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WEYL-TYPE QUANTIZATION RULES AND N-PARTICLE CANONICAL REALIZATION OF THE POINCARÉ ALGEBRA IN THE TWO-DIMENSIONAL SPACE-TIME

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Квантування Вайлівського типу та N-частинкова реалізація агебри Пуанкаре у двовимірному просторі-часі

В. Шпитко

Анотація. У рамах фронтальної форми динаміки у двовимірному просторі-часі розглядається квантування канонічної реалізації агебри Пуанкаре, що відповідає системі N взаємодіючих частинок. За допомогою правил квантування Вайлівського типу побудовано унітарні представлення групи $\mathcal{P}(1,1)$. Показано, що вимога збереження алгебри Лі цієї групи обмежує множину правил квантування, але не усуває сама собою неоднозначності процедури квантування. Множина правил квантування розбивається на класи еквівалентности. Правила квантування із того ж самого класу приводять до того ж самого спектру мас та еволюції квантованої системи і дають унітарно еквівалентні представлення групи $\mathcal{P}(1,1)$. Правила квантування із різних класів дають унітарно нееквівалентні представлення та приводять до різних виразів для спостережуваних величин.

Weyl-type quantization rules and N-particle canonical realization of the Poincaré algebra in the two-dimensional space-time

V. Shpytko

Abstract. The quantization of canonical realization of Poincaré algebra corresponding to N-particle interacting system in the two-dimensional space-time $\mathbb{M}_2$ in the front form of dynamics is considered. Unitary realizations of the group $\mathcal{P}(1,1)$ are obtained by means of a set of Weyl-type quantization rules. We demonstrate that the requirement of preservation of Lie algebra of this group restricts the set of quantization rules but does not by itself remove the ambiguity of quantization procedure. The set of quantization rules fall apart into equivalence classes. The quantization rules from the same equivalence class give the same mass spectrum, evolution of the quantized system and lead to equivalent unitary representations of the group $\mathcal{P}(1,1)$. The quantizations which belong to different classes lead to non-equivalent unitary representations and may result in different values for observable quantities.

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I. Introduction

The problem of quantization of classical theory occupies a prominent place in the theoretical physics in 20-th century. Since the very beginning of appearance of the quantum mechanics it has been known about the deep difference between the classical and the quantum description – the deterministic character of the classical mechanics contrasts with the probabilistic interpretation of the quantum mechanics.

At the same time it has been realized that the quantum and classical mechanics are only different levels of the description of the physical reality and therefore they must have some common features. Various attempts to coordinate this two contrary facts and construct quantum description the structure of which "remembers" some essential features of the classical mechanics have led to various treatments in quantization problem [1–6].

The basic structure of the classical Hamiltonian mechanics for an unconstrained system is a 2N-dimensional phase space $\mathbb{P} \simeq \mathbb{R}^{2N}$ (in general case a symplectic manifold) with symplectic form ω. Phase space can be (locally) parametrized by canonical variables $q_a,\ p_a,\ a = \overline{1,N}$. The state of the classical system is described by a point in $\mathbb{P}$. Observable quantities are identified with smooth functions on $\mathbb{P}$. They form the space $C^\infty(\mathbb{P})$. Symplectic form determines on $C^\infty(\mathbb{P})$ the structure of Lie algebra (which is called Poisson algebra) by means of the Poisson bracket [7].

In the quantum mechanics a state is described by a vector $|\psi\rangle$ in some Hilbert space $\mathcal{H}$ and physical observables are self-adjoint operators in $\mathcal{H}$. Correspondence between the classical and quantum pictures is established within the framework of certain quantization procedure which is meant as a linear map $\mathcal{Q} : f \mapsto \hat{f}$ of the Poisson algebra into the set of self-adjoint operators in the Hilbert space $\mathcal{H}$.

The maximal group of automorphisms of the phase space which preserve the symplectic structure is the infinite dimensional group $Symp(\mathbb{P},\ \omega)$ of canonical transformations (symplectomorphisms). Quantum counterpart of such transformation is the group $U(\mathcal{H})$ of unitary transformations. It is well known that not every canonical transformation of a classical system leads after quantization to a unitary transformation in the quantum case. This shows that these two groups are not isomorphic [2].

For every symmetry group, which is some Lie group G, the classical Hamiltonian description provides a canonical realization of this group. The symmetry group of a classical system generates canonical transformations and as we have mentioned above, after quantization such classical transformations do not necessarily lead to unitary transformations on the quantum level of description. Besides, they may violate commutation relations of the Lie algebra of G. Thus, we cannot *a priori* be sure that any classical symmetry leads after quantization to the quantum one. Moreover, different quantization rules may preserve some types of symmetries and break out other ones. It is natural to demand the preservation of physically important symmetries. Therefore, we shall require for a quantization procedure the fulfilment of condition

$$\mathcal{Q}(\{f, g\}) = i[\hat{f}, \hat{g}] \tag{1.1}$$

only for some subalgebra of the Poisson algebra. It is clear that canonical generators corresponding to physically important symmetries have to belong to this subalgebra. This means that after quantization we come to a unitary representation of the symmetry group G acting in the Hilbert space $\mathcal{H}$.

