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A generalized hydrodynamical Gurevich-Zybin equation of Riemann type and its Lax type integrability

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This paper is devoted to the study of a hydrodynamical equation of Riemann type, generalizing the remarkable Gurevich–Zybin system. This multi-component non-homogenous hydrodynamic equation is characterized by the only characteristic flow velocity. The compatible bi-Hamiltonian structures and Lax type representations of the 3-and 4-component generalized Riemann type hydrodynamical system are analyzed. For the first time the obtained results augment the theory of integrability of hydrodynamic type systems, originally developed only for distinct characteristic velocities in homogenous case.

Key words: *Riemann type hydrodynamical equations, Lax type integrability, conservation laws*

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1. Introduction

Evolution differential equations of special type are capable of describing [1–3] many important problems of wave propagation in nonlinear media with distributed parameters, for instance, invisible non-dissipative dark matter, playing a decisive role [4, 5] in the formation of large scale structure in the Universe like galaxies, clusters of galaxies, super-clusters. In particular, if the nonlinear medium is endowed with some regularity no-blow up properties, the propagation of the corresponding waves can be modeled by means of the so-called Gurevich-Zybin dynamical system

$$D_t u = z, \qquad D_t z = 0, \tag{1}$$

where, by definition, $D_t := \partial/\partial t + u\partial/\partial x$, $u := \mathrm{d}x/\mathrm{d}t$, $(x,t) \in \mathbb{R}^2$, is the corresponding characteristic flow velocity and $z \in C^\infty(\mathbb{R}^2;\mathbb{R})$, is the related self-dual inhomogeneity magnitude. It was amazing to see that the *inhomogeneous* Riemann type hydrodynamic system (1) can be integrated, up to the first singularity, using the hodograph method (see [4, 6]).

Below in the second section we first construct a general solution to (1) using the method of reciprocal transformations. In the third and fourth sections we will analyze the related infinite hierarchies of conservation laws, the bi-Hamiltonian structures and Lax type integrability of a Riemann type hydrodynamical system $D_t^N u = 0$ at $N = 3$ and $N = 4$, naturally generalizing the system (1). In the Conclusion, the obtained results subject to the well-known C- and S-integrability schemes of nonlinear dynamical systems are discussed.

2. A general solution to the Gurevich-Zybin hydrodynamical system of Riemann type

Let us introduce the auxiliary field variable $\rho := z_x$ for some smooth mapping $\rho \in C^\infty(\mathbb{R}^2;\mathbb{R})$. Then the second equation of (1) reduces to the so-called continuity equation

$$\rho_t + \partial_x(\rho u) = 0, \tag{2}$$

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which can be utilized in the construction of a simple reciprocal transformation

$$\mathrm{d}z = \rho \mathrm{d}x - \rho u \mathrm{d}t, \qquad \mathrm{d}y = \mathrm{d}t. \tag{3}$$

From (3) one easily obtains that $\partial_x = \rho\partial_z$ and $\partial_t = \partial_y - \rho u \partial_z$. Thus, system (1) reduces to the form

$$\left(\rho^{-1}\right)_y = u_z, \qquad u_y = z. \tag{4}$$

The second equation can be easily integrated to

$$u = yz + f(z), \tag{5}$$

where $f \in C^\infty(\mathbb{R};\mathbb{R})$ is an arbitrary smooth function. Then, the first equation in (4) reduces to the form

$$\left(\rho^{-1}\right)_y = y + f'(z),$$

which can be easily integrated as

$$\rho^{-1} = y^2/2 + f'(z)y + \varphi'(z), \tag{6}$$

where $\varphi \in C^\infty(\mathbb{R};\mathbb{R})$ is another arbitrary smooth function. Taking into account (5) and (6), an independent spatial variable $x \in \mathbb{R}$ can be found by integrating the inverse to (3) reciprocal transformation

$$\mathrm{d}x = \rho^{-1}\mathrm{d}z + u\mathrm{d}y, \qquad \mathrm{d}t = \mathrm{d}y. \tag{7}$$

Thus, the general solution to the dynamical system (1) is given implicitly in the following parametric form:

$$u = zt + f(z), \qquad x = \frac{zt^2}{2!} + f(z)t + \varphi(z).$$

3. An N-component generalized one-dimensional Riemann type hydrodynamical equation

The inhomogeneous hydrodynamic type system (1) can be written in a compact form (thanks to Darryl Holm for this observation)

$$D_t^2 u = 0, \tag{8}$$

where the flow operator $D_t = \partial_t + u\partial_x, (x,t) \in \mathbb{R}^2$, is well known in fluid dynamics [10]. Indeed, the aforementioned equation can be written as two interrelated equations of the first order

$$D_t u = z, \qquad D_t z = 0, \tag{9}$$

which is nothing else but exactly (1). Thus, an obvious generalization of (1) for an N-component case is written as

$$D_t^N u = 0, \tag{10}$$

where $N \in \mathbb{Z}_+$ is arbitrary.

Thus, we can formulate our result as the following proposition.

Proposition 3.1 *The generalized dynamical system (10) is also integrable by means of a suitable reciprocal transformation (see (3)), possessing the related infinite hierarchies of conservation laws.*

Proof. Indeed, let us write (10) as $N-$component quasilinear system of the first order

$$D_t u_1 = u_2, \qquad D_t u_2 = u_3, \qquad \ldots, \qquad D_t u_{N-1} = u_N, \qquad D_t u_N = 0, \tag{11}$$

where mappings $u_1 = u, u_2, u_3, \ldots, u_{N-1}, u_N \in C^\infty(\mathbb{R}^2;\mathbb{R})$ are the corresponding intermediate smooth field variables.

Let us introduce an auxiliary field variable $\rho = z_x$, where $z := u_N$. The inhomogeneous hydrodynamic Riemann type system (11), upon its rewriting as

$$\partial_t u_N + u_1 \partial_x u_N = 0, \qquad \ldots, \qquad \partial_t u_k + u_1 \partial_x u_k = u_{k+1}\,, \qquad \partial_t u_1 + u_1 \partial_x u_1 = u_{1+1}\,, \tag{12}$$

reduces (by means of the reciprocal transformation (3), based on the continuity equation (2)), to the following form

$$\left(\rho^{-1}\right)_y = \partial_z u_1\,, \qquad \partial_y u_1 = u_2\,, \qquad \partial_y u_2 = u_3\,, \qquad \ldots, \qquad \partial_y u_{N-2} = u_{N-1}\,, \qquad \partial_y u_{N-1} = z,$$

which is, evidently, equivalent to a pair of simple equations

$$\left(\rho^{-1}\right)_y = \partial_z u, \qquad \partial_y^{N-1} u = z. \tag{13}$$

The last equation can be easily integrated to

$$u = \frac{z y^{N-1}}{(N-1)!} + \sum_{n=0}^{N-2} f_{n+1}(z) \frac{y^n}{n!}\,,$$

where $f_n \in C^\infty(\mathbb{R}^2;\mathbb{R})$, $n = \overline{1, N-1}$, are arbitrary smooth functions. Then, the first equation reduces to the form

$$\left(\rho^{-1}\right)_y = \frac{y^{N-1}}{(N-1)!} + \sum_{n=0}^{N-2} f'_{n+1}(z) \frac{y^n}{n!}$$

one can easily integrate as follows:

$$\rho^{-1} = \frac{y^N}{N!} + \sum_{n=0}^{N-2} f'_{n+1}(z) \frac{y^{n+1}}{(n+1)!} + f'_0(z),$$

where $f_0 \in C^\infty(\mathbb{R}^2;\mathbb{R})$ is also an arbitrary smooth function. Thus, an independent spatial variable $x \in \mathbb{R}$ can be found from the inverse reciprocal transformation (7) as

$$x = \frac{z y^N}{N!} + \sum_{n=0}^{N-2} f_{n+1}(z) \frac{y^{n+1}}{(n+1)!} + f_0(z).$$

As a result, a general solution to (10) is given implicitly by the parametric form

$$\begin{aligned} x &= \frac{z t^N}{N!} + \sum_{n=0}^{N-2} f_{n+1}(z) \frac{t^{n+1}}{(n+1)!} + f_0(z), \\ u &= \frac{z t^{N-1}}{(N-1)!} + \sum_{n=0}^{N-2} f_{n+1}(z) \frac{t^n}{n!}\,, \end{aligned} \tag{14}$$

finishing the proof.

The next remark demonstrates a very deep symmetry degeneracy of the generalized Riemann type hydrodynamical equation (10).

Remark: The inhomogeneous hydrodynamic type system (12) with a common characteristic velocity $\mathrm{d}x/\mathrm{d}t = u_1 := u$ can be generalized for the case of an arbitrary characteristic velocity $\mathrm{d}x/\mathrm{d}t = a(u_1, u_2, \ldots, u_N) := a(\hat{u})$, still preserving the reciprocal transformation integrability described above. Indeed, such an inhomogeneous hydrodynamic type system

$$\partial_t u_N + a(\hat{u}) \partial_x u_N = 0, \qquad \ldots, \qquad \partial_t u_k + a(\hat{u}) \partial_x u_k = u_{k+1}\,,$$

under the reciprocal transformation

$$\mathrm{d}z = \rho \mathrm{d}x - \rho a(\hat{u}) \mathrm{d}t, \qquad \mathrm{d}y = \mathrm{d}t,$$

reduces to the form

$$\left(\rho^{-1}\right)_y = \partial_z a(\hat{u}), \qquad u_k = \frac{zy^{N-k}}{(N-k)!} + \sum_{n=k-1}^{N-2} f_{n+1}(z) \frac{y^{n+1-k}}{(n+1-k)!}\,,$$

where $k = \overline{1,N}$. Since all functions $u_k \in C^\infty(\mathbb{R}^2;\mathbb{R}), k = \overline{1,N}$, are found explicitly in terms of the new independent variables $z \in \mathbb{R}$ and $y \in \mathbb{R}$, the first equation can be easily integrated for any functional dependence $a \in C^\infty(\mathbb{R}^N;\mathbb{R})$. Then, the functional dependence $x := x(z,y)$ can be also found in quadratures.

