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ON THE NONLINEAR DYNAMICS OF PAIR HYDRODYNAMICAL FLUCTUATIONS

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Nonlinear evolution equations of pair hydrodynamical fluctuations are obtained. Quasi stationary solutions to the equations are found in approximation of an ideal gas state equation if dissipative processes to be absent. The existance time of such solutions is evaluated. The expression of a fluctuation propagation velocity depending on pair hydrodynamical fluctuation amplitudes is obtained by a nonlinear way. The problem on stability of arising quasi stationary states is studied. Connection of the problem solved with the theory of "long hydrodynamical tails" is ascertained.

Introduction

Problems of the turbulence theory [1,2], and the "long hydrodynamical tails" theory [3,4], and interacting modes theory [5] stimulated an advanced interest in studying long wave fluctuation hydrodynamics. It is natural, first of all, a question arose to derive the very fluctuation hydrodynamics equations. In traditional phenomenological aproaches, the fluctuation hydrodynamics equations of both incompressible and compressible liquid were derived by averaging the usual hydrodynamics equations over random initial conditions or external random force applied to a system (see, for instance, [6,7]). In connection with this the question came up to derive the fluctuation hydrodynamics equations by an unified approach like, for instance, the Chapman - Enskog approach to derivation of the usual hydrodynamics equations, proceeding from the kinetic Boltzmann equation. It is clear that in the case of dilute gases a fluctuationally kinetic stage of system evolution [8] must also precede the fluctuationally hydrodynamic one. It is why the problem on the regular derivation of long wave fluctuation kinetics equations is set up. The task is solved in the work [8] by the microscopic approach based on modification of the reduced description Bogolyubov method [9], the latter being applied to a fluctuating system. In this work, a transition from the fluctuationally kinetic stage of evolution to the fluctuationally hydrodynamic one is studied too.

Here, the long wave fluctuation hydrodynamic equations obtained in [8] are solved in approximation when a system state is described by pair correlation functions, and it is possible to neglect an effect of higher than second fluctuations order on system dynamics. Besides, at solving this problem, dissipative processes are disregarded, and the system state is admitted to be described by the ideal gas state equations.

1. Basic equations

The general nonlinear equations of long wave fluctuation hydrodynamics have the form (see [8]):

$$\frac{\partial}{\partial t}\zeta_\alpha(\vec{x},t) = \exp\left\{\mathcal{G}(\frac{\delta}{\delta\zeta};\xi_a(t))\right\}\frac{\partial}{\partial x_k}\zeta_{\alpha k}(\vec{x};\zeta), \tag{1.1}$$

$$\frac{\partial}{\partial t}\mathcal{G}(v;\xi_a(t)) = \Big\{\exp\Big\{\mathcal{G}(v+\frac{\delta}{\delta\zeta};\xi_a(t)) - \mathcal{G}(v;\xi_a(t))\Big\} -$$

$$-\exp\Big\{\mathcal{G}(\frac{\delta}{\delta\zeta};\xi_a(t))\Big\}\Big\}\int d\vec{x}\frac{\partial v_\alpha(\vec{x})}{\partial x_k}\zeta_{\alpha k}(\vec{x};\zeta),$$

where $\zeta_\alpha(\vec{x},t), (\alpha = 0, i, 4; i = 1, 2, 3)$ are densities of additive motion integrals, ($\zeta_0(\vec{x};t) = \varepsilon(\vec{x};t)$ is the energy density, $\zeta_i(\vec{x};t) = \pi_i(\vec{x};t)$ - movement density, and $\zeta_4(\vec{x};t) = \rho(\vec{x};t)$ - mass density); $\zeta_{\alpha k}(\vec{x};\zeta)$ - densities for flaxes of motion integral additive, and the generating functional $\mathcal{G}(v;\xi_a(t))$ of the hydrodynamic correlations functions $\xi_{\alpha_1\ldots\alpha_s}(\vec{x}_1,\ldots,\vec{x}_s;t)$ ($s \geq 2$ is correlation function number) is defined by the expression

$$\mathcal{G}(v;\xi_a) = \sum_{s=2}^{\infty}\frac{1}{s!}\int d\vec{x}_1\ldots\int d\vec{x}_s v_{\alpha_1}(\vec{x}_1)\ldots v_{\alpha_1}(\vec{x}_s)\xi_{\alpha_1\ldots\alpha_s}(\vec{x}_1,\ldots,\vec{x}_s;t), \tag{1.2}$$

where $v_\alpha(\vec{x})$ is a functional argument.