On the other hand, it is well known that there exist a lot of different treatments to the quantization problem. Starting from some classical system different quantization procedures may result in non-equivalent quantum systems. From this point of view it is very important to understand whether there exists such a subalgebra of the Poisson algebra which minimizes the ambiguities of the quantization procedure.

In the relativistic mechanics of interacting particles the main algebraic structure is Lie algebra $\mathfrak{p}(1,3)$ of Poincaré group $\mathcal{P}(1,3)$ and the description of a system of N interacting particles must be Poincaré invariant in the classical case as well as in the quantum one. The Hamiltonian formalism leads to a canonical realization of $\mathfrak{p}(1,3)$. Therefore, the quantization procedure must preserve the structure of the Poincaré algebra, i.e. canonical generators of the Poincaré group have to be transformed into Hermitian operators which satisfy commutation relations of this algebra. In the relativistic case, the quantization problem is of special interest because Poincaré invariance conditions lead to the complicated dependence of interaction potentials on canonical coordinates and momenta. In most cases classical relativistic Hamiltonians depend on the products of non-commutative (in terms the of the Poisson bracket) quantities. This raises the question of symmetrization of non-commutative operators in the quantum description. Different ordering methods may result in different expressions for physical observable quantities [8].

The ten generators of the Poincaré group are realized in terms of canonical coordinates and momenta. As a rule, canonical coordinates do not coincide with the covariant particle coordinates owing to the no-interaction theorem [9]. Poincaré group is split into two parts – kinematical and dynamical. The kinematical part contains the generators independent of interaction and is called stability group $\mathcal{G}_\Sigma$. It determines the Hamiltonian form of relativistic dynamics and is automorphism group of simultaneity hypersurface Σ [10]. The hypersurface Σ determines geometrical form of dynamics [11]. The generators which constitute the dynamical part depend on interaction. Dirac called them Hamiltonians [10]. Different forms of dynamics for N-particle system lead to different numbers of generators independent of interaction. The *front form of dynamics* [10,12,13] has the largest possible stability group $\mathcal{G}_\Sigma$ for N-particle system – only three generators are Hamiltonians.

To construct the relativistic description which contains only one Hamiltonian it is necessary to choose in the four-dimensional Minkowsky space $\mathbb{M}_4$ a Poincaré-invariant hypersurface Σ. Unfortunately such a hypersurface does not exist. In other words, $\mathcal{G}_\Sigma \neq \mathcal{P}(1,3)$ [11]. This fact means that Lagrangian of N-particle system with an interaction must be a function $L : J^\infty\pi \to \mathbb{R}$ defined on the infinite order jet space of fibre bundle $\pi : \mathbb{F} \to \mathbb{R}, (t, x^i_a) \mapsto t$ [14]. The last statement is the Lagrangian variant of no-interaction theorem [15]. To avoid the difficulty related with the presence of time derivatives of infinite order in Lagrangian function we have to go beyond the class of geometrical forms of dynamics. Determining simultaneity conditions only for points of particle world lines (not for the whole space-time) we come to *isotropic forms of dynamics* [16].

In the two-dimensional space-time $\mathbb{M}_2$ the hypersurface Σ which determines the front form of dynamics for arbitrary N-particle system becomes isotropic [17]. The front form corresponds to the foliation of $\mathbb{M}_2$ by isotropic hyperplanes Σ_F:

$$x^0 + x = t. \tag{1.2}$$

The Poincaré group $\mathcal{P}(1,1)$ is the automorphism group of the foliation (1.2). In this form of dynamics Poincaré-invariance conditions for N-particle system allow the existence of interaction Lagrangians which do not contain derivatives higher than the first order. Only one generator of $\mathfrak{p}(1,1)$ contains an interaction and mechanical description is in some sense similar to the nonrelativistic one. The two-dimensional variant of the front form permits the construction of the number of exactly solvable classical and quantum relativistic models [18]-[21].

Due to the certain simplicity of the relativistic description in the front form in $\mathbb{M}_2$, we are able to elucidate the peculiarities of the quantization procedure in the relativistic case [18]-[20] and to understand how the

Poincaré invariance reduces the quantization ambiguities.

The aim of this article is the quantization of the canonical realization of the Poincaré algebra corresponding to N-particle relativistic system with an interaction (Sec. II) within the framework of the two-dimensional variant of the front form of dynamics. Using the set of Weyl-type quantization rules we construct in Sec. III unitary representations of the group $\mathcal{P}(1,1)$. We study the influence of different quantization rules on quantized system and propose some classification method of non-equivalent quantizations of the canonical realization of the Lie algebra of $\mathcal{P}(1,1)$. In Sec.IV we apply the obtained results to N-particle relativistic system with oscillator-like interaction.

