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HAMILTONIAN DYNAMICS OF CONDENSED MEDIA WITH SPONTANEOUSLY BROKEN PHASE INVARIANCE

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The Hamiltonian formalism to the dynamics of the condensed media with spontaneously broken phase invariance has been built. The method for finding of Poisson brackets for the dynamic variables has been formulated. It is based on the consideration of the transformations that leave invariant the kinematic part of the action and on the interpretation of the outside integral terms as the generators of these transformations. Important role in the developed approach plays the construction of the translation operators and their densities for the different physical fields, figuring in the action. The order parameter phases that correspond to the broken symmetry are defined in terms of the quantities canonically conjugate to the additive motion integral densities. In capacity of specific objects the quantum crystal, the superfluid phase of ^{3}He–B and the quantum spin crystal are considered.

The determination of the Poisson brackets (PB) for dynamic variables plays a principal role in the Hamiltonian approach to the theory of various physical systems. In [1] PB were derived on the basis of consideration of the variables' variations at transformations of the group corresponding to the broken symmetry of the system. In the suggesting approach it is shown how the PB for different physical systems with broken phase invariance can be obtained proceeding from the transformations that leave invariant the kinematic part of the action (or Lagrangian). The outside integral terms in the action variation are interpreted as the generators of these transformations. Because the setting of the kinematic part of the Lagrangian plays a principal role in our approach we stop in detail on the way of construction of the kinematic part. An important part of this procedure is the construction of the translation operators and their densities for different physical fields figuring in the action. As an examples of the specific physical objects the dynamics of the quantum crystal, superfluid liquids and the quantum spin crystal are studied.

1. Fundamentals of the formalism

Let us represent the system Lagrangian in the form

$$L = L_k(\varphi, \dot{\varphi}) - H(\varphi) \equiv \int \mathrm{d}x\, F_\alpha(x;\varphi)\dot{\varphi}_\alpha(x) - H(\varphi), \qquad (1.1)$$

where $L_k(\varphi, \dot{\varphi})$ is the kinematic part of the Lagrangian, $H(\varphi)$ is the Hamiltonian, $F_\alpha(x; \varphi(x'))$ is an arbitrary functional of the dynamic variables $\varphi_\alpha(x)$. We consider the infinitesimal transformations of the field variables $\varphi_\alpha(x)$:

$$\varphi_\alpha(x,t) \longrightarrow \varphi'_\alpha(x,t) = \varphi_\alpha(x,t) + \delta\varphi_\alpha(x,t), \tag{1.2}$$

(below we do not write an argument t in dynamic variables and their variations). Variation of the action $W = \int_{t_1}^{t_2} L\mathrm{d}t$ under the transformations (1.2) is equal

$$\delta W = G(t_2) - G(t_1) + \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}x'\ \delta\varphi_\beta(x') \cdot$$
$$\cdot \left(\int \mathrm{d}x\ J_{\beta\alpha}(x', x; \varphi)\dot{\varphi}_\alpha(x) - \frac{\delta H}{\delta\varphi_\beta(x')} \right), \tag{1.3}$$

where

$$G(\varphi) = \int \mathrm{d}x\ F_\alpha(x,\varphi)\delta\varphi_\alpha(x);\ J_{\alpha\beta}(x,x';\varphi) = \frac{\delta F_\beta(x';\varphi)}{\delta\varphi_\alpha(x)} - \frac{\delta F_\alpha(x;\varphi)}{\delta\varphi_\beta(x')}.$$

The equations of motion for the field components $\varphi_\alpha(x)$ taking into account the principle of the stationarial action have the form

$$\dot{\varphi}_\alpha(x) = \int \mathrm{d}x'\ J^{-1}_{\alpha\beta}(x,x';\varphi)\frac{\delta H}{\delta\varphi_\beta(x')}. \tag{1.4}$$

We define PB for the arbitrary functionals A and B of the dynamic variables φ_α by the equality

$$\{A, B\} = \int \mathrm{d}x\mathrm{d}x'\ \frac{\delta A}{\delta\varphi_\alpha(x)} J^{-1}_{\alpha\beta}(x,x';\varphi)\frac{\delta B}{\delta\varphi_\beta(x')}. \tag{1.5}$$

