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On the discontinuity of the specific heat of the Ising model on a scale-free network

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We consider the Ising model on an annealed scale-free network with node-degree distribution characterized by a power-law decay $P(K) \sim K^{-\lambda}$. It is well established that the model is characterized by classical mean-field exponents for $\lambda > 5$. In this note we show that the specific-heat discontinuity δc_h at the critical point remains λ-dependent even for $\lambda > 5$: $\delta c_h = 3(\lambda-5)(\lambda-1)/[2(\lambda-3)^2]$ and attains its mean-field value $\delta c_h = 3/2$ only in the limit $\lambda \to \infty$. We compare this behaviour with recent measurements of the d dependency of δc_h made for the Ising model on lattices with $d > 4$ [Lundow P.H., Markström K., Nucl. Phys. B, 2015, **895**, 305].

Key words: *Ising model, scale-free networks, annealed network*

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In the Ehrenfest classification, a second-order phase transition is manifest by a discontinuity of the second derivative of the free energy at the transition temperature T_c [1]. However, derivatives taken with respect to different thermodynamic variables may demonstrate qualitatively different behaviour. For magnetic systems, it is well known that the isothermal susceptibility χ_T and magnetocaloric coefficient m_T (a mixed derivative of the free energy with respect to magnetic field and temperature) are strongly diverging quantities, whereas the specific heat c_h often does not diverge at T_c. Considered in the mean-field approximation, the first two quantities are singular at $\tau = |T-T_c|/T_c = 0$: $\chi_T \sim \tau^{-\gamma}$, $m_T \sim \tau^{-\omega}$ with $\gamma^{\mathrm{mfa}} = 1$, $\omega^{\mathrm{mfa}} = 1/2$. However, the third quantity displays a jump at T_c:

$$\delta c_h = c_h(T \to T_c^-) - c_h(T \to T_c^+), \qquad (1)$$

with $\delta c_h^{\mathrm{mfa}} = 3/2$ and hence $c_h \sim \tau^{-\alpha}$ with $\alpha^{\mathrm{mfa}} = 0$.

For the Ising model in d dimensions, the singularity of the specific heat is d-dependent: the famous Onsager solution [2] predicted $c_h(d=2) \sim \ln\tau$ (a weak singularity with $\alpha = 0$) while $\alpha(d=3) \simeq 0.109(4)$ [3] and α attains its mean-field value in dimensions higher than the upper critical value, $\alpha(d>4) = 0$. Strictly at $d = 4$, the scaling is affected by the logarithmic correction [4]

$$c_h \sim \tau^{-\alpha^{\mathrm{mfa}}}(\ln\tau)^{\hat{\alpha}}. \qquad (2)$$

Since $\alpha^{\mathrm{mfa}} = 0$ and the logarithmic correction-to-scaling exponent $\hat{\alpha} = 1/3$ is positive [4], the specific heat of the Ising model diverges at $d = 4$.

Although the critical exponents attain their mean-field values above the upper critical dimension, this is not the case for critical amplitudes. For $d > 4$, the latter determine the value of the specific heat discontinuity in equation (1). As has been shown recently [5], δc_h for the Ising model at $d > 4$ remains a d-dependent quantity that reaches the mean-field result only in the limit $\delta c_h(d \to \infty) = 3/2$. Inspired by this observation, which was produced using Monte Carlo simulations for 5, 6, and 7-dimensional lattices [5], in this note we analyze the behaviour of the specific-heat discontinuity of the Ising model on complex networks. Recent interest in structures of numerous natural and man-made systems [6–9] lead, in particular, to the development of phase transition theory on complex networks [10]. Of particular interest are scale-free networks, where the node-degree distribution is characterized by a power-law decay:

$$P(K) = c/K^{\lambda}. \tag{3}$$

Here, $P(K)$ is the probability that the number of nearest neighbours of a node (node degree) is K and c is a normalizing constant. It appears that many real-world complex networks (e.g., the internet, www, transportation networks, social networks of communication between people and many others) are scale-free [6–9]. In turn, studying properties of phase transitions on scale-free networks may also explain peculiarities of processes occurring on such networks too. To give just two examples, the analysis of percolation phenomena on scale-free networks is directly related to the stability of the network to random breakdowns or targeted attacks, whereas the onset of an ordered phase (e.g., ferromagnetic ordering in a spin model on a network) may correspond to a unanimous opinion formation in a social network.