4. The generalized Riemann type hydrodynamical equation at $N = 2$: conservation laws, bi-Hamiltonian structure and Lax type representation

Consider the generalized Riemann type hydrodynamical equation (10) at $N = 2$:

$$D_t^2 u = 0, \tag{15}$$

where $D_t = \partial/\partial t + u\partial/\partial x$, which is equivalent to the following dynamical system:

$$\left.\begin{array}{c} u_t = v - uu_x \\ v_t = -uv_x \end{array}\right\} := K[u,v], \tag{16}$$

where $K : \mathcal{M} \to T(\mathcal{M})$ is a related vector field on the 2π-periodic smooth nonsingular functional phase space $\mathcal{M} := \{(u,v)^\intercal \in C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^2) : u_x^2 - 2v_x \neq 0, x \in \mathbb{R}\}$. As we are interested first in the conservation laws for the system (16), the following proposition holds.

Proposition 4.1 *Let $H(\lambda) := \int_0^{2\pi} h(x;\lambda)\mathrm{d}x \in D(\mathcal{M})$ be an almost everywhere smooth functional on the manifold $\mathcal{M}$, depending parametrically on $\lambda \in \mathbb{C}$, and whose density satisfies the differential condition*

$$h_t = \lambda(uh)_x \tag{17}$$

for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ on the solution set of dynamical system (16). Then the following iterative differential relationship

$$(f/h)_t = \lambda(uf/h)_x \tag{18}$$

holds, if a smooth function $f \in C^\infty(\mathbb{R};\mathbb{R})$ (parametrically depending on $\lambda \in \mathbb{C}$) satisfies for all $t \in \mathbb{R}$ the linear equation

$$f_t = 2\lambda u_x f + \lambda u f_x\,. \tag{19}$$

Proof. We have from (17)-(20) that

$$\begin{aligned}(f/h)_t &= f_t/h - fh_t/h^2 = f_t/h - \lambda f u_x/h - \lambda f u h_x/h^2 \\ &= f_t/h + \lambda f u(1/h)_x - \lambda u_x f/h \\ &= \lambda(uf)_x/h + \lambda u f(1/h)_x = \lambda(uf/h)_x\,,\end{aligned} \tag{20}$$

proving the proposition.

The obvious generalization of the previous proposition is read as follows.

Proposition 4.2 *If a smooth function $h \in C^\infty(\mathbb{R};\mathbb{R})$ satisfies the relationship*

$$h_t = ku_x h + uh_x\,, \tag{21}$$

where $k \in \mathbb{R}$, then

$$H = \int_0^{2\pi} h^{1/k}\mathrm{d}x \tag{22}$$

is a conservation law for the Riemann type hydrodynamical system (16).

Remark 4.3 *Let $\hat{h} \in C^{\infty}(\mathbb{R};\mathbb{R})$ satisfy the differential relationship $\hat{h}_t = (\hat{h}u)_x$, then $f = \hat{h}^2$ is a solution to equation (19).*

Remark 4.4 *If functions $h_j \in C^{\infty}(\mathbb{R};\mathbb{R}), j \in \mathbb{Z}_+$, satisfy the relationships $h_{j,t} = \lambda(h_j u)_x, j \in \mathbb{Z}_+$, $\lambda \in \mathbb{C}$, then the functionals*

$$H_{(i,j)} = \sum_{n\in \quad \mathbb{Z}_+} k_n^{(i,j)} \int_0^{2\pi} h_i^{2^n} h_j^{(1-2^n)} \tag{23}$$

with $k_n^{(i,j)} \in \mathbb{R},\ \ n \in \mathbb{Z}_+,\ i,j \in \mathbb{Z}_+$, being arbitrary constants, are conserved quantities to equation (16). This formula, in particular, makes it possible to construct an infinite hierarchy of non-polynomial conserved quantities for the Riemann type hydrodynamical system (16).

Example 4.5 *The following non-polynomial functionals*

$$\begin{aligned}
H_4^{(\frac{1}{3})} &= \int_0^{2\pi} \sqrt{u_x^2 - 2v_x}\mathrm{d}x, \qquad\qquad H_7^{(\frac{1}{3})} = \int_0^{2\pi} (u_x v_{xx} - u_{xx} v_x)^{1/3}\,\mathrm{d}x,\\
H_7^{(\frac{1}{2})} &= \int_0^{2\pi} \sqrt{v(u_x^2 - 2v_x)}\mathrm{d}x,\\
H_8^{(\frac{1}{3})} &= \int_0^{2\pi} \left(k_1 u(u_{xx}v_x - u_x v_{xx}) + k_1 v_{xx} v + k_2(u_x^2 v - 2v_x^2)\right)^{1/3}\mathrm{d}x,\\
H_9^{(\frac{1}{6})} &= \int_0^{2\pi} (u_{xx}v_{xxx} - u_{xxx}v_{xx})^{\frac{1}{6}}\,\mathrm{d}x,\\
H_9^{(\frac{1}{4})} &= \int_0^{2\pi} (u_x(u_{xx}v_x - u_x v_{xx}) + v_{xx}v_x))^{\frac{1}{4}},\\
H_{10}^{(\frac{1}{6})} &= \int_0^{2\pi} \left(2u_{xx}(u_x v_{xx} - u_{xx}v_x) - v_{xx}^2\right)^{\frac{1}{6}}
\end{aligned} \tag{24}$$

are conservation laws for the Riemann type dynamical system (16).

Quite different conservation laws have been obtained in [21–23] using the recursion operator technique. The corresponding recursion operator proves to generate no new conservation law, if one applies it to the non-polynomial conservations laws (24).

We also notice that dynamical system (16), as it was shown before in [21, 22], can be transformed via the substitution

$$v = \frac{1}{2}\partial^{-1}(u_x^2 + \eta^2) \tag{25}$$

into the generalized two-component Hunter-Saxton equation:

$$\begin{aligned}
u_{x,t} &= -\frac{1}{2}u_x^2 - uu_{xx} + \frac{1}{2}\eta^2,\\
\eta_t &= -(u\eta)_x.
\end{aligned} \tag{26}$$

This equation allows a simple reduction to the Hunter-Saxton dynamical system [13, 16, 21, 22, 24] at $\eta = 0$:

$$u_{xt} = -\frac{1}{2}u_x^2 - uu_{xx}\,. \tag{27}$$

The non-polynomial conservation laws (24), upon rewriting with respect to the substitution (25), give rise to the related non-polynomial conservation laws for a generalized two-component Hunter-Saxton dynamical system (27). Moreover, if we further apply the reduction $\eta = 0$, we obtain, respectively, new non-polynomial conservation laws for the Hunter-Saxton dynamical system (27), supplementing those found before in [22, 24].

Example 4.6 *The following functionals*

$$\begin{aligned} H_7^{(\frac{1}{3})} &= \int_0^{2\pi} \left(u_{xx}u_x^2\right)^{\frac{1}{3}} \mathrm{d}x, \qquad H_9^{(\frac{1}{6})} = \int_0^{2\pi} \frac{u_{xxx}u + 2u_{xx}u_x}{\sqrt{u_{xx}}} \mathrm{d}x, \\ H_8^{(\frac{1}{3})} &= \int_0^{2\pi} \left[u_{xx}u_x(\partial^{-1}u_x^2) - u_{xx}u_x^2 u\right]^{\frac{1}{3}} \mathrm{d}x \end{aligned} \tag{28}$$

are the conservation laws for the Hunter-Saxton dynamical system (27).

All of these and many other non-polynomial conservation laws can be easily obtained using Proposition (4.2). For example, the following functionals

$$\begin{aligned} H^{(n,m)} &= \int_0^{2\pi} \left(u_{xx}^n u_x^m\right)^{\frac{2}{m+4n}} \mathrm{d}x, \qquad H_1^{(2)} = \int_0^{2\pi} u_x^2(\partial^{-1}u_x^2)^2 \mathrm{d}x, \\ H_2^{(1/2)} &= \int_0^{2\pi} \sqrt{u_{xx}} \mathrm{d}x, \qquad H_3^{(1/3)} = \int_0^{2\pi} \sqrt{u_{xx}(\partial^{-1}u_x^2)} \mathrm{d}x, \\ H_4^{(2/9)} &= \int_0^{2\pi} \left[(\partial^{-1}u_x^2)(uu_xu_{xx}^2 - u_{xx}^2(\partial^{-1}u_x^2))\right]^{\frac{2}{9}} \mathrm{d}x \end{aligned} \tag{29}$$

are also conservation laws for the Hunter-Saxton dynamical system (27), where $m \neq -4n$ and $n, m \in \mathbb{Z}$.