Then the equations of motion (1.4) have the form

$$\dot{\varphi}_\alpha(x) = \{\varphi_\alpha(x), H\}. \tag{1.6}$$

Let us consider the infinitesimal transformations (1.2) that leave invariant the kinematic part of Lagrangian. Such transformations satisfy the relationship

$$\int \mathrm{d}x'\ J_{\alpha\beta}(x,x';\varphi)\delta\varphi_\beta(x') = \frac{\delta G}{\delta\varphi_\alpha(x)}, \tag{1.7}$$

which, taking into account (1.5), can be written in the form $\delta\varphi_\alpha(x) = \{\varphi_\alpha(x), G\}$, where G is the generator of the infinitesimal transformations, coinsiding with the introduced in (1.3).

In the frame of the Hamiltonian approach it is easy to formulate the differential conservation laws connected with the different symmetry properties of Hamiltonian. Equation of motion for density $a(x)$ of arbitrary physical quantity $A = \int \mathrm{d}^3x\ a(x)$ can be represented, accordingly to [2], in the form

$$\dot{a}(x) = \{a(x), H\} \equiv \{A, \varepsilon(x)\} - \frac{\partial a_k(x)}{\partial x_k}, \tag{1.8}$$

where

$$a_k(x) = \int \mathrm{d}^3x'\ x'_k \int_0^1 \mathrm{d}\lambda\ \{a(x+\lambda x'), \varepsilon(x-(1-\lambda)x')\}.$$

If quantity A is the generator G of the symmetry group of Hamiltonian then equation (1.8) has the form of the differential conservation law.

2. PB for variables of the classical continuum

At first we consider the way of the construction of the kinematic part of Lagrangian for the classical continuum and the derivation on its basis of the PB of the continuum variables. It is useful for illustration how the different translation operators of physical fields are introduced in the theory. We start from the kinematic part [2]

$$L_k(x) = \pi_i(x) b_{ij}^{-1}(x) \dot{u}_j(x), \quad b_{ij} \equiv \delta_{ij} - \nabla_j u_i, \tag{2.1}$$

where $\pi_i(x)$ is the density of momentum, $u_i(x)$ is the displacement vector. With the use of the kinematic part (2.1) PB for variables $u_i(x), \pi_i(x)$ may be obtained. For the derivation of the adiabaticity equation it is necessary to know also the PB of the entropy density $\sigma(x)$ with other variables. For finding these PB we write the Lagrangian kinematic part in the form

$$\mathcal{L}_k(x) = p_j(x)\dot{u}_j(x) - \sigma(x)\dot{\psi}(x), \tag{2.2}$$

where

$$p_j(x) = (\pi_i(x) - \sigma(x)\nabla_i\psi(x)) b_{ij}^{-1}(x).$$

Variable $\psi(x)$ which is conjugated to variable $\sigma(x)$ has been introduced in the kinematic part formally and at the obtaining of the motion equation will be considered as cyclic. Let us explain the structure of the quantity p_j. Note that the density of momentum $\pi_i(x)$ in (2.1) is connected with the translations in the space of variables $u_i(x), \pi_i(x)$. After the new variables $\sigma(x), \psi(x)$ have been introduced the density of momentum $\pi_i(x)$ is connected already with the translations in the total space of variables $u_i(x), \pi_i(x), \sigma(x), \psi(x)$ and can be represented in the form

$$\pi_i(x) = \tilde{\pi}_i(x) + \pi_i^\sigma(x),$$

where $\tilde{\pi}_i(x)$ is the momentum density connected with the translations only in the space of variables $u_i(x)$ and $\pi_i(x)$, and $\pi_i^\sigma(x)$ is connected with translations in the space of $\sigma(x), \psi(x)$. Because the quantities $\psi(x)$ and $-\sigma(x)$ are the generalized coordinates and momenta then it follows $\pi_i^\sigma(x) = \sigma(x)\nabla_i\psi(x)$. Hence, the momentum density $\tilde{\pi}_i(x)$ connected with the translations in the space of variables $u_i(x), \pi_i(x)$ has the form