We shall study solutions to the general nonlinear equations (1.1) of fluctuations hydrodynamics in the spatially homogeneous case, neglecting the influence of higher than second order correlation functions on system dynamics. We shall limit ourselves by investigation of such solutions in approximation of ideal gas when the quantities $\zeta_{\alpha k}(\vec{x},\zeta)$ after neglecting dissipative processes take the form:

$$\zeta_{0k}(\zeta(\vec{x})) = -\frac{\pi_k(\vec{x})}{3\rho(\vec{x})}\left(5\varepsilon(\vec{x}) - \frac{\pi^2(\vec{x})}{\rho(\vec{x})}\right), \tag{1.3}$$

$$\zeta_{ij}(\zeta(\vec{x})) = \frac{2}{3}\varepsilon(\vec{x})\delta_{ij} + \frac{1}{\rho(\vec{x})}\left(\pi_i(\vec{x})\pi_j(\vec{x}) - \frac{1}{3}\delta_{ij}\pi^2(\vec{x})\right),$$

$$\zeta_{4k}(\zeta(\vec{x})) = \pi_k(\vec{x}).$$

In the spatially homogeneous and isotropical case, the additive motion integral densities $\zeta_\alpha(\vec{x},t)$ are independent of $\vec{x}$ and t, and the quantities $\xi_{\alpha_1\alpha_2}(\vec{x}_1,\vec{x}_2;t)$ are functions of the coordinate difference $\vec{x}_1 - \vec{x}_2$.

Then we introduce the Fourier components $\xi_{\alpha\beta}(\vec{k},t)$ of the pair hydrodynamic correlation functions $\xi_{\alpha\beta}(\vec{x}_1 - \vec{x}_2,t)$ and note that by virtue of assumption on spatial homogeneity and isotropy of a studied system

$$\xi_{ab}(\vec{k},t) = \xi_{ab}(k,t) = \xi_{ba}(k,t); a,b = (0,4), \tag{1.4}$$

$$\xi_{ia}(\vec{k},t) = -\xi_{ai}(\vec{k},t) = k_i\xi_a(k,t),$$

$$\xi_{ij}(\vec{k},t) = \xi_{tr}(k,t)\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right) + \xi_l(k,t)\frac{k_ik_j}{k^2}; i,j = (1,2,3).$$

Then from Eqs.(1.1) in approximation of pair functions, we obtain

$$\dot{\xi}_l(k,t) = \frac{4}{3} i k^2 \xi_0(k,t), \qquad \dot{\xi}_{00}(k,t) = -\frac{3}{4} i k^2 s^2(t) \xi_0(k,t),$$

$$\dot{\xi}_0(k,t) = \frac{3}{8} i s^2(t) \xi_l(k,t) - \frac{2}{3} i \xi_{00}(k,t), \tag{1.5}$$

$$\dot{\xi}_4(k,t) = i \xi_l(k,t) - \frac{2}{3} i \xi_{04}(k,t), \qquad \dot{\xi}_{44}(k,t) = -2 i k^2 \xi_4(k,t),$$

$$\dot{\xi}_{04}(k,t) = -\frac{3}{8} i k^2 s^2(t) \xi_4(k,t) - i k^2 \xi_0(k,t).$$

Upon that the quantity $\xi_{tr}(k,t)$ entering the definition (1.4) does not depend on time: $\xi_{tr}(k,t) = \xi_{tr}(k)$.