Now we proceed to the analysis of the Hamiltonian properties of the dynamical system (16), for which we will search for solutions [7–9] of Nöther equation

$$L_K\vartheta = \vartheta_t - \vartheta K^{\prime,*} - K^\prime\vartheta = 0. \tag{30}$$

where L_K denotes the corresponding Lie derivative on $\mathcal{M}$ subject to the vector field $K : \mathcal{M} \to T(\mathcal{M})$, $K' : T(\mathcal{M}) \to T(\mathcal{M})$ is its Frechet derivative, $K'^* : T^*(\mathcal{M}) \to T^*(\mathcal{M})$ is its conjugation with respect to the standard bilinear form $(\cdot,\cdot)$ on $T^*(\mathcal{M})\times T(\mathcal{M})$, and $\vartheta : T^*(\mathcal{M}) \to T(\mathcal{M})$ is a suitable implectic operator on $\mathcal{M}$, with respect to which the following Hamiltonian representation

$$K = -\vartheta \ \mathrm{grad} \ H_\vartheta \tag{31}$$

for some smooth functional $H_\vartheta \in D(\mathcal{M})$ holds. To show this, it is enough to find, for instance by means of the small parameter method [7, 8], a non-symmetric ($\psi' \neq \psi^{\prime,*}$) solution $\psi \in T^*(\mathcal{M})$ to the following Lie-Lax equation:

$$\psi_t + K^{\prime,*}\psi = \mathrm{grad} \ \mathcal{L} \tag{32}$$

for some suitably chosen smooth functional $\mathcal{L} \in \mathcal{D}(M)$. As a result of easy calculations one obtains that

$$\psi = (v, 0)^\intercal, \qquad L = \frac{1}{2}\int_0^{2\pi} v^2 \mathrm{d}x. \tag{33}$$

Making use of (32) jointly with the classical Legendrian relationship

$$H_\vartheta := (\psi, K) - \mathcal{L} \tag{34}$$

for the suitable Hamiltonian function, one easily obtains the corresponding symplectic structure

$$\vartheta^{-1} := \psi' - \psi'^{,*} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{35}$$

and the non-singular Hamilton function

$$H_\vartheta := \frac{1}{2}\int_0^{2\pi} (v^2 + v_x u^2)\mathrm{d}x. \tag{36}$$

Since the operator (35) is nonsingular, we obtain the corresponding implectic operator

$$\vartheta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{37}$$

necessarily satisfying the Nöther equation (43).

Here it is worth to observe that the Lie-Lax equation (32) possesses another solution

$$\psi = \left(\frac{u_x}{2}, -\frac{u_x^2}{2v_x}\right), \qquad \mathcal{L} = \frac{1}{4}\int_0^{2\pi} u v_x \mathrm{d}x, \tag{38}$$

giving rise, owing to expressions (35) and (34), to the new co-implectic (singular "symplectic") structure

$$\eta^{-1} := \psi' - \psi'^{,*} = \begin{pmatrix} \partial & -\partial u_x v_x^{-1} \\ -u_x v_x^{-1}\partial & u_x^2 v_x^{-2}\partial + \partial u_x^2 v_x^{-2} \end{pmatrix} \tag{39}$$

on the manifold $\mathcal{M}$, subject to which the Hamiltonian functional equals

$$H_\eta := \frac{1}{2}\int_0^{2\pi} (u_x v - v_x u)\mathrm{d}x, \tag{40}$$

supplying the second Hamiltonian representation

$$K = -\eta \ \mathrm{grad} \ H_\eta \tag{41}$$

of the Riemann type hydrodynamical system (16). The co-implectic structure (39) is singular, since $\hat{\eta}^{-1}(u_x, v_x)^\intercal = 0$, nonetheless one can calculate its inverse expression

$$\eta := \begin{pmatrix} -\partial^{-1} & u_x\partial^{-1} \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x \end{pmatrix}. \tag{42}$$

Moreover, the corresponding implectic structure $\eta : T^*(\mathcal{M}) \to T^*(\mathcal{M})$ satisfies the Nöther equation

$$L_K\eta = \eta_t - \eta K'^{,*} - K'\eta = 0, \tag{43}$$

whose solutions can also be obtained by means of the small parameter method [7, 8]. We also remark that, owing to the general symplectic theory results [7–9] for nonlinear dynamical systems on smooth functional manifolds, operator (39) defines on the manifold $\mathcal{M}$ a closed functional differential two-form. Thereby it is *a priori* co-implectic (in general, singular symplectic), satisfying on $\mathcal{M}$ the standard Jacobi brackets condition.

As a result, the second implectic operator (42), being compatible [7, 9] with the implectic operator (37), gives rise to a new infinite hierarchy of polynomial conservation laws

$$\gamma_n := \int_0^1 \mathrm{d}\lambda \left\langle (\vartheta^{-1}\eta)^n \ \text{grad} \ H_\vartheta[u\lambda\}, u \right\rangle \tag{44}$$

for all $n \in \mathbb{Z}_+$. Having defined the recursion operator $\Lambda := \vartheta^{-1}\eta$, one also finds from (44), (30) and (43) that the following Lax type relationship

$$L_K \Lambda = \Lambda_t - [\Lambda, K^{',*}] = 0 \tag{45}$$

holds. If we construct the asymptotical expansion $\varphi(x;\lambda) \simeq \sum_{j \in \mathbb{Z}_+} \lambda^{1-2j} \ \text{grad} \ \ \gamma_{j-1}[u,v]$ as $\lambda \to \infty$, it is easy to obtain from (44) that the gradient like relationship

$$\lambda^2 \vartheta \varphi(x;\lambda = \eta \varphi(x;\lambda) \tag{46}$$

holds. The latter relationship, making use of the implectic operators (37) and (42), can be represented in the two factorized forms:

$$\varphi(x;\lambda) := \left(\begin{array}{c} \varphi_1(x;\lambda) \\ \varphi_2(x;\lambda) \end{array} \right) = \left(\begin{array}{c} -4\lambda^3 f_1^2 + 2\lambda v_x f_2^2 \\ -4\lambda^2 f_1 f_2 - 2\lambda u_x f_2^2 \end{array} \right) = \left(\begin{array}{c} -2\lambda (f_1 f_2)_x \\ -(f_2^2)_x \end{array} \right), \tag{47}$$

where a vector $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^2)$ lies in an associated to manifold $\mathcal{M}$ vector bundle $\mathcal{L}(\mathcal{M};\mathbb{E}^2)$, whose fibers are isomorphic to the complex Euclidean vector space $\mathbb{E}^2$. Take now into account [7, 8] that the Lie-Lax equation

$$L_K \varphi(x;\lambda) = \mathrm{d}\varphi(x;\lambda)/\mathrm{d}t + K^{\prime,*}\varphi(x;\lambda) = 0 \tag{48}$$

can be transformed equivalently for all $x, t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ into the following evolution system:

$$D_t \varphi = \left(\begin{array}{cc} 0 & v_x \\ -1 & -u_x \end{array} \right) \varphi, \qquad D_t = \partial/\partial t + u\partial/\partial x. \tag{49}$$

The equation (49), owing to the relationship (46) and the obvious identity

$$D_t f_x + u_x f_x = (D_t f)_x \,, \tag{50}$$

can be further split into the adjoint to (49) system

$$D_t f = q(\lambda) f, \qquad q(\lambda) := \left(\begin{array}{cc} 0 & 0 \\ -\lambda & 0 \end{array} \right), \tag{51}$$

where a vector $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^2)$ satisfies the following linear equation

$$f_x = \ell[u,v;\lambda] f, \qquad \ell[u,v;\lambda] := \left(\begin{array}{cc} -\lambda u_x & -v_x \\ 2\lambda^2 & \lambda u_x \end{array} \right), \tag{52}$$

compatible with (51). Moreover, as a result of (51) and (50), the general solution to (52) allows the following functional representation:

$$\begin{aligned} f_1(x,t) &= \tilde{g}_1(u - tv, x - tu + vt^2/2), \\ f_2(x,t) &= -t\lambda \tilde{g}_1(u - tv, x - tu + vt^2/2) + \tilde{g}_2(u - tv, x - tu + vt^2/2), \end{aligned} \tag{53}$$

where $\tilde{g}_j \in C^\infty(\mathbb{R}^2;\mathbb{C}), j = \overline{1,2}$, are arbitrary smooth complex valued functions. Now combining together the obtained relationships (51) and (52), we can formulate the following proposition.

Proposition 4.7 *The Riemann type hydrodynamical system (16) is equivalent to a completely integrable bi-Hamiltonian flow on the functional manifold $\mathcal{M}$, allowing the Lax type representation*

$$f_x = \ell[u,v;\lambda]f, \qquad f_t = p(\ell)f, \qquad p(\ell) := -u\ell[u,v;\lambda] + q(\lambda),$$

$$\ell[u,v;\lambda] := \begin{pmatrix} -\lambda u_x & -v_x \\ 2\lambda^2 & \lambda u_x \end{pmatrix}, \qquad q(\lambda) := \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix},$$

$$p(\ell) = \begin{pmatrix} \lambda u_x u & v_x u \\ -\lambda - 2\lambda^2 u & -\lambda u_x u \end{pmatrix}, \tag{54}$$

where $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^2)$ and $\lambda \in \mathbb{C}$ is an arbitrary spectral parameter.