$$\tilde{\pi}_i(x) = \pi_i(x) - \sigma(x)\nabla_i\psi(x)$$

and we obtain the structure of the quantity p_j in (2.2). It is easy to see that variations

$$\delta p_i(x) = \delta\sigma(x) = 0, \ \delta u_i(x) = f_i(x), \ \delta\psi(x) = \chi(x), \tag{2.3}$$

where functions $f_i(x), \chi(x)$ do not depend on $u_i(x), p_i(x), \sigma(x), \psi(x)$, leave invariant the kinematic part $L_k = \int \mathrm{d}x \, \mathcal{L}_k(x)$ and may be represented as

$$\delta\phi(x) = \{\phi(x), G\}, \ \phi(x) = (p_i(x), u_i(x), \sigma(x), \psi(x)). \tag{2.4}$$

Here G is the generator of the transformations (2.3) which according to (1.3) equals

$$G = \int \mathrm{d}x \, \Big(p_i(x) f_i(x) - \sigma(x)\chi(x)\Big). \tag{2.5}$$

From (2.4), (2.5) it follows that

$$\{u_i(x), p_j(x')\} = \delta_{ij}\delta(x-x'),\ \{\sigma(x), \psi(x')\} = \delta(x-x'). \qquad (2.6)$$

Let us find now the PB for variables $u_i(x), p_i(x), \sigma(x)$ and $\psi(x)$. It is easy to convince that from formulae

$$\{\tilde{\pi}_i(x), \psi(x')\} = \{\tilde{\pi}_i(x), \sigma(x')\} = 0,$$

where

$$\tilde{\pi}_i(x) \equiv \pi_i(x) - \sigma(x)\nabla_i\psi(x) = p_j(x)b_{ji}(x),$$

follow the brackets

$$\{\pi_i(x), \sigma(x')\} = -\sigma(x)\nabla_i\delta(x-x'),\ \{\pi_i(x), \psi(x')\} = \nabla_i\psi(x)\delta(x-x').$$

With the taking into account of (2.6) for the quantities $\pi_i^\sigma(x) \equiv \sigma(x)\nabla_i\psi(x)$ we have

$$\{\pi_i^\sigma(x), \pi_k^\sigma(x')\} = \pi_k^\sigma(x)\nabla_i'\delta(x-x') - \pi_i^\sigma(x')\nabla_k\delta(x-x').$$

From this formula and from the bracket $\{p_i(x), p_k(x')\} = 0$ we obtain

$$\{\pi_i(x), \pi_k(x')\} = \pi_k(x)\nabla_i'\delta(x-x') - \pi_i(x')\nabla_k\delta(x-x'). \qquad (2.7)$$

The first formula in (2.6) leads to the equality $\{u_i(x), \pi_k(x')\} = b_{ik}(x)\delta(x-x')$. Thus nontrivial PB for variables of continuum have the form

$$\{\sigma(x), \psi(x')\} = \delta(x-x'), \{\pi_i(x), \sigma(x')\} = -\sigma(x)\nabla_i\delta(x-x'),$$

$$\{\pi_i(x), \psi(x')\} = \nabla_i\psi(x)\delta(x-x'),\ \{u_i(x), \pi_k(x')\} = b_{ik}(x)\delta(x-x')\,,$$

$$\{\pi_i(x), \pi_k(x')\} = \pi_k(x)\nabla_i'\delta(x-x') - \pi_i(x')\nabla_k\delta(x-x'). \qquad (2.8)$$

Dynamic equations of continuum [2] can be obtained with the use of (2.8) if we take into account the general structure of the system Hamiltonian

$$H = \int \mathrm{d}^3x\ \varepsilon(x), \qquad \varepsilon(x) = \varepsilon\Big(x; \sigma(x'), \pi_i(x'), b_{ij}(x')\Big)$$

Here $\varepsilon(x)$ is the energy density which is, in general case, some functional of variables $\sigma(x), \pi_i(x), b_{ij}(x)$ (variable ψ is cyclic). Note that due to the property of invariance of $\varepsilon(x)$ with respect to the translations of the Euler variables the density of energy $\varepsilon(x)$ does not depend on the quantities $u_i(x)$, but only on their derivatives $\dfrac{\partial u_i}{\partial x_k}$ or, it is the same, on the quantities $b_{ik}(x)$.