II. Hamiltonian description of N-particle system in the front form of dynamics in $\mathbb{M}_2$

The classical Hamiltonian description of the system of N structureless particles with masses m_a $(a = \overline{1,N})$ in the two-dimensional Minkowsky space $\mathbb{M}_2$ in the framework of the front form of dynamics (1.2) leads to the canonical realization of the Lie algebra of the Poincaré group $\mathcal{P}(1,1)$ with generators H, P, K [17]. They correspond to energy, momentum, and boost integral. Due to the positiveness of the momentum variables $(p_a > 0)$ [12,17] in the front form of dynamics, the phase space of N-particle Hamiltonian system is $\mathbb{P} = \mathbb{R}^N_+ \times \mathbb{R}^N$ with standard Poisson bracket

$$\{f, g\} = \sum_{a=1}^{N} \left(\frac{\partial f}{\partial x_a}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial x_a}\frac{\partial f}{\partial p_a} \right). \tag{2.1}$$

We use more convenient in the front form quantities $P_\pm = H \pm P$ with the following Poisson bracket relations of the Poincaré algebra $\mathfrak{p}(1,1)$

$$\{P_+, P_-\} = 0, \qquad \{K, P_\pm\} = \pm P_\pm. \tag{2.2}$$

The generators are determined in terms of particle canonical variables x_a, p_a [17] as follows:

$$P_+ = \sum_{a=1}^{N} p_a, \qquad K = \sum_{a=1}^{N} x_a p_a, \tag{2.3}$$

$$P_- = \sum_{a=1}^{N} \frac{m_a^2}{p_a} + \frac{1}{P_+} V(rp_b, r_{1c}/r) \tag{2.4}$$

and we can see that only one generator, namely P_-, depends on interaction. The Poincaré-invariant function V describes the particles interaction and depends on $2N-1$ indicated arguments, where $r_{ac} = x_a - x_c; r = r_{12}; a, b = \overline{1, N}, c = \overline{2, N}$. In this case, the particle canonical coordinates coincide with covariant ones and we can pass to the Lagrangian description by means of the inverse of usual Legendre transformation. In contrast to the nonrelativistic mechanics, this transformation is nonlinear and, therefore, purely paired interactions in one formalism do not correspond to such ones in the other formalism [17].

Generators (2.3), (2.4) determine the square of the mass function of the system

$$M^2 = P_+ P_- = P_+ \sum_{a=1}^{N} \frac{m_a^2}{p_a} + V(rp_b, r_{1c}/r). \tag{2.5}$$

The first terms in Eqs. (2.4), (2.5) corresponds to the free-particle system.

The description of motion of the system as a whole may be performed by choosing P_+ and $Q = K/P_+$ as new (external) variables. There exist a lot of possibilities of the choice of inner variables. If $(\eta,\ q) = (\eta_1, ..., \eta_{N-1}, q_1, ..., q_{N-1})$ determine the set of inner canonical variables,

$$\begin{aligned} \{q_a, \eta_b\} &= \delta_{ab}; \qquad \{Q, P_+\} = 1, \qquad a, b = \overline{1, N-1}, \\ \{q_a, Q\} &= \{\eta_a, P_+\} = 0, \end{aligned} \tag{2.6}$$

such that the inner momenta η_b do not depend on the particle canonical coordinates, then

$$M^2 = M_f^2(\eta) + F(q, \eta),\ \ F(q, \eta) = V(rp_b, r_{1c}/r) \tag{2.7}$$

and Hamiltonian equations of motion take the form

$$\dot{Q} = 1/2 - \frac{M^2}{2P_+}, \quad \dot{P}_+ = 0, \tag{2.8}$$

$$\dot{q}_a = \frac{1}{2P_+} \frac{\partial M^2}{\partial \eta_a}, \quad \dot{\eta}_a = -\frac{1}{2P_+} \frac{\partial M^2}{\partial q_a}. \tag{2.9}$$

One of the possible choices of inner variables is [19]:

$$\eta_a = \frac{P_{a+} - p_{a+1}}{2P_{(a+1)+}},\ \ q_a = P_{(a+1)+}(Q_a - x_{a+1}); \tag{2.10}$$

where $a, b = \overline{1, N-1}$ and we use the following notations

$$\begin{aligned} P_{a+} &= \sum_{i=1}^{a} p_i,\ \ Q_a = P_{a+}^{-1} \sum_{i=1}^{a} x_i p_i, \\ P_{N+} &= P_+, \quad Q_N = Q. \end{aligned} \tag{2.11}$$

Variables (2.10) satisfy (2.6) and the free-particle squared mass function in Eq. (2.7) has the form

$$M_f^2(\eta) = \sum_{a=1}^{N} \frac{m_a^2}{1/2 - \eta_{a-1}} \prod_{i=a}^{N} (1/2 + \eta_i)^{-1}. \tag{2.12}$$

In the two-particle case variables (2.10) coincide with the variables proposed in Ref.[12].

III. Quantization of N-particle canonical realization of the Poincaré algebra in $\mathbb{M}_2$

To perform the quantization procedure, we have to determine firstly quantum operators corresponding to the particular canonical variables $x_a,\ p_a$. Then for a given set of classical observables $a = a(x, p)$ we construct corresponding quantum operators $\hat{A}$. Let $\hat{x}_a, \hat{p}_a$ be Hermitian operators corresponding to the classical particle coordinates and momenta with the following commutation relations

$$[\hat{x}_a, \hat{p}_b] = i\delta_{ab}. \tag{3.1}$$

The original Weyl application [22] is a basis for the whole set of quantization rules $W_{\mathcal{F}} : a \mapsto \hat{A}$, which map bijectively a family of classical real functions $a(x, p) \in C^\infty(\mathbb{P})$ to a family of Hermitian operators $\hat{A}$ in some Hilbert space $\mathcal{H}$. For $\mathbb{P} \approx \mathbb{R}^{2N}$, the formal definition is given in the explicit form [5,23] as follows