Remark 4.8 *It is worth to mention here that equation (51) is equivalent on the solution set of the Riemann type hydrodynamical system (16) to the single equation*

$$D_t^2 f_2 = 0 \Longleftrightarrow D_t f_1 = 0, D_t f_2 = -\lambda f_1, \tag{55}$$

where vector $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^2)$ satisfies for all $\lambda \in \mathbb{C}$ the compatibility condition (52) and whose general solution is represented in the functional form (53).

Concerning the set of conservation laws $\{H_2^{(1/2)}, H_3^{(1/2)}\}$, constructed above in (29), they can be extended to an infinite hierarchy $\{\gamma_j^{(1/2)} \in D(\mathcal{M}) : j \in \mathbb{Z}_+\}$, where

$$\gamma_j^{(1/2)} := \int_0^{2\pi} \sigma_{2j-1}[u,v]\mathrm{d}x, \tag{56}$$

and the affine generating function $\sigma(x;\lambda) := \mathrm{d}/\mathrm{d}x \ln f_2(x;\lambda) \simeq \sum_{j=\mathbb{Z}_+\cup\{-1\}} \sigma_j[u,v]\lambda^{-j}$ as $\lambda \to \infty$ satisfies the following functional equation:

$$(\sigma - \lambda u_x)_x + \sigma^2 + \lambda^2(2v_x - u_x^2) = 0. \tag{57}$$

In addition, the gradient functional $\varphi(x;\lambda) := \operatorname{grad} \gamma(x;\lambda) \in T^*(\mathcal{M})$, where $\gamma(\lambda) := \int_0^{2\pi} \sigma(x;\lambda)\mathrm{d}x$, satisfies for all $\lambda \in \mathbb{C}$ the gradient relationship (46).

4.1. The Lax type representation

Here we proceed to the analysis of conservation laws and bi-Hamiltonian structure of the generalized Riemann type equation (10) at $N = 3$:

$$\left.\begin{aligned} u_t &= v - uu_x \\ v_t &= z - uv_x \\ z_t &= -uz_x \end{aligned}\right\} := K[u,v,z], \tag{58}$$

where $K : \mathcal{M} \to T(\mathcal{M})$ is a suitable vector field on the periodic functional manifold $\mathcal{M} := C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^3)$ and $t \in \mathbb{R}$ is an evolution parameter. The system (58) also proves to possess infinite hierarchies of polynomial conservation laws, being suspicious for complete and Lax type integrability.

Namely, the following polynomial functionals, found by means of the algorithm described in section 2, are conserved with respect to the flow (58):

$$
\begin{aligned}
H_n^{(1)} &:= \int_0^{2\pi} \mathrm{d}x z^n \left(vu_x - v_x u - \frac{n+2}{n+1} z \right), \\
H^{(4)} &:= \int_0^{2\pi} \mathrm{d}x \left[-7v_x v^2 u + z(6zu + 2v_x u^2 - 3v^2 - 4vuu_x) \right], \\
H^{(5)} &:= \int_0^{2\pi} \mathrm{d}x (z^2 u_x - 2zvv_x), \qquad H^{(6)} := \int_0^{2\pi} \mathrm{d}x (z_z v^3 + 3z^2 v_x u + z^3), \\
H^{(7)} &:= \int_0^{2\pi} \mathrm{d}x (z_x v^3 + 3z^2 vu_x - 3z^3), \\
H^{(8)} &:= \int_0^{2\pi} \mathrm{d}x z (6z^2 u + 3zv_x u^2 - 3zv^2 - 4zvu_x - 2v_x v^2 u + 2v^3 u_x), \\
H^{(9)} &:= \int_0^{2\pi} \mathrm{d}x [1001 v_x v^4 u + (1092 z^2 u^2 + 364 z v_x u^3 \\
&\quad -1092 z v^2 u - 728 z v u_x u^2 - 364 v_x v^2 u^2 + 273 v^4 + 728 v^3 u_x u]), \\
H_n^{(2)} &:= \int_0^{2\pi} \mathrm{d}x z_x v z^n, \qquad H_n^{(3)} := \int_0^{2\pi} \mathrm{d}x z_x (v^2 - 2zu)^n,
\end{aligned} \tag{59}
$$

where $n \in \mathbb{Z}_+$. In particular, as $n = 1, 2, \ldots$, from (59) one obtains that

$$
\begin{aligned}
H_0^{(2)} &:= \int_0^{2\pi} \mathrm{d}x z_x v, \qquad H_1^{(2)} := \int_0^{2\pi} \mathrm{d}x z_x z v, \ldots, \\
H_1^{(3)} &:= \int_0^{2\pi} \mathrm{d}x z_x (v^2 - 2uz), \\
H_2^{(3)} &:= \int_0^{2\pi} \mathrm{d}x z_x (v^4 + 4z^2 u^2 - 4zv^2 u), \ldots,
\end{aligned} \tag{60}
$$

and so on.

Making use of the iterative property, similar to that, formulated above in Proposition 4.1, one can construct the following hierarchy of non-polynomial dispersive and dispersionless conservation laws:

$$
\begin{aligned}
H_1^{(1/4)} &= \int_0^{2\pi} \mathrm{d}x (-2u_{xx} u_x z_x + u_{xx} v_x^2 + 2u_x^2 z_{xx} - u_x v_{xx} v_x + 3v_{xx} z_x - 3v_x z_{xx})^{1/4}, \\
H_2^{(1/3)} &= \int_0^{2\pi} \mathrm{d}x (-v_{xx} z_x + v_x z_{xx})^{1/3}, \\
H_3^{(1/3)} &= \int_0^{2\pi} \mathrm{d}x (v_{xx} u_x - v_x u_{xx} - z_{xx})^{1/3},
\end{aligned}
$$

$$
\begin{aligned}
H_1^{(1/2)} &= \int_0^{2\pi} \mathrm{d}x[-2vu_xz_x + v_x^2 + z(-u_xv_x + 3z_x)]^{1/2},\\
H_2^{(1/2)} &= \int_0^{2\pi} \mathrm{d}x(8u_x^3z_x - 3u_x^2v_x^2 - 18u_xv_xz_x + 6v_x^3 + 9z_x)^{1/2},\\
H_1^{(1/5)} &= \int_0^{2\pi} \mathrm{d}x(-2u_{xxx}u_xz_x + u_{xxx}v_x^2 + 6u_{xx}^2z_x - 6u_{xx}u_xz_{xx}\\
&\quad - 3u_{xx}v_{xx}v_x + 2u_x^2z_{xxx} - u_xv_{xxx}v_x + 3u_xv_{xx}^2 + 3v_{xxx}z_x - 3v_xz_{xxx})^{1/5},\\
H^{(1/3)} &= \int_0^{2\pi} \mathrm{d}x[k_1u(-v_{xx}z_x + v_xz_{xx}) + k_1v(u_{xx}z_x - u_xz_{xx})\\
&\quad + z(k_2u_{xx}v_x - k_2u_xv_{xx} + k_1z_{xx} + k_2z_{xx}) + k_3(-3u_xv_xz_x + v_x^3 + 3z_x^2)]^{1/3}, \quad (61)
\end{aligned}
$$

where $k_j \in \mathbb{R}, j = \overline{1,3}$, are arbitrary real numbers. Below we will attempt to generalize the crucial relationship (51) from section 2 on the case of the Riemann type hydrodynamical system (58). Namely, we will assume, based on the Remark (4.3), that there exists its following linearization:

$$D_t^3 f_3(\lambda) = 0, \tag{62}$$

modeling the generalized Riemann type hydrodynamical equation (10) at $N = 3$, and where $f_3(\lambda) \in C^\infty(\mathbb{R}^2;\mathbb{C})$ for all values of the parameter $\lambda \in \mathbb{C}$. The scalar equation (62) can be easily rewritten as the system of three linear equations

$$D_tf_1 = 0, \qquad D_tf_2 = \mu_1 f_1, \qquad D_tf_3 = \mu_2 f_2 \tag{63}$$

where we have defined a vector $f := (f_1, f_2, f_2)^\intercal \in C^\infty(\mathbb{R}^2;\mathbb{C}^3)$ and naturally introduced constant numbers $\mu_j := \mu_j(\lambda) \in \mathbb{C},\ j = \overline{1,2}$. It is easy to observe now that, owing to the former result (14), the system of equations (63) allows the following solution representation:

$$
\begin{aligned}
f_1(x,t) &= \tilde g_1(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6),\\
f_2(x,t) &= t\mu_1\tilde g_1(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6)\\
&\quad + \tilde g_2(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6),\\
f_3(x,t) &= \mu_1\mu_2\frac{t^2}{2}\tilde g_1(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6)\\
&\quad + t\mu_2\tilde g_2(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6)\\
&\quad + \tilde g_3(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6),
\end{aligned} \tag{64}
$$

where $\tilde g_j \in C^\infty(\mathbb{R}^3;\mathbb{C}), j = \overline{1,3}$, are arbitrary smooth complex valued functions. The system (63) transforms into the equivalent vector equation

$$D_tf = q(\mu)f, \qquad q(\lambda) := \begin{pmatrix} 0 & 0 & 0\\ \mu_1(\lambda) & 0 & 0\\ 0 & \mu_2(\lambda) & 0 \end{pmatrix}, \tag{65}$$

which should be compatible both with a suitably chosen equation for derivative

$$f_x = \ell[u,v,z;\lambda]f \tag{66}$$

with some matrix $\ell[u,v,z;\lambda] \in SL(3;\mathbb{C})$, defined on the functional manifold $\mathcal{M}$, and with the Lie-Lax equation (48), rewritten as the following system of equations