3. Quantum crystal

Now we shall study the superfluid systems for which the ground state is characterized by the spontaneously broken symmetry with respect to the phase transformations. At first we consider the quantum crystal. Phenomena of it superfluidity [3] is connected with the possibility of two kinds of motion. The first one is the motion of lattice points as in solids, the second

one is the transfer of mass by the quasiparticles under the condition of the fixed lattice points as in the superfluid liquid.

Variables which describe the violation of phase and translational invariances of the quantum crystal ground state are the superfluid phase $\phi(x)$ and the displacement vector $u_i(x)$ in the configurational space. Therefore the energy density $\varepsilon(x)$ is, in general case, the functional of the entropy $\sigma(x)$, mass $\varrho(x)$, and momentum $\pi_i(x)$ densities and of the superfluid phase $\phi(x)$ and displacement vector $u_i(x)$:

$$\varepsilon(x) = \varepsilon\Big(x; \sigma(x'), \varrho(x'), \pi_i(x'), \phi(x'), u_i(x')\Big).$$

Now, let us obtain the PB of the dynamic variables. We write the density of the Lagrangian kinematic part in the form

$$\mathcal{L}_k(x) = \tilde{\pi}_i(x) b_{ij}^{-1}(x) \dot{u}_j(x) - \sigma(x)\dot{\psi}(x) - \varrho(x)\dot{\phi}(x), \tag{3.1}$$

where

$$\tilde{\pi}_i = \pi_i - \sigma\nabla_i\psi - \varrho\nabla_i\phi.$$

Expression (3.1) is built analogously to the case of the classical continuum (see the explanation of the introduction of term $\sigma\dot{\psi}$). Namely, the last term in (3.1) represents itself the account of superfluidity; correspondingly, in the density momentum $\tilde{\pi}_i$, which is connected with translations in the space of variables u_i, π_i, it is necessary to subtract additional term $\varrho\nabla_i\phi$. For derivation of PB, as in the case of classical continuum, the variations (2.3) with momentum $p_j = \tilde{\pi}_i b_{ij}^{-1}$ as well as variations $\delta\varrho(x) = 0, \delta\phi(x) = g(x)$ with generator

$$G = \int \mathrm{d}^3x \, \{p_i(x) f_i(x) - \sigma(x)\chi(x) - \varrho(x) g(x)\}$$

should be cousidered. Making further computations analogously to Section 2 we come to the following algebra of the PB for variables of quantum crystal

$$\{\pi_i(x), u_k(x')\} = -(\delta_{ik} - \nabla_i u_k(x))\delta(x - x'),$$

$$\{\pi_i(x), \sigma(x')\} = -\sigma(x)\nabla_i\delta(x - x'), \; \{\pi_i(x), \varrho(x')\} = -\varrho(x)\nabla_i\delta(x - x'),$$

$$\{\pi_i(x), \phi(x')\} = \delta(x - x')\nabla_i\phi(x), \; \{\sigma(x), \psi(x')\} = \{\varrho(x), \phi(x')\} = \delta(x - x'),$$

$$\{\pi_i(x), \pi_k(x')\} = \pi_k(x)\nabla_i'\delta(x - x') - \pi_i(x')\nabla_k\delta(x - x'),$$