$$\hat{A} = \int (dk)(ds)\tilde{a}(k, s)\mathcal{F}(k, s) \exp\left[i \sum_a (k_a \hat{x}_a + s_a \hat{p}_a)\right], \tag{3.2}$$

where $a(k, s)$ is Fourier transform of the function $a(p, q)$. Function $\mathcal{F}(k, s)$ determines the type of quantization. Different choices of $\mathcal{F}(k, s)$ correspond to different ordering conventions. We shall call the elements of the family of quantizations (3.2) Weyl-type quantization rules. For the original Weyl quantization $\mathcal{F}(k, s) = 1$. Let us restrict ourselves to real functions $\mathcal{F}(k,s) \in C^\infty(\mathbb{R}^{2N})$, i.e. $\mathcal{F}(k,s) = \mathcal{F}^*(k,s)$. Every quantization rule must obey the following condition:

$$\mathcal{Q}(1) = \hat{1}, \tag{3.3}$$

where the unity means function on $\mathbb{P}$ and in r.-h. side $\hat{1}$ is the unit operator. As a result, for the family of quantizations (3.2) we obtain

$$\mathcal{F}(0,0) = 1 \ . \tag{3.4}$$

Hermiticity condition means:

$$\mathcal{F}(k,s) = \mathcal{F}(-k,-s). \tag{3.5}$$

For a system of N spinless particles we shall work with a momentum space basis given by

$$|p\rangle = |p_1\rangle \otimes |p_2\rangle \otimes \cdots \otimes |p_N\rangle \ , \tag{3.6}$$

where $|p_a\rangle$ is eigenvector of operator $\hat{p}_a \ : \ \hat{p}_a|p_a\rangle = p_a|p_a\rangle$. The wave functions $\psi(p) = \langle p|\psi\rangle$ describing the physical (normalized) states in the front form of dynamics constitute the Hilbert space $\mathcal{H}_N^F = \mathcal{L}^2(\mathbb{R}_+^N, d\mu_N^F)$ with the inner product [18]

$$(\psi_1,\psi) = \int d\mu_N^F(p)\psi_1^*(p)\psi(p), \tag{3.7}$$

where

$$d\mu_N^F(p) = \prod_{a=1}^N \frac{dp_a}{2p_a}\Theta(p_a) \tag{3.8}$$

is the Poincaré-invariant measure and $\Theta(p_a)$ is Heaviside step function. Operators act on wave functions $\psi(p) \in \mathcal{H}_N^F$ as integral operators:

$$(\hat{A}\psi)(p) = \int d\mu_N^F(p')\widetilde{A}(p,p')\psi(p'). \tag{3.9}$$

The kernel corresponding to operator (3.2) has the form

$$\widetilde{A}(p,p') = \frac{1}{(2\pi)^N}\int (dx)(dz)\exp(i\sum_{a=1}^N\Big(p'_a - p_a)x_a\Big) \times$$
$$\left(\prod_{a=1}^N \delta\Big(z_a - \frac{p_a+p'_a}{2}\Big)2\sqrt{p_a p'_a}\right)\mathcal{F}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial z}\right)a(x,z) \ , \tag{3.10}$$

where we take into account Eq.(3.5). We understand the expression $\mathcal{F}(i\partial/x, i\partial/z)$ as a formal series:

$$\mathcal{F}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial z}\right) = \sum_{\substack{i_1,\ldots,i_N \\ j_1,\ldots,j_N}}^\infty \frac{\partial^{i_1+\ldots+i_N+j_1+\ldots+j_N}\mathcal{F}(0,0)}{\partial^{i_1}k_1\ldots\partial^{i_N}k_N\partial^{j_1}s_1\ldots\partial^{j_N}s_N} \times$$
$$\times\prod_{a=1}^N \frac{1}{i_a!j_a!}\left(i\frac{\partial}{\partial x_a}\right)^{i_a}\left(i\frac{\partial}{\partial z_a}\right)^{j_a} . \tag{3.11}$$

Condition (3.5) leads to the equalities

$$\frac{\partial\mathcal{F}(0,0)}{\partial k_a} = \frac{\partial\mathcal{F}(0,0)}{\partial s_a} = 0, \quad a = \overline{1,N} \ . \tag{3.12}$$

Now let us consider the quantization procedure of classical canonical generators (2.3), (2.4) of $\mathfrak{p}(1,1)$. Substituting expressions (2.3) of the generators K, P_+ into (3.10) and using (3.4), (3.12) we obtain the following operators

$$\hat{P}_+ = P_+ \ ; \quad \hat{K} = i\sum_{a=1}^N p_a\frac{\partial}{\partial p_a} - \sum_{a=1}^N \frac{\partial^2\mathcal{F}(0,0)}{\partial k_a \partial s_a} \ . \tag{3.13}$$

The Weyl-type quantization rules transform the generator P_- into integral operator (3.9) with the kernel

$$\widetilde{P}_-(p,p') = \frac{1}{(2\pi)^N}\int (dx)(dz)\exp(i\sum_{a=1}^N\Big(p'_a - p_a)x_a\Big) \times$$
$$\left(\prod_{a=1}^N \delta\Big(z_a - \frac{p_a+p'_a}{2}\Big)2\sqrt{p_a p'_a}\right) \times \tag{3.14}$$
$$\mathcal{F}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial z}\right)\left(\sum_{a=1}^N \frac{m_a^2}{z_a} + \frac{V(rz_b, r_{1c}/r)}{\sum_{a=1}^N z_a}\right) .$$