The quantity $s(t)$ of (1.5) is determined by the expressions:

$$\frac{s^2(t)}{s_0^2} = 8 \left(x(t) F(x(t)) + x^2(t) y(t) F'(x(t)) \right), \tag{1.6}$$

$$x(t) = \frac{\rho}{\sqrt{2\xi_{44}(0,t)}}, \qquad y(t) = \frac{1}{\rho\varepsilon} \left(\xi_{04}(0,t) + \frac{1}{3} \xi_{jj}(0,t) \right),$$

where s_0 is a sound velocity in an ideal gas, $s_0^2 = \frac{10}{9}\frac{\varepsilon}{\rho}$, and the function $F(x)$ is set by the formula $F(x) = \exp(-x^2) \int_0^x dt \exp(t^2)$ and appears in calculating the quantity $\exp\left(\frac{1}{2}\xi_{44}(0,t)\frac{\partial^2}{\partial\rho^2}\right)\frac{1}{\rho}$ (see Eq. (1.2)). (Throughout what follows we interpret $\xi_{\alpha\beta}(0,t)$ as $\xi_{\alpha\beta}(\vec{x}_1 = \vec{x}_2, t)$).

It is Eqs.(1.5) together with Eqs.(1.6) which describe evolution of the pair hydrodynamic fluctuations in spatially homogeneous and isotropic media having the ideal gas state equation.

2. Quasi stationary solutions of the pair fluctuatuion dynamics equations in dissipativeless approximation

As it is impossible to solve the equation system (1.6) in the general form, we study its solution behavior in the asymptotic region $t \to \infty$, assuming a finite limiting value s_∞^2 of the quantity $s^2(t)$: $s^2(t) \underset{t\to\infty}{\longrightarrow} s_\infty^2$ to exist. It is necessary to note that at $t \to \infty$ the limiting solutions to Eqs.(1.5), (1.6) cannot exist infinitely long because of involving dissipative processes in the real systems. And no matter how small these dissipative processes are, they will essentially influence the system evolution at the end of a certain time, resulting in relaxation to the total statistical equilibrium state of the system. In the other words, the limit $t \to \infty$ for nonlinear equation solutions exist for the certain time interval $\tau_0 \ll t \ll \tau_d$ where, how it will be shown below, time τ_0 of system reaching a limiting solution is evaluated by the formula $\tau_0 \sim \frac{L}{s_\infty}$, and the time τ_d at the end of which the dissipative processes influence system evolution is estimated by the expression $\tau_d \sim \frac{L^2}{v}$ (see, for instance, [1]). Here v is kinematic viscosity while the quantity L determines spatial localization of the limiting pair hydrodynamic correlation functions

$$\xi_{\alpha\beta}^\infty(\vec{x}_1 - \vec{x}_2) = \lim_{t\to\infty} \xi_{\alpha\beta}(\vec{x}_1 - \vec{x}_2; t) = \xi_{\alpha\beta}^\infty(\frac{\vec{x}_1 - \vec{x}_2}{L}).$$

Upon that accordingly (1.6) the quantity s_∞^2, in turn, is determined by the amplitudes of the limiting pair correlation functions. Allowing for the inequality $\tau_0 \ll t \ll \tau_d$ we will call the limiting at $t \to \infty$ for solutions to Eqs.(1.5), (1.6) as quasi stationary.

Assuming $s^2(t) \underset{t\to\infty}{\longrightarrow} s_\infty^2$ the solution to Eqs.(1.5), (1.6) can be represented in the asymptotic region $t \ll \tau_0$ as

$$\xi_l(k,t) = A(k)\cos(s_\infty kt) + D(k)\sin(s_\infty kt) + C(k), \tag{2.1}$$

$$\xi_{44}(k,t) = M(k)\cos(\frac{1}{2}s_\infty kt) + N(k)\sin(\frac{1}{2}s_\infty kt) + F(k) -$$
$$-4s_\infty^{-2}\left(\xi_l(k,t) - C(k)\right),$$