$$\{\pi_i(x), \psi(x')\} = \nabla_i\psi(x)\delta(x - x'). \tag{3.2}$$

Further variable ψ will be considered as cyclic. Note that due to the invariance of $\varepsilon(x)$ with respect to the phase transformations and space translations the density $\varepsilon(x)$ depends not on quantities $\phi(x), u_i(x)$ themselves, but only on their derivatives $\nabla_i\phi \equiv p_i, \nabla_k u_i$ (or, in the last case, on the quantities $b_{ik} = \delta_{ik} - \nabla_k u_i$): $\varepsilon(x) = \varepsilon\Big(x; \sigma(x'), \varrho(x'), \pi_i(x'), p_i(x'), b_{ik}(x')\Big)$. Vector $\vec{p}$ has the sense of the superfluid momentum. Considering that energy density is the function of the dynamic variables at point x we write now the dynamic equations in the local form. We use for this the results of the first section. Putting in formula (1.8) consistently $a(x) = \Big(\varrho(x), \varepsilon(x), \pi_i(x)\Big)$,

and supposing the invariance of the energy density $\varepsilon(x)$ with respect to the phase and translation transformations, we obtain

$$j_k = \varrho\frac{\partial\varepsilon}{\partial\pi_k} + \frac{\partial\varepsilon}{\partial p_k}, \qquad t_{ik} = p\delta_{ik} + \pi_i\frac{\partial\varepsilon}{\partial\pi_k} + p_i\frac{\partial\varepsilon}{\partial p_k} + b_{ji}\frac{\partial\varepsilon}{\partial b_{jk}},$$

$$q_k = \frac{\partial\varepsilon}{\partial\pi_k}\left(\sigma\frac{\partial\varepsilon}{\partial\sigma} + \varrho\frac{\partial\varepsilon}{\partial\varrho} + \pi_l\frac{\partial\varepsilon}{\partial\pi_l}\right) + \frac{\partial\varepsilon}{\partial p_k}\left(\frac{\partial\varepsilon}{\partial\varrho} + p_i\frac{\partial\varepsilon}{\partial\pi_i}\right) + b_{ij}\frac{\partial\varepsilon}{\partial b_{ik}}\frac{\partial\varepsilon}{\partial\pi_j}, \quad (3.3)$$

$$p \equiv -\varepsilon + \sigma\frac{\partial\varepsilon}{\partial\sigma} + \pi_l\frac{\partial\varepsilon}{\partial\pi_l} + \varrho\frac{\partial\varepsilon}{\partial\varrho}.$$

Here p is the pressure and j_k, q_k, t_{ik} are the fluxes densities of the mass, energy and momentum, respectively. These equations coincide with the obtained earlier [5] on the basis of microscopic approach. Note that formulae (3.3) for the densities fluxes can be written in the compact form by introducing the density of the thermodynamic potential

$$\omega = Y_a\zeta_a - \sigma, \quad (3.4)$$

where Y_a are the thermodynamic forces defined by the equalities

$$\frac{\partial\varepsilon}{\partial\sigma} \equiv \frac{1}{Y_0}, \quad \frac{\partial\varepsilon}{\partial\pi_i} \equiv -\frac{Y_i}{Y_0}, \quad \frac{\partial\varepsilon}{\partial\varrho} \equiv -\frac{Y_4}{Y_0}. \quad (3.5)$$

From formulae (3.4),(3.5) follows that $\dfrac{\partial\omega}{\partial Y_a} = \zeta_a$ and, since $\omega = \omega(Y_a, p_i, b_{ik})$, the second principle of thermodynamics in terms of ω has the form

$$d\omega = \zeta_a dY_a + \frac{\partial\omega}{\partial p_i}dp_i + \frac{\partial\omega}{\partial b_{ik}}db_{ik}. \quad (3.6)$$

With the account of equations (3.3), (3.6) we present the fluxes densities in the form

$$\zeta_{ak} = -\frac{\partial}{\partial Y_a}\frac{\omega Y_k}{Y_0} + \frac{\partial\omega}{\partial p_k}\frac{\partial}{\partial Y_a}\frac{Y_4 + Y_i p_i}{Y_0} + \frac{\partial\omega}{\partial b_{jk}}\frac{\partial}{\partial Y_a}\frac{b_{jl}Y_l}{Y_0}. \quad (3.7)$$