After quantization we want to obtain a unitary realization of the group $\mathcal{P}(1,1)$. Therefore Hermitian operators (3.13), (3.14) have to satisfy the quantum commutation relations of the Poincaré algebra $\mathfrak{p}(1,1)$

$$[\hat{P}_+, \hat{P}_-] = 0 \ , \qquad [\hat{K}, \hat{P}_\pm] = \pm i\hat{P}_\pm. \tag{3.15}$$

The last term in the expression (3.13) of the boost operator $\hat{K}$ has no influence on commutation relations (3.15). Thus, the quantization problem reduces in fact to the construction of Hermitian operator $\hat{P}_-$. That in its turn determines the form of the function $\mathcal{F}$.

Proposition 1. *So that the Weyl-type quantizations could lead to unitary realizations of the group $\mathcal{P}(1,1)$, the function $\mathcal{F}$ has to be of the following form:*

$$\mathcal{F} = \mathcal{F}(ks) \ , \tag{3.16}$$

where the function $\mathcal{F}$ on the right-hand side depends on the all possible products of arguments: $k_1 s_1, ..., k_1 s_N, k_2 s_1, ..., k_2 s_N, ...$.

Proof: In order to satisfy relations (3.15) the kernel (3.14) must obey the following conditions:

$$\sum_{a=1}^{N}\Big(p_a \frac{\partial}{\partial p_a} + p'_a \frac{\partial}{\partial p'_a}\Big)\widetilde{P}_-(p,p') = -\widetilde{P}_-(p,p') \ , \tag{3.17}$$

$$\sum_{a=1}^{N}(p_a - p'_a)\widetilde{P}_-(p,p') = 0 \ . \tag{3.18}$$

Equation (3.18) is obtained as a result of commutation of the operators $\hat{P}_+$ and $\hat{P}_-$. It holds if the kernel $\widetilde{P}_-(p,p')$ is proportional to δ-function $\delta(P_+ - P'_+)$. The classical expression for P_- is translational invariant function. Therefore equation (3.18) holds for arbitrary Weyl-type quantization rule.

Equation (3.17) is the consequence of commutation of the operators $\hat{K}$ and $\hat{P}_-$. It means that the kernel $\widetilde{P}_-(p,p')$ must be homogeneous function of the order -1. To satisfy this condition the function $\mathcal{F}$ must obey the following homogeneity equation

$$\mathcal{F}(\beta k, \beta^{-1} s) = \mathcal{F}(k,s) \ . \tag{3.19}$$

The only possibility to satisfy this equation is (3.16).

Thus, we see that not every Weyl-type quantization rule preserves commutation relations (3.15) of the Lie algebra of the Poincaré group $\mathcal{P}(1,1)$. The set of quantization rules with the function $\mathcal{F}$ of the form (3.16) transforms arbitrary classical function depending only on momentum variables ($A = A(p_1, ..., p_N)$) into operator which acts on wave functions of momentum representation (3.7) as multiplication operator and has the classical form: $A(\hat{p}_1, ..., \hat{p}_N) = A(p_1, ..., p_N)$. Such quantization rules transform the interaction function V from (2.4) into operator which commutes with $\hat{K}$ and $\hat{P}_+$;

$$\Big[\hat{K}, \hat{V}\Big] = \Big[\hat{P}_+, \hat{V}\Big] = 0 \ . \tag{3.20}$$

It should be noted that the set of Weyl-type quantization rules with the function $\mathcal{F}$ of the form (3.16) does not include normal rule of ordering of noncommuting operators (see [23]).

Let us introduce the following variables

$$y_1 = x_1 \ , \ y_2 = P_+ = \sum_{a=1}^{N} p_a \ ,$$
$$\varkappa_a = r p_a \ , \quad y_b = \frac{r_{1b}}{r}, \ b = \overline{3,N} \ . \tag{3.21}$$

In terms of variables (3.21) the differential operators $\partial/\partial x_a$, $\partial/\partial p_a$ take the form

$$\frac{\partial}{\partial x_b} = -\frac{P_+}{\varkappa_+}\frac{\partial}{\partial y_b} \ , \quad b = \overline{3,N} \ , \ \frac{\partial}{\partial p_a} = \frac{\varkappa_+}{P_+}\frac{\partial}{\partial \varkappa_a} + \frac{\partial}{\partial P_+} \ ,$$
$$\frac{\partial}{\partial x_2} = -\frac{P_+}{\varkappa_+}\left(\sum_{a=1}^{N}\varkappa_a\frac{\partial}{\partial \varkappa_a} - \sum_{d=3}^{N} y_d \frac{\partial}{\partial y_d}\right) \ , \tag{3.22}$$
$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1} + \frac{P_+}{\varkappa_+}\left(\sum_{a=1}^{N}\varkappa_a\frac{\partial}{\partial \varkappa_a} + \sum_{d=3}^{N}(1-y_d)\frac{\partial}{\partial y_d}\right) \ ;$$

where $\varkappa_+ = \sum\limits_{a=1}^{N}\varkappa_a$. As a result of translational invariance of the classical generator P_-, it does not depend on y_1:

$$P_- = \frac{\varkappa_+}{P_+}\sum_{a=1}^{N}\frac{m_a^2}{\varkappa_a} + \frac{1}{P_+}V(\varkappa_1, ..., \varkappa_N, y_3, ..., y_N) \ . \tag{3.23}$$