$$D_t\varphi = \begin{pmatrix} 0 & v_x & z_x\\ -1 & -u_x & 0\\ 0 & -1 & -u_x \end{pmatrix}\varphi, \qquad D_t = \partial/\partial t + u\partial/\partial x, \tag{67}$$

where the vector $\varphi := \varphi(x;\lambda) \in T^*(\mathcal{M})$ is considered as the one factorized by means of a solution $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^3)$ to (66), satisfying the identity (50). Namely, it is assumed that the following quadratic trace-relationship

$$\varphi(x;\lambda) = tr(\Phi\, f \otimes f^{\intercal}) \tag{68}$$

holds for some vector valued matrix functional $\Phi := \Phi[\lambda; u, v, z] \in \mathbb{E}^3 \otimes End\ \mathbb{E}^3$, defined on the manifold $\mathcal{M}$, where "$\otimes$" means the standard tensor product of vectors from the Euclidean space $\mathbb{E}^3$. Making now use of the expressions (50), (68) and (65), one can find by means of somewhat cumbersome and tedious calculations that $\mu_1(\lambda) = \lambda$, $\mu_2(\lambda) = \lambda$, $\lambda \in \mathbb{C}$, and the matrix representation of the derivative (66)

$$\ell[u,v,z;\lambda] = \begin{pmatrix} \lambda^2 u_x & -\lambda v_x & z_x \\ 3\lambda^3 & -2\lambda^2 u_x & \lambda v_x \\ 6\lambda^4 r[u,v,z] & -3\lambda^3 & \lambda^2 u_x \end{pmatrix}, \tag{69}$$

compatible with equation (67), where a smooth mapping $r : \mathcal{M} \to \mathbb{R}$ satisfies the differential relationship

$$D_t r + u_x r = 1. \tag{70}$$

The latter possesses a wide set $\mathcal{R}$ of different solutions amongst which there are the following:

$$\begin{aligned} r \in \mathcal{R} \quad := \quad & \Big\{ \big[(xv - u^2/2)/z\big]_x, (v_x - u_x^2/6)z_x^{-1}, \frac{u_x^3/3 - u_x v_x + 3z_x/2}{2u_x z_x - v_x^2}, \\ & (v_x v^3/6 - u_x v^2 z/2 + u z_x(uz - v^2)/6 + v z^2) z^{-3} \Big\}. \end{aligned} \tag{71}$$

Note here that only the third element from the set (71) allows the reduction $z = 0$ to the case $N = 2$. Thus, the resulting Lax type representation for the Riemann type dynamical system (58) ensues in the form:

$$f_x = \ell[u,v,z;\lambda] f, \qquad f_t = p(\ell) f, \qquad p(\ell) := -u\ell[u,v,z;\lambda] + q(\lambda),$$

$$\ell[u,v,z;\lambda] = \begin{pmatrix} \lambda^2 u_x & -\lambda v_x & z_x \\ 3\lambda^3 & -2\lambda^2 u_x & \lambda v_x \\ 6\lambda^4 r[u,v,z] & -3\lambda^3 & \lambda^2 u_x \end{pmatrix}, \qquad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix},$$

$$p(\ell) = \begin{pmatrix} -\lambda^2 u u_x & \lambda u v_x & -u z_x \\ -3u\lambda^3 + \lambda & 2\lambda^2 u u_x & -\lambda u v_x \\ -6\lambda^4 u \quad r[u,v,z] & \lambda + 3u\lambda^3 & -\lambda^2 u u_x \end{pmatrix}, \tag{72}$$

where $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^3)$ and $\lambda \in \mathbb{C}$ is a spectral parameter.

The next problem, which is of great interest, consists in proving that the generalized hydrodynamical system (58) is a completely integrable bi-Hamiltonian flow on the periodic functional manifold $\mathcal{M}$, as it was proved above for the system (16).

That dynamical system (58) is bi-Hamiltonian which easily follows as a simple corollary from the fact that it possesses the Lax type representation (72) and from the general Lie-algebraic integrability theory [7–9]. Taking into account that dynamical system (58) possesses many (at least 4) Lax type representations, one derives that it possesses many (at least 4) different pairs of compatible co-symplectic structures, each of which generates its own infinite hierarchy of conservation laws commuting to each other. Moreover, the involution of conservation laws belonging to different hierarchies fails owing to their non-compatibility. As the procedure of finding these structures is adjoint with quite cumbersome analytical calculations, hereinbelow we present only one pair of related co-symplectic structures, making use of the standard properties of determining Lie-Lax equation (32).

To tackle with the related task of retrieving the Hamiltonian structure of the dynamical system (58), it is enough, as in section 2, to construct [7, 8] exact non-symmetric solutions to the Lie-Lax equation

$$\psi_t + K'^{,*}\psi = \text{ grad } \mathcal{L}, \qquad \psi' \neq \psi'^{,*}, \tag{73}$$

for some functional $\mathcal{L} \in D(\mathcal{M})$, where $\psi \in T^*(\mathcal{M})$ is, in general, a quasi-local vector, such that the system (58) allows for the following Hamiltonian representation:

$$K[u,v,z] = -\eta \text{ grad } H_\eta[u,v,z], \qquad H_\eta = (\psi_\eta, K) - \mathcal{L}, \qquad \eta^{-1} = \psi_\eta^{'} - \psi_\eta^{',*}. \tag{74}$$

As a test solution to (73) one can take the one

$$\psi_\eta = (u_x/2, 0, -z_x^{-1}u_x^2/2 + z_x^{-1}v_x)^\intercal, \qquad \mathcal{L} = \frac{1}{2}\int_0^{2\pi}(2z + vu_x)\mathrm{d}x,$$

which gives rise to the following co-implectic operator:

$$\eta^{-1} := \psi_\eta' - \psi_\eta'^{,*} = \begin{pmatrix} \partial & 0 & -\partial u_x z_x^{-1} \\ 0 & 0 & \partial z_x^{-1} \\ -u_x z_x^{-1}\partial & z_x^{-1}\partial & \frac{1}{2}(u_x^2 z_x^{-2}\partial + \partial u_x^2 z_x^{-2}) - (v_x z_x^{-2}\partial + \partial v_x z_x^{-2}) \end{pmatrix}. \tag{75}$$

This expression is not strictly invertible, as its kernel possesses the translation vector field $d/\mathrm{d}x : \mathcal{M} \to T(\mathcal{M})$ with components $(u_x, v_x, z_x)^\intercal \in T(\mathcal{M})$, that is $\eta^{-1}(u_x, v_x, z_x)^\intercal = 0$.

Nonetheless, upon formally inverting the operator expression (75), using quite simple, but a bit cumbersome, direct calculations, we obtain that the Hamiltonian function is equal to:

$$H_\eta := \int_0^{2\pi} \mathrm{d}x(u_x v - z) \tag{76}$$

and the implectic η-operator looks as follows:

$$\eta := \begin{pmatrix} \partial^{-1} & u_x\partial^{-1} & 0 \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x & \partial^{-1}z_x \\ 0 & z_x\partial^{-1} & 0 \end{pmatrix}. \tag{77}$$

The same way, representing the Hamiltonian function (76) in the scalar form

$$H_\eta = (\psi_\vartheta, (u_x, v_x, z_x)^\intercal), \qquad \psi_\vartheta = \tfrac{1}{2}(-v, \qquad u + \cdots, -\tfrac{1}{\sqrt{z}}\partial^{-1}\sqrt{z})^\intercal,$$
$$\mathrm{d}\psi_\vartheta/\mathrm{d}t + K'^{,*}\psi_\vartheta = \text{ grad } \mathcal{L}_\vartheta \tag{78}$$

for some functional $\mathcal{L}_\vartheta \in D(\mathcal{M})$, one can construct a second implectic (co-symplectic) operator $\vartheta : T^*(\mathcal{M}) \to T(\mathcal{M})$, looking up to $O(\mu^2)$ terms, as follows:

$$\vartheta = \begin{pmatrix} \mu\big(\frac{(u^{(1)})^2}{z^{(1)}}\partial + \partial\frac{(u^{(1)})^2}{z^{(1)}}\big) & \begin{matrix} 1 + \frac{2\mu}{3}\big(\frac{u^{(1)}v^{(1)}}{z^{(1)}}\partial \\ +2\partial\frac{u^{(1)}v^{(1)}}{z^{(1)}}\big) \end{matrix} & \frac{2\mu}{3}\big(\partial\frac{(v^{(1)})^2}{z^{(1)}} + \partial u^{(1)}\big) \\ \begin{matrix} -1 + \frac{2\mu}{3}\big(\partial\frac{u^{(1)}v^{(1)}}{z^{(1)}} \\ +2\frac{u^{(1)}v^{(1)}}{z^{(1)}}\partial\big) \end{matrix} & \begin{matrix} \frac{2\mu}{3}\big(\frac{(v^{(1)})^2}{z^{(1)}}\partial + \partial\frac{(v^{(1)})^2}{z^{(1)}}\big) \\ +\frac{2\mu}{3}\big(u^{(1)}\partial + \partial u^{(1)}\big) \end{matrix} & 2\mu\partial v^{(1)} \\ \frac{2\mu}{3}\big(\frac{(v^{(1)})^2}{z^{(1)}}\partial + u^{(1)}\partial\big) & 2\mu v^{(1)}\partial & \mu(\partial z^{(1)} + z^{(1)}\partial) \end{pmatrix} + O(\mu^2), \tag{79}$$

where we put, by definition, $\vartheta^{-1} := (\psi_\vartheta' - \psi_\vartheta'^{,*})$, $u := \mu u^{(1)}, v := \mu v^{(1)}, z := \mu z^{(1)}$ as $\mu \to 0$, and whose exact form needs some additional simple but cumbersome calulations, which will be presented in a work under preparation.