$$\xi_{04}(k,t) = \frac{3}{16}\left\{s_\infty^2\left(\xi_{44}(k,t) - F(k)\right) - 8\left(\xi_l(k,t) - 2C(k)\right)\right\},$$

$$\xi_{00}(k,t) = \frac{9}{8}s_\infty^2\left(-\frac{1}{2}\xi_l(k,t) + C(k)\right),$$

$$\xi_0(k,t) = -\frac{3}{4}\frac{i}{k^2}\dot{\xi}_l(k,t), \qquad \xi_4(k,t) = \frac{i}{2k^2}\dot{\xi}_{44}(k,t),$$

where $A(k), D(k), M(K), N(k), C(k), F(k)$ are the arbitrary functions of k, being integration constants, and the quantity s_∞, as before, is given by Eq.(1.6) in which, however, the quantities $x(t), y(t)$ must be replaced by their limiting values x_∞, y_∞ at $t \to \infty$.

From Eqs.(2.1), rewritten in the coordinate representation is seen that quantity s_∞ has meaning of a pulsation propagation velocity depending in a complex nonlinear way on amplitudes of the pulsation $\xi_{\alpha\beta}^\infty(\vec{x}_1 = \vec{x}_2)$ (see (1.6)), the Fourier components of the quantities $\xi_{\alpha\beta}^\infty(\vec{x}_1 - \vec{x}_2)$ being determined by

$$\xi_{i0}^\infty(k) = 0, \qquad \xi_{i4}^\infty(k) = 0, \tag{2.2}$$

$$\xi_{04}^\infty(k) = \frac{3}{2}\xi_l^\infty(k), \qquad \xi_{00}^\infty(k) = \frac{9}{16}s_\infty^2\xi_l^\infty(k),$$

$$\xi_{ij}^\infty(k) = \xi_{tr}^\infty(k)\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right) + \xi_l^\infty(k)\frac{k_ik_j}{k^2}, \qquad i,j = (1,2,3),$$

where $\xi_{44}^\infty(k) = F(k), \xi_l^\infty(k) = C(k), \xi_{tr}^\infty = \xi_{tr}(k)$ - are the arbitrary functions of k. For the small amplitudes of the quasi stationary solutions, the pulsation propagation velocity dependence, in accordance with (1.6), (2.2), is given by

$$s_\infty^2 \approx 4s_0^2\left\{1 - \frac{1}{\rho\varepsilon}\left(\frac{11}{6}\xi_l^\infty(0) + \frac{2}{3}\xi_{tr}^\infty(0)\right) + \frac{1}{\rho^2}\xi_{44}^\infty(0)\right\}, \tag{2.3}$$

$$\frac{1}{\rho^2}\xi_{44}^\infty 0 \ll 1, \qquad \frac{1}{\rho\varepsilon}\left(\frac{11}{6}\xi_l^\infty(0) + \frac{2}{3}\xi_{tr}^\infty(0)\right) \ll 1.$$

It is necessery to note that by virtue of positivity of the quantities x_∞, y_∞, how it is easy to believe starting from (1.6), (2.2), in the region $x_\infty > 0.92$ (where the derivative of $F(x_\infty)$ is negative, $F'(x_\infty) < 0$) the quantity s_∞^2 can take negative values that is why the limiting solutions

(2.2) are certainly unstable. Moreover, in this region the solutions (2.2) become ineligible, as in this case the quantity $s(t)$ has no finite limit at $t \to \infty$ what was as indispensable condition of existance for the above solutions.