As it follows from (3.11), (3.22) formal series $\mathcal{F}(-i\partial/x, -i\partial/z)$ expressed in terms of variables (3.21) contains differentiations with respect to P_+. Therefore not every Weyl-type quantization rule with the arbitrary function $\mathcal{F}$ of the form (3.16) will transform the product $P_+P_- = M^2$ ($\{P_+, P_-\} = 0$) of classical functions into the corresponding product of quantum (commutating) operators $\hat{P}_+\hat{P}_- = \hat{M}^2$. This means that not every quantization rule $W_{\mathcal{F}}$, preserving the structure of Lie algebra of the group $\mathcal{P}(1,1)$, preserves commutability of the following diagram

$$\begin{array}{ccc} P_+, \ P_- & \xrightarrow{M^2 = P_+P_-} & M^2 \\ \Big\downarrow W_{\mathcal{F}} & & \Big\downarrow W_{\mathcal{F}} \\ \hat{P}_+, \ \hat{P}_- & \xrightarrow{\hat{M}^2 = \hat{P}_+\hat{P}_-} & \hat{M}^2 \ . \end{array} \tag{3.24}$$

In the classical case the squared total mass function M^2 is an invariant of the group $\mathcal{P}(1,1)$. Thus, to obtain in the quantum case the algebraic structure which is most closely related to the classical one, the quantum Kasimir operator $\hat{M}^2 = \hat{P}_+\hat{P}_-$ should be a quantization result of the classical function $M^2 = P_+P_-$.

Proposition 2. *If the function* $\mathcal{F}$ *has the following form*

$$\mathcal{F} = \mathcal{F}(\Delta_1, \Delta_2) \ , \tag{3.25}$$

where

$$\Delta_1 = \sum_{a=1}^{N} k_a s_a \ , \quad \Delta_2 = \sum_{\substack{a=1 \\ a\neq b}}^{N}\sum_{b=1}^{N} k_a s_b, \tag{3.26}$$

then diagram (3.24) *is commutative.*

Proof: If $\mathcal{F} = \mathcal{F}(\Delta_1, \Delta_2)$, then kernel (3.10) takes the form:

$$\widetilde{A}(p,p') = \frac{1}{(2\pi)^N}\int (dx)(dz)\exp(i\sum_{a=1}^{N}\Big(p'_a - p_a)x_a\Big) \times$$
$$\left(\prod_{a=1}^{N}\delta\Big(z_a - \frac{p_a+p'_a}{2}\Big)2\sqrt{p_a p'_a}\right)\mathcal{F}\left(\hat{\Delta}_1, \hat{\Delta}_2\right) a(x,z) \ , \tag{3.27}$$

where

$$\mathcal{F}(\hat{\Delta}_1, \hat{\Delta}_2) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\partial^{i+j}\mathcal{F}(0,0)}{(\partial\Delta_1)^i(\partial\Delta_2)^j}\frac{1}{i!j!}(\hat{\Delta}_1)^i(\hat{\Delta}_2)^j \tag{3.28}$$

and

$$\hat{\Delta}_1 = -\sum_{a=1}^{N}\frac{\partial^2}{\partial x_a\partial z_a} \ , \quad \hat{\Delta}_2 = -\sum_{\substack{a=1 \\ a\neq b}}^{N}\sum_{b=1}^{N}\frac{\partial^2}{\partial x_a \partial z_b} \ . \tag{3.29}$$

Let us consider the action of the sum of partial derivatives $\sum\limits_{a=1}^{N}\partial^2/(\partial x_a\partial p_a)$ on a translation-invariant function

$$\sum_{a=1}^{N}\frac{\partial^2 f(P_+, \varkappa, y_3, ..., y_N)}{\partial x_a \partial p_a} =$$

$$\left[\left(1+\sum_{c=1}^{N}\varkappa_c\frac{\partial}{\partial\varkappa_c}\right)\left(\frac{\partial}{\partial\varkappa_1} - \frac{\partial}{\partial\varkappa_2}\right) + \right.$$
$$+\sum_{d=3}^{n}\left[(1-y_d)\frac{\partial}{\partial\varkappa_1} + \right.$$
$$\left.\left. y_d\frac{\partial}{\partial\varkappa_2} - \frac{\partial}{\partial\varkappa_d}\right]\frac{\partial}{\partial y_d}\right] f(P_+, \varkappa, y_3, ..., y_N) \ . \tag{3.30}$$

As follows from the last equation, such an action does not contain differentiations with respect to P_+. Moreover

$$\sum_{a=1}^{N}\frac{\partial^2 f(P_+, \varkappa, y_3, ..., y_N)}{\partial x_a \partial p_a} =$$
$$-\sum_{\substack{a=1 \\ a\neq b}}^{N}\sum_{b=1}^{N}\frac{\partial^2 f(P_+, \varkappa, y_3, ..., y_N)}{\partial x_a \partial p_a} \ . \tag{3.31}$$

Thus, the proposition follows from the translation-invariance of P_-.