Here, the subject of our analysis is the Ising model on a complex scale-free network. In particular, we will consider the behaviour of the specific heat on an annealed network. This has been widely used to analyze properties of various spin models (see e.g., [11–13] and references therein). For annealed networks, the links fluctuate on the same time scale as the spin variables [11–13], therefore, the partition function is averaged both with respect to the link distribution and the Boltzmann distribution. This is achieved by assigning to each node i a hidden variable k_i. In our particular case of a scale-free network, the distribution of k_i is given by (3) too. The probability of a link between any pair of nodes (i, j) is chosen to be proportional to the product $k_i k_j$ of k-variables on these nodes. One can check that the expected node-degree value is then $\mathrm{E}[K_i] = k_i$. This choice leads to the Hamiltonian which, in the absence of an external magnetic field, reads:

$$\mathscr{H} = -\frac{1}{N\langle k\rangle}\sum_{i>j} k_i k_j S_i S_j. \tag{4}$$

Here, $S_i = \pm 1$ is a spin variable, the sum spans all pairs of N nodes and $\langle k\rangle = \sum_{i=1}^{N} k_i/N$.

The prominent feature of (4) is that the interaction term attains a separable form. In turn, this allows for an exact representation of the partition function via e.g., Stratonovich-Hubbard transformation, as it is usually done for the Ising model on a complete graph [14], see [15, 16] and references therein. It is straightforward to get thermodynamic functions and, in particular, to arrive at the conclusion that universal behaviour of the specific heat depends on the node-degree distribution exponent λ [17–20][1]:

$$\alpha = (\lambda-5)/(\lambda-3), \qquad 3<\lambda\leqslant 5; \qquad \alpha = 0, \qquad \lambda>5. \tag{5}$$

The negativity of the exponent α in the region $3<\lambda<5$ means that $\delta c_h = 0$ there. Moreover, directly at $\lambda = 5$ the logarithmic correction-to-scaling exponent governs the behaviour, similar as for lattices at $d = 4$, see equation (2). However, in contrast to the lattice case, the value of the exponent for scale-free networks is negative: $\hat{\alpha} = -1$ [17–20]. This means that $\delta c_h = 0$ at $\lambda = 5$ too.

Here, we are interested in the behaviour of the specific heat in the region $\lambda > 5$, where usual mean-field results for the critical exponents hold. Keeping terms leading in N for the partition function, one can represent it in the form (see [15, 16] for more details)

$$Z_N(T) = \int_{-\infty}^{+\infty} \exp\left\{N\left[\frac{-\langle k\rangle x^2}{2}(T-T_{\mathrm{c}}) - \frac{\langle k^4\rangle x^4}{12}\right]\right\}\mathrm{d}x, \qquad \lambda>5, \tag{6}$$

where $T_{\mathrm{c}} = \langle k^2\rangle/\langle k\rangle$ and we have omitted a prefactor which is not important for our analysis.

Using the method of steepest descent one finds points of maxima ($x_\star$) of the function under integration at $T > T_{\mathrm{c}}$ ($x_\star = 0$) and $T < T_{\mathrm{c}}$ ($x_\star = [-(3\langle k\rangle/\langle k^4\rangle)(T-T_{\mathrm{c}})]^{1/2}$). The free energy reads:

$$f(T) = \begin{cases} 0, & T > T_{\mathrm{c}}, \\ -\frac{3\langle k\rangle^2}{4\langle k^4\rangle} T(T-T_{\mathrm{c}})^2, & T < T_{\mathrm{c}}. \end{cases} \tag{7}$$

[1] The system remains ordered at any finite temperature for $2 < \lambda \leqslant 3$.