The operator (79) satisfies the Hamiltonian vector field condition:

$$(u_x, v_x, z_x)^{\intercal} = -\vartheta \ \text{grad} \ H_\eta \,, \tag{80}$$

following easily from (78). Now, making use of the expressions (74) and (78), one can derive that

$$\vartheta^{-1}\eta \ \text{grad} \ H_\eta = -\vartheta^{-1}K := \varphi_\vartheta \,, \qquad \varphi'_\vartheta = \varphi'^{,*}_\vartheta . \tag{81}$$

Owing to the second equality of (81) and the classical homology relationship $\varphi_\vartheta = \text{grad}\ H_\vartheta$ for some function $H_\vartheta \in D(\mathcal{M})$, one can calculate the expression

$$H_\vartheta = \int_0^1 (\varphi_\vartheta[su, sv, sz], (u, v, z)^{\intercal})\mathrm{d}s, \tag{82}$$

satisfying the Hamiltonian condition

$$K = -\vartheta \ \text{grad} \ H_\vartheta \,. \tag{83}$$

Remark 4.9 *We mention here that the exact expression of the Hamiltonian function (82) can be easily calculated modulo the exact form of the element* $\varphi_\vartheta \in T^*(\mathcal{M})$ *and the co-implectic operator* $\vartheta^{-1} : T(\mathcal{M}) \to T^*(\mathcal{M})$, *constructed by formulae (81) and (79), respectively.*

The results obtained above can be formulated as the following proposition.

Proposition 4.10 *The Riemann type hydrodynamical system (10) at* $N = 3$ *is equivalent to a completely integrable bi-Hamiltonian flow on the functional manifold* $\mathcal{M}$, *allowing the Lax type representation (72) and the compatible pair of co-symplectic structures (77) and (79).*

4.2. The hierarchies of conservation laws and their origin analysis

The infinite hierarchy of conservation laws like (61) and related recurrent relationships can be regularly reconstructed, if we compute the asymptotical solutions to the following Lie-Lax equation:

$$\begin{aligned} L_{\tilde{K}}\tilde{\varphi} &= \tilde{\varphi}_\tau + \tilde{K}'^{,*}\tilde{\varphi} = 0, \\ \tilde{\varphi} &\simeq \tilde{a}(x;\lambda)\exp\{\lambda^2\tau + \partial^{-1}\tilde{\sigma}(x;\lambda)\}, \end{aligned} \tag{84}$$

where, by definition, $\tilde{a}(x;\lambda) \simeq \sum_{j\in\mathbb{Z}_+} \tilde{a}_j[u, v, z]\lambda^{-j}$, $\tilde{\sigma}(x;\lambda) \simeq \sum_{j\in\mathbb{Z}_+\cup\{-2\}} \tilde{\sigma}_j[u, v, z]\lambda^{-j}$ as $\lambda \to \infty$, and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}(u, v, z)^{\intercal} &:= -3\eta \ \text{grad} \ H_3^{(1/3)}[u, v, z] = \left.\begin{array}{c} -(u_x^2h^{-2})_x + v_x^{-1}(v_x^2h^{-2})_x \\ -v_xu_x^{-1}(u_x^2h^{-2})_x + z_x^{-1}(z_x^2h^{-2})_x \\ -z_xu_x^{-1}(z_x^2h^{-2})_x \end{array}\right\} := \tilde{K}[u, v, z], \\ H_3^{(1/3)} &:= \int_0^{2\pi} h[u, v, z]\mathrm{d}x, \qquad h[u, v, z] = (v_{xx}u_x - u_{xx}v_x - z_{xx})^{1/3}, \end{aligned} \tag{85}$$

is a Hamiltonian vector field on the functional manifold $\mathcal{M}$ with respect to a suitable evolution parameter $\tau \in \mathbb{R}$. Since the vector fields (85) and (58) are commuting to each other on the whole manifold $\mathcal{M}$, the functionals

$$\tilde{H}_j^{(1/3)} := \int_0^{2\pi} \tilde{\sigma}_{j-2}[u, v, z]\mathrm{d}x, \tag{86}$$

$j \in \mathbb{Z}_+$,will be functionally independent conservation laws for both these dynamical systems. Moreover, as one can check by means of quite cumbersome calculations, the conservation laws

$\tilde{H}_j^{(1/3)}, j \in \mathbb{Z}_+$, coincide up to constant coefficients with the conservation laws $H_j^{(1/3)}, j \in \mathbb{Z}_+$, given by suitable elements of (61). But here a question arises – how they are related with the Lax pair (72), strongly depending on the r-solutions (71) to the differential-functional equation (70)? (We cordially thank a Referee of the article for posing this question.) To reply to this question, it is enough to construct the corresponding hierarchy of conservation laws making use of the standard Riccati type procedure, applied to the first equation of (72). Namely, having put, by definition,

$$\partial f_3/\partial x := \sigma(x;\lambda) f_3\,, \qquad f_2 := b(x;\lambda) f_3\,, \qquad f_1 := a(x;\lambda) f_3\,, \tag{87}$$

where the following asymptotical expansions

$$\begin{aligned} \sigma(x;\lambda) &\simeq \sum_{j\geqslant -2}^{\infty} \sigma_j[u,v,z;r]\lambda^{-j}, a(x;\lambda) \simeq \sum_{j\geqslant 2}^{\infty} a_j[u,v,z;r]\lambda^{-j}, \\ b(x;\lambda) &\simeq \sum_{j\geqslant 1}^{\infty} b_j[u,v,z;r]\lambda^{-j}, \end{aligned} \tag{88}$$

hold as $|\lambda| \to \infty$ and whose coefficients satisfy the sequences of reccurent differential-functional equations

$$\begin{aligned} \partial a_j/\partial x + \sum_k a_{j-k}\sigma_k &= u_x a_{j+2} - v_x b_{j+1} + z_x \delta_{j,0}\,, \\ \partial b_j/\partial x + \sum_k b_{j-k}\sigma_k &= 3a_{j+3} - 2u_x b_{j+2} + v_x \delta_{j,-1}\,, \\ \sigma_j &= r a_{j+4} - 3b_{j+3} + u_x \delta_{j,-2}\,, \end{aligned} \tag{89}$$

for all integers $j+4 \in \mathbb{Z}_+$, we easily obtain that the initial local functionals $\sigma_{-2}[u,v,z;r], a_2[u,v,z;r]$ and $b_1[u,v,z;r]$ solve the system of equations

$$\begin{aligned} \sigma_{-2} + 3b_1 - r a_2 &= u_x\,, \\ b_1(3u_x + r a_2 - 3b_1) - 3a_2 &= v_x\,, \\ a_2(r a_2 - 3b_1) + v_x b_1 &= z_x\,, \end{aligned} \tag{90}$$

easily reducing to a one cubic equation on the local functional $\sigma_{-2}[u,v,z;r]$. Since the latter makes it possible, owing to (89), to recurrently calculate all other functionals $\sigma_j[u,v,z;r], j \geqslant 1$, we can obtain this way an infinite hierarchy of functionals

$$\gamma_j^{(1/3)} := \int_0^{2\pi} \sigma_{j-2}[u,v,z;r]\mathrm{d}x \tag{91}$$

for all $j \in \mathbb{Z}_+$, being, owing to the first equation of (87) and the second one of (72), conservation laws for a dynamical system (58). Moreover, these conservation laws at $r := (v_x - u_x^2/6)z_x^{-1}$ coincide, up to constant coefficients, with those (86) constructed above. Similar calculations can be also performed for other r-solutions of (71), but owing to their cumbersomeness, we do not present them in detail.

Remark 4.11 *Based on the Lax type representation (87) one can state on the manifold $\mathcal{M}$ by means of direct analytical calculations the well known gradient-like identity (46)*

$$\lambda^2 \vartheta\varphi(x;\lambda) = \eta\varphi(x;\lambda) \tag{92}$$

for the gradient functional $\varphi(x;\lambda) := \mathrm{grad}\lambda[u,v,z;r] \in T^*(\mathcal{M})$, where the implectic operators $\eta, \vartheta : T^*(\mathcal{M}) \to T(\mathcal{M})$ coincide at some $r \in R$ with those, given by expressions (77) and (79).

The Lax type integrability of the Riemann type hydrodynamical equation (10) at $N = 2$ and $N = 3$, stated above, allows one to speculate that this property holds for arbitrary $N \in \mathbb{Z}_+$.

Concerning the evident difference between analytical properties of the cases $N = 2$ and $N = 3$, we can easily observe that it is related with structures of the corresponding Lax type operators (52) and (72): in the first case the corresponding r-equation (70) is trivial (that is empty), but in the second case it is already nontrivial, allowing many different solutions. This situation generalizes, as we will see below, to the case $N \geqslant 4$, thereby explaining the appearing diversity of the related Lax type representations.