We shall also consider partial cases with

$$\mathcal{F} = \mathcal{F}(\Delta_1, 0) = \mathcal{F}_1(\Delta_1) \ , \tag{3.32}$$

$$\mathcal{F} = \mathcal{F}(0, \Delta_2) = \mathcal{F}_2(\Delta_2) \ . \tag{3.33}$$

Quantization rule with $\mathcal{F} = \mathcal{F}_1$ has been considered, for example, in Ref. [24].

If

$$\mathcal{F} = \mathcal{F}(\Delta_0) = \mathcal{F}_0 \ , \quad \Delta_0 = \Delta_1 + \Delta_2 \tag{3.34}$$

then, as it follows from (3.31), (3.26), (3.28)

$$\mathcal{F}(\hat{\Delta}_0) f(P_+, \varkappa_1, ..., \varkappa_N, y_3, ..., y_N) =$$
$$f(P_+, \varkappa_1, ..., \varkappa_N, y_3, ..., y_N) \ . \tag{3.35}$$

Thus, the quantization rule $W_{\mathcal{F}_0}$ leads to the same operators $\hat{P}_-, \hat{P}_+$ as well as the original Weyl quantization does. The operator $\hat{K}$ takes the form

$$\hat{K} = i\sum_{a=1}^{N} p_a\frac{\partial}{\partial p_a} - \frac{d\mathcal{F}_0(0)}{d\Delta_0} \ . \tag{3.36}$$

If $\mathcal{F} = \mathcal{F}_0$ and

$$\frac{d\mathcal{F}_0(0)}{d\Delta_0} = 0 \ , \tag{3.37}$$

then we obtain the same operators $\hat{K}, \hat{P}_\pm$ as in the case of the original Weyl quantization. This means that quantization rules connected with the classes of functions $\mathcal{F}(\Delta_1, \Delta_2)$ and $\mathcal{F}(\Delta_1, \Delta_2)\mathcal{F}_0$, where $\mathcal{F}_0$ has property (3.37), lead exactly to the same unitary realization of the group $\mathcal{P}(1,1)$.

If condition (3.37) is not satisfied, then quantization rules $W_\mathcal{F}$ and $W_{\mathcal{F}\mathcal{F}_0}$ give us different expressions for $\hat{K}$ but they lead to the same realization of commutative ideal $\mathfrak{h} = (\hat{P}_+, \ \hat{P}_-)$.

In the front form of dynamics the evolution of the quantum system is described by the Schrödinger-type equation

$$i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi, \tag{3.38}$$

where $\Psi \in \mathcal{H}_N^F$ and

$$\hat{H} = \frac{1}{2}(\hat{P}_+ + \hat{P}_-) = \frac{1}{2}(\hat{P}_+ + \hat{M}^2/\hat{P}_+). \tag{3.39}$$

Putting $\Psi = \chi(t, P_+)\psi$, where ψ is a function of some Poincaré-invariant inner variables, we obtain the stationary eigenvalue problem for the operator $\hat{M}^2$:

$$\hat{M}^2\psi = \hat{P}_+\hat{P}_-\psi = M^2_{n,\lambda}\psi. \tag{3.40}$$

The ideal $\mathfrak{h}$ generates by means of the Eqs (3.38), (3.40) the evolution of the system and the mass spectrum.

Boost operator (3.13) obtained from classical expression (2.3) by means of arbitrary Weyl-type quantization rule $W_\mathcal{F}$ preserving commutation relation of the Poincaré algebra $\mathfrak{p}(1,1)$ generates Lorentz transformation

$$\left(e^{-i\lambda\hat{K}}\psi\right)(p) = \exp\left(i\sum_{a=1}^{N}\frac{\partial^2\mathcal{F}(0,0)}{\partial k_a \partial s_a}\right)\psi(e^{-\lambda}p). \tag{3.41}$$

We see that different quantization rules $W_\mathcal{F}, \ W_{\mathcal{F}'}$ preserving commutation relation of $\mathfrak{p}(1,1)$ lead to boost transformations which distinguish on phase factor:

$$\left(e^{-i\lambda\hat{K}'}\psi\right)(p) = e^{i\alpha}\left(e^{-i\lambda\hat{K}}\psi\right)(p), \tag{3.42}$$

where

$$\alpha = \sum_{a=1}^{N}\left(\frac{\partial^2\mathcal{F}(0,0)}{\partial k_a \partial s_a} - \frac{\partial^2\mathcal{F}'(0,0)}{\partial k_a \partial s_a}\right). \tag{3.43}$$

These Lorentz transformations are physically equivalent, because $\exp(-i\lambda\hat{K}')\psi(p)$ and $\exp(-i\lambda\hat{K})\psi(p)$ belong to the some ray. Thus, quantizations which lead to the same realization of the ideal $\mathfrak{h}$ give equivalent unitary representations of the group $\mathcal{P}(1,1)$. Therefore, it is natural to introduce the following

Definition 1. *Quantizations $W_\mathcal{F}$, $W_{\mathcal{F}'}$ which lead to the same realization of the ideal $\mathfrak{h}$ are called* **equivalent***:*

$$W_\mathcal{F} \simeq W_{\mathcal{F}'} \ . \tag{3.44}$$

Proposition 3. *Quantization rules $W_\mathcal{F}$, $W_{\mathcal{F}'}$ preserving the commutation relations of $\mathfrak{p}(1,1)$, where $\mathcal{F} = \mathcal{F}(ks, \Delta_0)$, $\mathcal{F}' = \mathcal{F}(ks, 0)$, are equivalent:*

$$W_{\mathcal{F}(ks,\Delta_0)} \simeq W_{\mathcal{F}(ks,0)}. \tag{3.45}$$

Proof: This follows immediately from (3.35) and (3.23).