Here $\zeta_{0k} = q_k, \zeta_{ik} = t_{ik}, \zeta_{4k} = j_k$. Dynamics of the additive motion integral densities and of the parameters, describing the broken symmetry, is defined according to (3.7) by equations

$$\dot\zeta_a = -\nabla_k\zeta_{ak}, \quad \dot p_i = \nabla_i\left(\frac{Y_4 + Y_j p_j}{Y_0}\right), \quad \dot b_{ik} = \nabla_k\left(\frac{b_{ij}Y_j}{Y_0}\right). \quad (3.8)$$

4. Superfluid ^{4}He

On the ground state of the superfluid ^{4}He the phase invariance is broken. The energy density $\varepsilon(x)$ is the functional of the entropy $\sigma(x)$, mass $\varrho(x)$ and momentum $\pi_i(x)$ densities as well as of the superfluid phase $\phi(x)$ playing the role of the order parameter: $\varepsilon(x) = \varepsilon\Big(x; \sigma(x'), \varrho(x'), \pi_i(x'), \phi(x')\Big)$. Algebra of the superfluid ^{4}He dynamic variables forms the subalgebra of the quantum crystal algebra (3.2). Hence, fluxes densities and equations of motion have the form (3.3),(3.8) where it is necessary to put down the terms containing quantity b_{ik}. This fact corresponds to the one that variable u_i is cyclic.

5. Quantum spin crystal

At consideration of the quantum crystal dynamics we did not took into account the influence of the spin degrees of freedom that can be essential for quantum solid ^{3}He. Taking into account, that in the superfluid liquid phases of ^{3}He the symmetry with respect to the uniform spin rotations is broken, we consider the case of the total violation of the spin invariance. The parameter describing such breaking is the real rotation matrix that corresponds to the B–phase of ^{3}He. Dynamical variables of the quantum spin crystal with the broken symmetry with respect to the spin rotations are the entropy $\sigma(x)$, mass $\varrho(x)$ and spin $s_\alpha(x)$ densities as well as the orthogonal matrix of rotation $a_{\alpha\beta}(x)$, superfluid phase $\phi(x)$ and the displacement vector $u_i(x)$. Therefore the energy density $\varepsilon(x)$ is, in general case, presented as a functional

$$\varepsilon(x) = \varepsilon\Big(x; \sigma(x'), \varrho(x'), \pi_i(x'), s_\alpha(x'), a_{\alpha\beta}(x'), \phi(x'), u_i(x')\Big).$$

The density of the Lagrangian kinematic part for the quantum spin crystal is written in the form

$$\mathcal{L}_k(x) = \tilde{\pi}_i(x) b^{-1}_{ij}(x) \dot{u}_j(x) - \sigma(x)\dot{\psi}(x) - \varrho(x)\dot{\phi}(x) - s_\alpha(x)\omega_\alpha(x), \qquad (5.1)$$

where

$$\tilde{\pi}_i = \pi_i - \sigma\nabla_i\psi - \varrho\nabla_i\phi - s_\alpha\omega_{\alpha k},\ \omega_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}a_{\lambda\gamma}\dot{a}_{\lambda\beta},\ \omega_{\alpha k} = \frac{1}{2}\epsilon_{\alpha\beta\gamma}a_{\lambda\gamma}\nabla_k a_{\lambda\beta}.$$

Here the last term in (5.1) is the spin kinematic part of Lagarangian [4,6], the term $s_\alpha\omega_{\alpha k}$ in quantity $\tilde{\pi}_i$ is the corresponding translation operator in the space of variables $s_\alpha, a_{\alpha\beta}$ such that $\tilde{\pi}_i$ is as ordinary the translation operator in the space of u_i, π_i. If we consider, in the same way as above, the variations of the dynamic variables leaving invariant $\mathcal{L}_k(x)$ then the algebra for the PB of dynamic variables is obtained which contains the algebra (3.2) and additional brackets [4]

$$\{\pi_i(x), s_\alpha(x')\} = -s_\alpha(x)\nabla_i\delta(x-x')\,, \{\pi_i(x), a_{\alpha\beta}(x')\} = \delta(x-x')\nabla_i a_{\alpha\beta}(x),$$

$$\{s_\alpha(x), s_\beta(x')\} = \epsilon_{\alpha\beta\gamma}s_\gamma(x)\delta(x-x'),\ \{s_\alpha(x), a_{\beta\gamma}(x')\} = \epsilon_{\alpha\gamma\rho}a_{\beta\rho}(x)\delta(x-x'). \qquad (5.2)$$