To support this hypothesis we will prove below that also at $N = 4$ it is equivalent to a Lax type integrable bi-Hamiltonian dynamical system on the suitable smooth 2π-periodic functional manifold $\mathcal{M} := C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^4)$, possesses infinite hierarchies of polynomial dispersionless and dispersive non-polynomial conservation laws.

5. The case N=4: conservation laws, bi-Hamiltonian structure and Lax type representation

The Riemann type hydrodynamical equation (10) at $N = 4$ is equivalent to the nonlinear dynamical system

$$\left.\begin{array}{c} u_t = v - uu_x \\ v_t = w - uv_x \\ w_t = z - uw_x \\ z_t = -uz_x \end{array}\right\} := K[u, v, w, z], \tag{93}$$

where $K : \mathcal{M} \to T(\mathcal{M})$ is a suitable vector field on the smooth 2π-periodic functional manifold $\mathcal{M} := C^\infty(R/2\pi\mathbb{Z}; \mathbb{R}^4)$. To state its Hamiltonian structure, we need to find a functional solution to the Lie-Lax equation (32):

$$\psi_t + K^{',*}\psi = \text{ grad } \mathcal{L} \tag{94}$$

for some smooth functional $\mathcal{L} \in D(\mathcal{M})$, where

$$K^{'} = \begin{pmatrix} -\partial u & 1 & 0 & 0 \\ -v_x & -u\partial & 1 & 0 \\ -w_x & 0 & -u\partial & 1 \\ -z_x & 0 & 0 & -u\partial \end{pmatrix}, \qquad K^{',*} = \begin{pmatrix} u\partial & -v_x & -w_x & -z_x \\ 1 & \partial u & 0 & 0 \\ 0 & 1 & \partial u & 0 \\ 0 & 0 & 1 & \partial u \end{pmatrix} \tag{95}$$

are, respectively, the Frechet derivative of the mapping $K : \mathcal{M} \to T(\mathcal{M})$ and its conjugate. The small parameter method [7], applied to equation (94), gives rise to the following exact solution:

$$\psi = \left(-w_x, v_x/2, 0, -\frac{v_x^2}{2z_x} + \frac{u_x w_x}{z_x}\right)^{\intercal}, \qquad \mathcal{L} = \int_0^{2\pi} (zu_x - vw_x/2)\mathrm{d}x. \tag{96}$$

As a result, right away from (94) we obtain that dynamical system (93) is a Hamiltonian system on the functional manifold $\mathcal{M}$, that is

$$K = -\vartheta \text{ grad } H, \tag{97}$$

where the Hamiltonian functional is equal to

$$H := (\psi, K) - \mathcal{L} = \int_0^{2\pi} (uz_x - vw_x)\mathrm{d}x \tag{98}$$

and the co-implectic operator is equal to

$$\vartheta^{-1} := \psi^{'} - \psi^{',*} = \begin{pmatrix} 0 & 0 & -\partial & \partial\frac{w_x}{z_x} \\ 0 & \partial & 0 & -\partial\frac{v_x}{z_x} \\ -\partial & 0 & 0 & \partial\frac{u_x}{z_x} \\ \frac{w_x}{z_x}\partial & -\frac{v_x}{z_x}\partial & \frac{u_x}{z_x}\partial & \begin{array}{c}\frac{1}{2}[z_x^{-2}(v_x^2 - 2u_xw_x)\partial + \\ +\partial(v_x^2 - 2u_xw_x)z_x^{-2}]\end{array} \end{pmatrix}. \tag{99}$$

The latter is degenerate: the relationship $\vartheta^{-1}(u_x, v_x, w_x, z_x)^{\intercal} = 0$ exactly on the whole manifold $\mathcal{M}$, but the inverse to (99) exists and can be calculated analytically.

In order to state the Lax type integrability of Hamiltonian system (93), we will apply the standard gradient-holonomic scheme of [7, 8] to it and find its following linearization:

$$D_t^4 f_4(\lambda) = 0, \tag{100}$$

where $f_4(\lambda) \in C^{\infty}(\mathbb{R}^2; \mathbb{C})$ for all $\lambda \in \mathbb{C}$. Having rewritten (100) in the form of a linear system

$$D_t f = q(\lambda) f, \qquad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix}, \tag{101}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter and the vector $f \in C^{\infty}(\mathbb{R}^2; \mathbb{C}^4)$ allows, owing to the relationship (100), the following functional representation:

$$\begin{aligned}
f_1(x,t) &= \tilde{g}_1\left(u - tv + \frac{wt^2}{2} - \frac{xt^3}{3!}, v - wt + \frac{zt^2}{2}, w - zt, x - tu + \frac{vt^2}{2} - \frac{wt^3}{3!} + \frac{zt^4}{4!}\right), \\
f_2(x,t) &= t\mu_1\tilde{g}_1\left(u - tv + \frac{wt^2}{2} - \frac{xt^3}{3!}, v - wt + \frac{zt^2}{2}, w - zt, x - tu + \frac{vt^2}{2} - \frac{wt^3}{3!} + \frac{zt^4}{4!}\right) \\
&\quad + \tilde{g}_2\left(u - tv + \frac{wt^2}{2} - \frac{xt^3}{3!}, v - wt + \frac{zt^2}{2}, w - zt, x - tu + \frac{vt^2}{2} - \frac{wt^3}{3!} + \frac{zt^4}{4!}\right), \\
f_3(x,t) &= \mu_1\mu_2\frac{t^2}{2}\tilde{g}_1\left(u - tv + \frac{wt^2}{2} - \frac{xt^3}{3!}, v - wt + \frac{zt^2}{2}, w - zt, x - tu + \frac{vt^2}{2} - \frac{wt^3}{3!} + \frac{zt^4}{4!}\right) \\
&\quad + t\mu_2\tilde{g}_2\left(u - tv + \frac{wt^2}{2} - \frac{xt^3}{3!}, v - wt + \frac{zt^2}{2}, w - zt, x - tu + \frac{vt^2}{2} - \frac{wt^3}{3!} + \frac{zt^4}{4!}\right) \\
&\quad + \tilde{g}_3\left(u - tv + \frac{wt^2}{2} - \frac{xt^3}{3!}, v - wt + \frac{zt^2}{2}, w - zt, x - tu + \frac{vt^2}{2} - \frac{wt^3}{3!} + \frac{zt^4}{4!}\right), \\
f_4(x,t) &= \mu_1\mu_2\mu_3\frac{t^3}{3!}\tilde{g}_1\left(u - tv + \frac{wt^2}{2} - \frac{xt^3}{3!}, v - wt + \frac{zt^2}{2}, w - zt, x - tu + \frac{vt^2}{2} - \frac{wt^3}{3!} + \frac{zt^4}{4!}\right) \\
&\quad + \mu_2\mu_3\frac{t^2}{2}\tilde{g}_2\left(u - tv + \frac{wt^2}{2} - \frac{xt^3}{3!}, v - wt + \frac{zt^2}{2}, w - zt, x - tu + \frac{vt^2}{2} - \frac{wt^3}{3!} + \frac{zt^4}{4!}\right) \\
&\quad + t\mu_3\tilde{g}_3\left(u - tv + \frac{wt^2}{2} - \frac{xt^3}{3!}, v - wt + \frac{zt^2}{2}, w - zt, x - tu + \frac{vt^2}{2} - \frac{wt^3}{3!} + \frac{zt^4}{4!}\right) \\
&\quad + \tilde{g}_4\left(u - tv + \frac{wt^2}{2} - \frac{xt^3}{3!}, v - wt + \frac{zt^2}{2}, w - zt, x - tu + \frac{vt^2}{2} - \frac{wt^3}{3!} + \frac{zt^4}{4!}\right),
\end{aligned} \tag{102}$$

where $\tilde{g}_j \in C^{\infty}(\mathbb{R}^4; \mathbb{C})$, $j = \overline{1,4}$, are arbitrary smooth complex valued functions.

Based on the expressions (101) and (102), one can construct the related linear representation of the expression $\partial f/\partial x \in C^{\infty}(\mathbb{R}^4; \mathbb{C}^4)$ in the following matrix form:

$$f_x = \ell[u, v, w, z; \lambda] f, \tag{103}$$

where

$$\ell[u, v, w, z; \lambda] := \begin{pmatrix} -\lambda^3 u_x & \lambda^2 v_x & -\lambda w_x & z_x \\ -4\lambda^2 & 3\lambda^3 u_x & -2\lambda^2 v_x & \lambda w_x \\ -10\lambda^5 r_1 & 6\lambda^4 & -3\lambda^3 u_x & \lambda^2 v_x \\ -20\lambda^6 r_2 & 10\lambda^5 r_1 & -4\lambda^4 & \lambda^3 u_x \end{pmatrix} \tag{104}$$

$f \in C^{\infty}(\mathbb{R}^2; \mathbb{C}^4)$ and which is compatible with (101). Thus, we can formulate the following theorem about the Lax integrability of the generalized Riemann type hydrodynamical system (93).