The quasi stationary states with parameters corresponding to the non-negative values of the quantity s_∞^2 can be both stable and unstable. The problem to investigate stability of such quasi stationary states consists of studying the time evolution of small deviations $\delta\xi_{\alpha\beta}(\vec{x}_1 - \vec{x}_2; t)$ of the exact solutions $\xi_{\alpha\beta}(\vec{x}_1-\vec{x}_2;t)$ from the solutions $\xi_{\alpha\beta}^\infty(\vec{x}_1-\vec{x}_2)$ (see (2.2)). According to (1.6) these deviations induce the expansion $s(t) \approx s_\infty + \delta s(t)$. Linearizing Eqs.(1.5) makes possible to obtain the evolution linear equations for the quantities $\delta\xi_{\alpha\beta}(\vec{x}_1-\vec{x}_2;t)$ and $\delta s(t)$. Analysis of the found solutions to these equations shows that quasi stationary solutions determined by Eqs.(2.1), (2.2) are stable in the region of small amplitudes $|\xi_{\alpha\beta}^\infty(0)| \ll |\zeta_\alpha\zeta_\beta|$ for these limiting solutions. Damping factor γ of deviations $\delta\xi_{\alpha\beta}(\vec{x}_1 - \vec{x}_2; t)$ is given by the formula

$$\gamma \sim \frac{s_\infty}{L}|z_1(\xi)|,$$

where the dimensionless quantity $z_1(\xi)$ is a certain functional of limiting solutions $\xi_{\alpha\beta}^\infty(\vec{x}_1-\vec{x}_2)$. For instance, when the correlation functions $\xi_{\alpha\beta}^\infty(\vec{x}_1-\vec{x}_2)$ have the Gauss dependence on $\frac{\vec{x}_1-\vec{x}_2}{L}$(see [1]), the quantity $z_1(\xi)$ is given by the expression:

$$z_1(\xi) \approx -4\left(\ln(\frac{25}{3\sqrt{\pi}}\frac{\varepsilon^2}{\xi_{00}(0)})\right)^{\frac{1}{2}}.$$

How it is easy to see it is the quantity $\tau_0 \sim \frac{1}{\gamma}$ that is a typical time of the system reaching the quasi stationary state (2.2) during its evolution process.

With rise in the amplitudes of the limiting solutions, accurring when the pair correlation functions becomes compared over a modulus with product of the corresponding hydrodynamic quantities $|\xi_{\alpha\beta}^\infty(0)| \sim |\zeta_\alpha\zeta_\beta|$, the quasi stationary solutions can lose stability. In this case an increase increment δ is given by the expression similar (2.1)

$$\delta \sim \frac{s_\infty}{L}|z_2(\xi)|,$$

where the quntity $z_2(\xi)$ also essentially depends on the explicit form of the functions $\xi_{\alpha\beta}^\infty(\vec{x}_1 - \vec{x}_2)$.

The final evolution stage (at $t \gg \tau_d, \tau_d \sim \frac{L^2}{v}$) of the pair hydrodynamic fluctuations can be completed by relaxation to the total statistical equilibrium state of the system. The evolution equations of the pair hydrodynamic fluctuations can be linearized due to smallness of the hydrodynamic fluctuations in the given system evolution stage. The solutions to such linearized equations [11, 12, 8] lead to the known results of the "long hydrodynamic tails" theory [3, 4, 11, 12].

Thus, accordingly the results of the present work the system with the pair hydrodynamic fluctuations can pass through two typical stages during its evolution. The first stage is completed by relaxation after the time of order of $\tau_0 \ll \tau_d$ to the quasi stationary state being stable for the small parameters of limiting circles or unstable for the limiting circle parameter values exceeding certain critical values. The second typical stage of the system evolution occurs at times $t \gg \tau_d$ and describes system relaxation to the total statistical equilibrium state.

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ДО НЕЛІНІЙНОЇ ДИНАМІКИ ПАРНИХ ГІДРОДИНАМІЧНИХ ФЛУКТУАЦІЙ

Ю.В.Слюсаренко

Отримуються рівняння нелінійної еволюції парних гідродинамічних флуктуацій. Квазістаціонарний розв'язок цих рівнянь знаходиться у наближенні рівняння стану ідеального газу при відсутності дисипативних процесів. Оцінюється час існування таких розв'язків. Отримуються вирази флуктуаційних швидкостей в залежності від парних гідродинамічних флуктуаційних амплітуд. Досліджується проблема стійкості та виникнення квазістаціонарних станів. Розглядається зв'язок з проблемою, що розв'язується в рамках теорії "довгих гідродинамічних хвостів".