Considering that the energy density is of the following form $\varepsilon(x) = \varepsilon\Big(\sigma(x), \varrho(x), \pi_i(x), p_i(x), b_{ik}(x), s_\alpha(x), a_{\alpha\beta}(x), \omega_{\alpha k}(x)\Big)$ we obtain dynamic equations in the local limit. Using formula (1.8) in which $a(x) = \{\varrho(x), s_\alpha(x),\ \varepsilon(x),\ \pi_i(x)\}$ and supposing that $\varepsilon(x)$ obeys the properties of the phase, translational and spin invariances we find for the fluxes densities of mass, spin, energy and momentum

$$j_k = \varrho\frac{\partial\varepsilon}{\partial\pi_k} + \frac{\partial\varepsilon}{\partial p_k},\ j_{\alpha k} = s_\alpha\frac{\partial\varepsilon}{\partial\pi_k} + \frac{\partial\varepsilon}{\partial\omega_{\alpha k}},$$

$$q_k = \frac{\partial\varepsilon}{\partial\pi_k}\left(\sigma\frac{\partial\varepsilon}{\partial\sigma} + \varrho\frac{\partial\varepsilon}{\partial\varrho} + \pi_l\frac{\partial\varepsilon}{\partial\pi_l} + s_\alpha\frac{\partial\varepsilon}{\partial s_\alpha}\right) + \frac{\partial\varepsilon}{\partial p_k}\left(\frac{\partial\varepsilon}{\partial\varrho} + p_i\frac{\partial\varepsilon}{\partial\pi_i}\right) +$$

$$+\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}\left(\frac{\partial\varepsilon}{\partial s_\alpha}+\omega_{\alpha i}\frac{\partial\varepsilon}{\partial\pi_i}\right)+b_{ij}\frac{\partial\varepsilon}{\partial b_{ik}}\frac{\partial\varepsilon}{\partial\pi_j}, \tag{5.3}$$

$$t_{ik}=p\delta_{ik}+\pi_i\frac{\partial\varepsilon}{\partial\pi_k}+p_i\frac{\partial\varepsilon}{\partial p_k}+\omega_{\alpha i}\frac{\partial\varepsilon}{\partial\omega_{\alpha k}}+b_{ji}\frac{\partial\varepsilon}{\partial b_{jk}}.$$

For the compact entry of the fluxes densities (5.3) we introduce thermodynamic potential

$$\omega=Y_a\zeta_a-\sigma,\quad a=(0,i,\alpha,4),\quad \zeta_a=(\varepsilon,\pi_i,s_\alpha,\varrho), \tag{5.4}$$

where Y_a are the thermodynamic forces, defined according to the equalities

$$\frac{\partial\varepsilon}{\partial\sigma}\equiv\frac{1}{Y_0},\ \frac{\partial\varepsilon}{\partial\pi_i}\equiv-\frac{Y_i}{Y_0},\ \frac{\partial\varepsilon}{\partial s_\alpha}\equiv-\frac{Y_\alpha}{Y_0},\ \frac{\partial\varepsilon}{\partial\varrho}\equiv-\frac{Y_4}{Y_0}. \tag{5.5}$$

With the account of (5.4), (5.5) we have $\dfrac{\partial\omega}{\partial Y_a}=\zeta_a$ and the second principle of thermodynamics for the Gibbs potential is written as

$$d\omega=\zeta_a dY_a+\frac{\partial\omega}{\partial p_i}+\frac{\partial\omega}{\partial b_{ik}}db_{ik}+\frac{\partial\omega}{\partial\omega_{\alpha k}}d\omega_{\alpha k}. \tag{5.6}$$