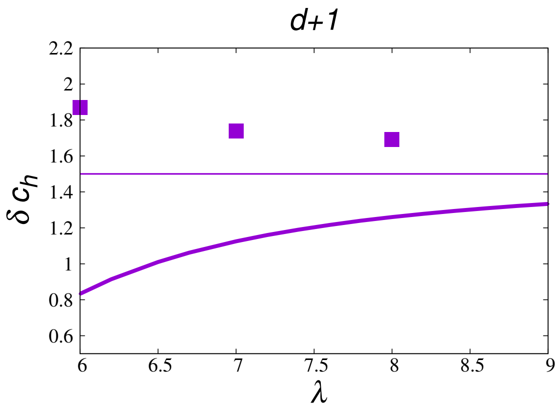

Figure 1. The jump in the specific heat of the Ising model on lattices at $d > 4$ (squares, results of MC simulations [5]) and on an annealed scale-free network for $\lambda > 5$, bold line equation (10). The thin line shows classical mean-field value $\delta c_h = 3/2$. Although $\delta c_h(\lambda \to \infty) = \delta c_h(d \to \infty) = 3/2$, the functions approach the mean-field limit from below and from above.

Correspondingly, for the specific heat one obtains

$$c_h = \begin{cases} 0, & T > T_{\rm c}, \\ -\frac{9\langle k\rangle^2}{2\langle k^4\rangle} T^2 + \frac{6\langle k\rangle^2}{2\langle k^4\rangle} T T_{\rm c}, & T < T_{\rm c}. \end{cases} \tag{8}$$

The jump of the specific heat at $T_{\rm c}$ is defined by the ratio

$$\delta c_h = \frac{3\langle k^2\rangle^2}{2\langle k^4\rangle}. \tag{9}$$

Substituting the averages calculated with the distribution (3) we obtain

$$\delta c_h = \frac{3(\lambda-5)(\lambda-1)}{2(\lambda-3)^2}, \qquad \lambda > 5. \tag{10}$$

In the limit of large λ this delivers $\delta c_h = 3/2$, which coincides with the corresponding value on a complete graph.

It is well known that Ising model on an annealed scale-free network is characterized by classical mean-field exponents at $\lambda > 5$. As we have shown in this note, the mean-field behaviour does not concern the specific heat jump δc_h at $\lambda > 5$. The jump remains λ-dependent and reaches the mean-field value $\delta c_h = 3/2$ only in the limit $\lambda \to \infty$. The function $\delta c_h(\lambda)$ is shown in figure 1. Similar effect has been observed for the Ising model on lattices at $d > 4$. We show the results of MC simulations of $d = 5, 6, 7$-dimensional lattices [5] in the figure too. Note, that although $\delta c_h(\lambda \to \infty) = \delta c_h(d \to \infty) = 3/2$, the functions approach the mean-field limit from below and from above. Another essential difference between the behaviour of δc_h in the Ising model on scale-free networks and on lattices is observed directly at the upper critical values of λ and of d, respectively. While $\alpha = 0$ in both cases, the overall behaviour of c_h remains singular on lattices at $d = 4$ (logarithmic singularity, $\hat{\alpha} = 1/3$) whereas $\hat{\alpha} = -1$ for networks at $\lambda = 5$ and hence $\delta c_h = 0$. This last case provides an example where the logarithmic correction to scaling leads to smoothing of behaviour of the thermodynamic function at $T_{\rm c}$.

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Стрибок теплоємності моделі Ізінга на безмасштабній мережі

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Ми розглядаємо модель Ізінга на відпаленій безмасштабній мережі зі степенево-спадною функцією розподілу вузлів $P(K) \sim K^{-\lambda}$. Відомо, що ця модель описується класичними критичними показниками середнього поля при $\lambda > 5$. Тут ми покажемо, що стрибок теплоємності δc_h при критичній температурі залишається λ-залежним навіть для $\lambda > 5$: $\delta c_h = 3(\lambda-5)(\lambda-1)/[2(\lambda-3)^2]$ і досягає свого середньопольового значення $\delta c_h = 3/2$ тільки в границі $\lambda \to \infty$. Ми порівнюємо цю поведінку із недавніми результатами залежності δc_h від d для моделі Ізінга на гратках з $d > 4$ [Lundow P.H., Markström K., Nucl. Phys. B, 2015, **895**, 305].

Ключові слова: *модель Ізінга, безмасштабна мережа, відпалена мережа*