Theorem 5.1 *The dynamical system (10) at $N = 4$, equivalent to the generalized Riemann type hydrodynamical system (93), possesses the Lax type representation*

$$f_x = \ell[u,v,z,w;\lambda]f, \qquad f_t = p(\ell)f, \qquad p(\ell) := -u\ell[u,v,w,z;\lambda] + q(\lambda), \tag{105}$$

where

$$\ell[u,v,w,z;\lambda] := \begin{pmatrix} -\lambda^3 u_x & \lambda^2 v_x & -\lambda w_x & z_x \\ -4\lambda^4 & 3\lambda^3 u_x & -2\lambda^2 v_x & \lambda w_x \\ -10\lambda^5 r_1 & 6\lambda^4 & -3\lambda^3 u_x & \lambda^2 v_x \\ -20\lambda^6 r_2 & 10\lambda^5 r_1 & -4\lambda^4 & \lambda^3 u_x \end{pmatrix}, \qquad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix},$$

$$p(\ell) = \begin{pmatrix} \lambda u u_x & -\lambda^2 u v_x & \lambda u w_x & -u z_x \\ \lambda + 4\lambda^4 u & -3\lambda^3 u u_x & 2\lambda^2 u v_x & -\lambda u w_x \\ 10\lambda^5 u r_1 & \lambda - 6\lambda^4 u & 3\lambda^3 u u_x & -\lambda^2 u v_x \\ 20\lambda^6 u r_2 & -10\lambda^5 u r_1 & \lambda + 4\lambda^4 u & -\lambda^3 u u_x \end{pmatrix}, \tag{106}$$

thus being a Lax type integrable dynamical system on the functional manifold $\mathcal{M}$.

Owing to the existence of the Lax type representation (105) and the related gradient such as relationship [7, 8], we can easily derive that the Hamiltonian system (93) is at the same time a bi-Hamiltonian flow on the functional manifold $\mathcal{M}$. In addition, making use of the above results and the approach of work [11], we can construct the infinite hierarchies of conservation laws for (93), both dispersionless polynomial and dispersive non-polynomial ones:

a) polynomial conservation laws:

$$\begin{aligned}
H_9 &= \int_0^{2\pi} \mathrm{d}x(vw_x - uz_x), \qquad H^{(13)} = \int_0^{2\pi} \mathrm{d}x(z_x w - z w_x), \\
H_{14} &= \int_0^{2\pi} \mathrm{d}x\Big(k_1\Big(z_x(v^2 - 2uw) - z^2\Big) + k_2\Big(- z_x v^2 + 2w_x(vw - uz) - z^2\Big) \\
&\quad + k_3\Big(2z_x v^2 + 4w_x(4z - vw) + 2z^2\Big)\Big), \\
H_{16} &= \int_0^{2\pi} \mathrm{d}x(3uz - vw)z_x, \qquad H^{18} = \int_0^{2\pi} \mathrm{d}x z_x(w^2 - 2vz), \\
H_{17} &= \int_0^{2\pi} \mathrm{d}x[12v_x uzw + z_x(9u^2 z + 16uvw - 2v^3) + 6ww_x(v^2 - 2uw) + 6z(2vz - w^2)], \\
H_{19} &= \int_0^{2\pi} \mathrm{d}x\Big[k_1\Big(10v_x uz^2 + z_x(12uvz - uw^2 - 2v^2 w) + 5ww_x(vw - 2uz)\Big) \\
&\quad + k_2\Big(z_x(6uvz - 3uw^2 - v^2 w) + 5w_x v(w^2 - vz)\Big)\Big];
\end{aligned} \tag{107}$$

b) non-polynomial conservation laws:

$$\begin{aligned}
H_{11} &= \int_0^{2\pi} \mathrm{d}x\Big(u_{xx}z_x - u_x z_{xx} + v_x w_{xx} - v_{xx}w_{xx}\Big)^{\frac{1}{3}}, \qquad H_{12}^{(1/2)} = \int_0^{2\pi} \mathrm{d}x\sqrt{w_x^2 - 2v_x z_x}, \\
H_{12}^{(1/3)} &= \int_0^{2\pi} \mathrm{d}x\Big(9u_x^2 z_x - 6u_x v_x w_x + 2v_x^3 - 12v_x z_x + 6w_x^2\Big)^{\frac{1}{3}},
\end{aligned}$$

$$
\begin{aligned}
H_{13}^{(1/3)} &= \int_0^{2\pi} \mathrm{d}x \Big(u(2v_x z_x - w_x^2) + v(v_x w_x - 3u_x z_x) + w(u_x w_x - v_x^2 + 2z_x) + z(u_x v_x - 2w_x) \Big)^{\frac{1}{3}}, \\
H_{15}^{(1/3)} &= \int_0^{2\pi} \mathrm{d}x \Big(z_x w_{xx} - z_{xx} w_x \Big)^{\frac{1}{3}}, \\
H_{15}^{(1/2)} &= \int_0^{2\pi} \mathrm{d}x \Big(k_1 \big(v(2v_x z_x - w_x^2) + z(4z_x - u_x w_x) + w(v_x w_x - 3u_x z_x) \big) \\
&\quad + k_2 z \big(2z_x + v_x^2 - u_x w_x \big) \Big)^{\frac{1}{2}}, \\
H_{16}^{(1/5)} &= \int_0^{2\pi} \mathrm{d}x \Big(u_{xxx}(2v_x z_x - w_x^2) + v_{xxx}(v_x w_w - 3u_x z_x) \\
&\quad + z_{xxx}(u_x v_x - 2w_x) + w_{xxx}(u_x w_x - v_x^2 + 2z_x) \\
&\quad + 3u_{xx}(v_{xx} z_x - 3v_x z_{xx} + w_{xx} w_x) + 3v_{xx}(2u_x z_{xx} \\
&\quad - v_{xx} w_x + v_x w_{xx}) - 3w_{xx}^2 u_x \Big)^{\frac{1}{5}}, \\
5H_{16}^{(1/4)} &= \int_0^{2\pi} \mathrm{d}x \Big(4u_x^2 w_x^2 - 4u_x v_x^2 w_x - 8u_x z_x w_x + v_x^4 - 4v_x^2 z_x + 4z_x^2 \Big)^{\frac{1}{4}}, \\
H_{16}^{(1/3)} &= \int_0^{2\pi} \mathrm{d}x \Big(k_1 \big(u(z_x w_{xx} - z_{xx} w_x) + v(v_x z_{xx} - v_{xx} z_x) + z z_{xx} + w(u_{xx} z_x - u_x z_{xx}) \big) \\
&\quad + k_2 \Big(z(u_{xx} w_x - u_x w_{xx} + 2z_{xx}) + w(u_{xx} z_x - u_x z_{xx} - v_{xx} w_x + v_x w_{xx}) \Big) \\
&\quad + k_3 z_x (v_x^2 - 2u_x w_x + 2z_x) \Big)^{\frac{1}{3}}, \qquad (108)
\end{aligned}
$$

where $k_j, j = \overline{1,3}$, are arbitrary constants. We also observe that the Hamiltonian functional (98) coincides exactly up to the sign with the polynomial conservation law $H_{(9)} \in D(\mathcal{M})$.

As concerns the general case $N \in \mathbb{Z}_+$, successively applying the above devised method, one can also obtain for the Riemann type hydrodynamical system (10) the corresponding Lax type representation, construct infinite hierarchies of dispersive and dispersionless conservation laws, their symplectic structures and the related Lax type representations, which is a topic of the next work under preparation.

6. Conclusion

As follows from the results obtained in this work, the generalized Riemann type hydrodynamical system (10) at $N = 2, 3$ and $N = 4$ possesses many infinite hierarchies of conservation laws, both dispersive non-polynomial and dispersionless polynomial ones. This fact can be easily explained by the fact that the corresponding dynamical systems (16), (58) and (93) possess many, plausibly, infinite sets of algebraically independent compatible implectic structures, which generate via the corresponding gradient like relationships [7, 8] the related infinite hierarchies of conservation laws, and as a by-product, infinite hierarchies of the associated Lax type representations. The existence of many Lax type representations for the generalized Riemann type equation 10 for $N \in \mathbb{Z}_+$ was recently justified by means of differential-algebraic tools in [12].

It is also worth to mention that the generalized Riemann type equation (10) is an example of integrable dynamical systems belonging [14, 15] at the same time to two different classes: C- and

S- integrable. Really, these systems are linearizable and have exact general solutions though in an unwieldy form. Thus, the Riemann type systems belong to the C-integrable class. Similar properties had been analyzed earlier for [17–20] for the case of the *Monge-Ampere* equations. Moreover, these systems have also infinite sets of compatible Hamiltonian structures, Lax type representations and respectively commuting flows. So, they belong to the S- integrable class too. Such a situation within the theory of Lax type integrable nonlinear dynamical systems is encountered, virtually, for the first time and is interesting from different points of view, both theoretically and practically. Keeping in mind these and some other important aspects of the Riemann type hydrodynamical systems (10), we consider that they deserve additional thorough investigation in the future.

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Узагальнене гідродинамічне рівняння Гуревича-Зибіна типу Рімана і його інтегровність типу Лакса

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Стаття присвячена дослідженню гідродинамічного рівняння типу Рімана, що узагальнює відому систему Гуревича-Зибіна. Це багатокомпонентне гідродинамічне рівняння характеризується єдиною характеристичною швидкістю потоку. Проаналізовані сумісні бі-гамільтонові структури та представлення Лакса для 3-та 4-компонентної узагальненої гідродинамічної системи типу Рімана. Отримані результати вперше доповнюють теорію інтегровності систем гідродинамічного типу, раніше розвинутої тільки для відмінних швидкостей в однорідному випадку.

Ключові слова: *гідродинамічні рівняння типу Рімана, інтегровність типу Лакса, закони збереження*