It follows from equations (5.3), (5.6) that

$$\zeta_{ak}=-\frac{\partial}{\partial Y_a}\frac{\omega Y_k}{Y_0}+\frac{\partial\omega}{\partial p_k}\frac{\partial}{\partial Y_a}\frac{Y_4+Y_ip_i}{Y_0}+\frac{\partial\omega}{\partial b_{jk}}\frac{\partial}{\partial Y_a}\frac{b_{jl}Y_l}{Y_0}+\frac{\partial\omega}{\partial\omega_{\alpha k}}\frac{\partial}{\partial Y_a}\frac{Y_\alpha+Y_l\omega_{\alpha l}}{Y_0}.$$

Here $\zeta_{0k}=q_k,\zeta_{ik}=t_{ik},\zeta_{\alpha k}=j_{\alpha k},\zeta_{4k}=j_k$. Dynamics of the additive motion integral densities and of the parameters describing violated symmetry is defined by equations

$$\dot\zeta_a=-\nabla_k\zeta_{ak},\qquad \dot a_{\alpha\beta}=\frac{1}{Y_0}\left(Y_i\nabla_i a_{\alpha\beta}-\epsilon_{\alpha\rho\gamma}Y_\alpha a_{\rho\beta}\right),$$

$$\dot p_i=\nabla_i\left(\frac{Y_4+Y_jp_j}{Y_0}\right),\ \dot b_{ik}=\nabla_k\left(\frac{b_{ij}Y_j}{Y_0}\right). \tag{5.7}$$

6. Superfluid ^{3}He–B

In the case of the superfluid ^{3}He–B phase the symmetry with respect to the phase transformations and spin rotations is spontaneously broken. The parameters describing such violation are the superfluid phase $\phi(x)$ and the orthogonal rotation matrix $a_{\alpha\beta}(x)$. Accodingly, the energy density is the functional of the form $\varepsilon(x)=\varepsilon\Big(x;\sigma(x'),\varrho(x'),\pi_i(x'),s_\alpha(x'),a_{\alpha\beta}(x'),\phi(x')\Big)$. PB for the dynamic variables of the superfluid ^{3}He–B phase form the subalgebra of the PB algebra (3.2),(5.2) for the quantum spin crystal. Thus, fluxes densities and equations of motion have the form (5.3),(5.7) where it is necessary to put down terms containing quantity b_{ik} (variable u_i is cyclic).

7. Conclusion

The Proposed approach for the construction of the dynamic variables PB permits us to study from the unique point of view different physical systems with spontaneously broken phase invariance. A principal role in the given method plays the structure of the kinematic part of Lagrangian L_k and the variations of dynamic variables that leave invariant the kinematic part L_k. The density of entropy is considered as a dynamic variable that leads to the necessity to introduce the corresponding conjugated variable which should be considered as cyclic. Further development of the present formalism can be connected with the introducing into the theory of the different gauge fields that corresponds to the consideration of the various types of defects in the systems studied above.

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ГАМІЛЬТОНОВА ДИНАМІКА КОНДЕНСОВАНИХ СЕРЕДОВИЩ ЗІ СПОНТАННО ПОРУШЕНОЮ ФАЗОВОЮ ІНВАРІАНТНІСТЮ

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Побудовано гамільтонів формалізм до динаміки конденсованих середовищ зі спонтанно порушеною фазовою інваріантністю. Сформульовано метод знаходження дужок Пуассона для динамічних змінних, який грунтується на розгляданні перетворень, що залишають інваріантною кінематичну частину дії, та інтерпретації позаінтегральних членів як генераторів цих перетворень. Важливу роль у розвинутому підході грає побудова операторів трансляцій та їх щільностей для різних фізичних полів, що фігурують в дії. Фази параметрів порядку,

що описуються порушеною симетрією, визначено в термінах щільностей адитивних інтегралів руху. У якості конкретних прикладів розглянуто квантовий кристал, надплинні фази ^{3}He-B та квантовий спіновий кристал.