \sum_{\mathbf{k}} \nu(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}_{ab}} \hat{P}_\perp(\mathbf{n})\hat{A}(\mathbf{n}), \tag{11}$$

$$\hat{A}(\mathbf{n}) = \frac{1}{\xi}\int e^{-\beta\sqrt{m^2+\mathbf{p}_b^2}} \hat{\Phi}_b(\mathbf{n}) \frac{d^3p_b}{(2\pi)^3}. \tag{12}$$

One can check that the matrix $\hat{A}$ is equal to

$$\hat{A}(\mathbf{n}) = a\hat{P}_\perp(\mathbf{n}),$$

$$a = \frac{\beta}{2\xi}\int e^{-\beta\sqrt{m^2+\mathbf{p}_b^2}} \frac{\mathbf{u}_b^2 - (\mathbf{n}\mathbf{u}_b)^2}{1-(\mathbf{n}\mathbf{u}_b)^2} \frac{d^3p_b}{(2\pi)^3} = \frac{1}{m}\frac{K_1(\beta m)}{K_2(\beta m)}. \tag{13}$$

Here K_α is the Macdonald function.

After that the partition function of the system of the charged particles can be presetned as follows

$$Z_N = Z_N^{\mathrm{id}} Q_N, \qquad Z_N^{\mathrm{id}} = \frac{(\xi V)^N}{N!}, \tag{14}$$

$$Q_N = \frac{1}{V^N}\int e^{-\beta U_0} \det\|\delta_{ab}\hat{I} + \hat{C}_{ab}\| \prod_{a=1}^{N} d^3x_a. \tag{15}$$

Now we can write that

$$W = \det\|\delta_{ab}\hat{I} + \hat{C}_{ab}\| = \exp\left[\mathrm{Tr}\ \ln\|\delta_{ab}\hat{I} + \hat{C}_{ab}\|\right]. \tag{16}$$

Using expansion for ln(1 + x), we come to expression

$$W = \exp\left[\mathrm{Tr}\left(\hat{C}_{ab} - \frac{1}{2}\sum_c \hat{C}_{ac}\hat{C}_{cb} + \frac{1}{3}\sum_{c,d}\hat{C}_{ac}\hat{C}_{cd}\hat{C}_{db} - ...\right)\right]. \tag{17}$$

Taking into account relations [8]

$$\sum_{a=1}^{N} e^{-i\mathbf{x}_a(\mathbf{k}-\mathbf{q})} = N\delta_{\mathbf{k},\mathbf{q}}, \qquad \hat{P}_\perp^s(\mathbf{n}) = \hat{P}_\perp(\mathbf{n}),$$

one has

$$\begin{aligned} W &= \exp\Bigg[\mathrm{Tr}\Bigg(\frac{1-\delta_{ab}}{2V} a \sum_{\mathbf{k}} \nu(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}_{ab}} \hat{P}_\perp(\mathbf{n}) \\ &\quad -\frac{1}{N}\sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}_{ab}}\hat{P}_\perp(\mathbf{n}) \sum_{s=2}^{\infty}\frac{1}{s}\left(-\frac{\kappa_r^2}{\mathbf{k}^2}\right)^s\Bigg)\Bigg], \end{aligned}$$

where

$$\kappa_r^2 = \frac{2\pi e^2 N}{V} a, \qquad \mathrm{Tr}\ \hat{P}_\perp(\mathbf{n}) = 2.$$

Then, W immediately becomes

$$W = \exp\left[-2\sum_{\mathbf{k}}\sum_{s=2}^{\infty}\frac{1}{s}\left(-\frac{\kappa_r^2}{\mathbf{k}^2}\right)^s\right]. \tag{18}$$

Since

$$\sum_{s=2}^{\infty}\frac{(-x)^s}{s} = x - \ln(1+x),$$

one can writes that

$$W = \exp\left[-2\sum_{\mathbf{k}}\left(\frac{\kappa_r^2}{\mathbf{k}^2} - \ln\left(1 + \frac{\kappa_r^2}{\mathbf{k}^2}\right)\right)\right]. \tag{19}$$

Summing, we obtain

$$W = \exp\left(-\frac{\kappa_r^3 V}{3\pi}\right). \tag{20}$$

Then, one finds

$$Q_N = W\int e^{-\beta U_0}\prod_{a=1}^{N}\frac{d^3x_a}{V}. \tag{21}$$

Substituting the well-known Coulomb contribution in the ring-diagram approximation into Q_N, it follows that

$$Q_N = e^{-\beta\Delta F}, \qquad \Delta F = -\frac{\kappa^3 V}{12\pi\beta} + \frac{\kappa_r^3 V}{3\pi\beta}, \tag{22}$$

here $\kappa^2 = 4\pi\beta N/V$.

Although we do not sum the ring diagrams by standard way, when the relavistic correction is calculated, they are accounting by means of preservation of the terms $e^2(e^2n)^p$ $(p = 0, 1, 2, ...)$ in the partition function (see (17)).

Finaly note that in the weakly relativistic approximation the interaction correction to the free energy has the form

$$\Delta F = -\frac{\kappa^3 V}{12\pi\beta}\left(1 - \frac{\sqrt{2}}{(\beta m)^{3/2}}\right). \tag{23}$$

Let us note that the weakly relativistic correction in [8] has the coefficient 2 instead of $\sqrt{2}$ in (23).

3. State equation

Now we can find the state equation. Pressure of the gas of relativistic particles with electromagnetic interaction is derived as

$$\begin{aligned} P &= -\left(\frac{\partial F}{\partial V}\right)_T \\ &= \frac{TN}{V}\left(1 - \frac{e^2\beta}{6}\kappa\left[1 - \sqrt{2}\left(\frac{1}{m\beta}\frac{K_1(\beta m)}{K_2(\beta m)}\right)^{3/2}\right]\right). \end{aligned} \tag{24}$$

It is easy to see that the second term in r.h.s. defines the Coulomb correction within the Debye-Hukkel theory. The last term corresponds to relativistic interaction.

References

1. L. D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, New York, 1971).
2. J. A. Wheeler and R. P. Feynman, *Rev. Mod. Phys.* **21**, 425 (1949).
3. R. P. Gaida, *Sov. J. Part. Nucl.* **13**, 179 (1982).
4. F. J. Kennedy, *J. Math. Phys.* **16**, 1844 (1975).
5. J. Llosa and J. Vives, *J. Math. Phys.* **35**, 2856 (1994).
6. C. G. Darwin, *Phil. Mag.* **39**, 537 (1920).
7. B. A. Trubnikov and V. P. Kosachev, *JETP* **27**, 501 (1958).
8. L. F. Blazhievskii, *Ukr. Fiz. Zh.* **20**, 1273 (1975).
9. A. Nazarenko, *Proceedings of Institute of Mathematics of Nat. Acad. Sci. of Ukraine* **30**, 343 (2000).

10. A. Duviryak, A. Nazarenko, and V. Tretyak, *Cond. Matter Phys.* **4**, 5 (2001).
11. A. Nazarenko, *Int. J. Mod. Phys.* (2001) (in print).

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