Corollary 1.

$$W_{\mathcal{F}(ks)\mathcal{F}_0} \simeq W_{\mathcal{F}(ks)}. \tag{3.46}$$

For the special class of quantization rules which preserve in addition to the commutation relation of $\mathfrak{p}(1,1)$ the commutability of the diagram (3.24), we have

$$W_{\mathcal{F}_1(\Delta_1)} \simeq W_{\mathcal{F}_2(-\Delta_2)}, \ W_{\mathcal{F}_2(\Delta_2)} \simeq W_{\mathcal{F}_1(-\Delta_1)}. \tag{3.47}$$

Hence, we see that the Weyl-type quantization rules which preserve the commutation relation of the Poincaré algebra $\mathfrak{p}(1,1)$ fall apart into equivalence classes. Rules from different classes give non-equivalent unitary representations of the group $\mathcal{P}(1,1)$ and may result in different expressions for such important observable quantity as the mass spectrum of the system. We shall demonstrate this fact by the example of N-particle system with oscillator-like interaction in the next section.

IV. Ambiguities of quantization of N-particle system with oscillator-like interaction

In the case of the free particle system ($V = 0$), arbitrary quantization rule of the type (3.16) transforms the classical canonical generators P_+, P_- into the quantum operators which in momentum representation (3.7) have the same form as the corresponding classical quantities. The expression for the boost operator $\hat{K}$ depends on the class of quantization rules $W_{\mathcal{F}}$, but different Weyl-type quantizations (3.16) of the free particle canonical realization of the Lie algebra $\mathfrak{p}(1,1)$ give equivalent unitary representations of the Poincaré group $\mathcal{P}(1,1)$.

Let us consider an example of N-particle system with interaction. Let us choose the interaction function V in the following form

$$V = \omega^2 \sum\sum_{a<b} r_{ab}^2 p_a p_b, \quad \omega^2 > 0. \tag{4.1}$$

The function (4.1) describes N-particle oscillator-like interaction [19]. In the nonrelativistic limit such a system is reduced to the nonrelativistic oscillator system. In terms of the variables (2.10) the interaction function V takes the form

$$V = F(q,\eta) = \omega^2 \sum_{a=1}^{N-1} (1/4 - \eta_a^2) q_a^2 \prod_{j=a+1}^{N} (1/2 + \eta_j)^{-1}. \tag{4.2}$$

The system with interaction (4.1) has $N-2$ additional integrals of motion λ_j, which mutually commute

$$\{\lambda_i, \lambda_k\} = 0, \quad i,k = \overline{2, N-1}. \tag{4.3}$$

In terms of the variables (2.10) they have the form

$$\lambda_{j+1}^2 = \sum_{d=1}^{j} \frac{m_d^2}{1/2-\eta_{d-1}} \prod_{i=d}^{j} (1/2+\eta_i)^{-1} + \frac{m_{j+1}^2}{1/2-\eta_j} +$$
$$+\omega^2 \sum_{d=1}^{j-1} (1/4-\eta_d^2) q_d^2 \prod_{i=d+1}^{j} (1/2+\eta_i)^{-1} + \omega^2 (1/4-\eta_j^2) q_j^2, \tag{4.4}$$

where $\lambda_N^2 = M^2, j = \overline{1, N-1}$. They can be represented by means of the recursive relations

$$\lambda_{j+1}^2 = \frac{\lambda_j^2}{1/2+\eta_j} + \frac{m_{j+1}^2}{1/2-\eta_j} + \omega^2 (1/4 - \eta_j^2) q_j^2, \tag{4.5}$$

where we denote $\lambda_1^2 = m_1^2$.

Quantum mechanical description for this system have been constructed by means of the ordinary Weyl quantization in Ref. [19]. Here we shall consider the Weyl-type quantization rules connected with the function $\mathcal{F}_1$ (see (3.32)). The Weyl-type quantization rules differ from the ordinary Weyl quantization in the presence of the nontrivial operator $\mathcal{F}(\hat{\Delta}_1)$ acting on the classical generator P_- in the expression for kernel (3.14). We consider in this section quantization rules which preserve the commutability of diagram (3.20). Therefore the quantization problem of the canonical generators reduces to the construction of quantum interaction operator $\hat{V}$. This gives us immediately the expression for the operator $\hat{P}_-$:

$$\hat{P}_- = \hat{M}^2 / \hat{P}_+. \tag{4.6}$$

Hence, first of all, we have to find the action of $\mathcal{F}(\hat{\Delta}_1)$ on classical interaction function (4.1). In terms of the variables (2.10) this gives us the following result

$$\mathcal{F}(\hat{\Delta}_1) V = {}^{\mathcal{F}_1}V = \omega^2 P_+ \sum_{a=1}^{N-1} \left(\frac{(1/4 - \eta_a^2) q_a^2}{P_{(a+1)+}} + \right.$$
$$2\mathcal{F}_1'(0) \frac{q_a \left[(a-1)/2 - (a+1)\eta_a \right]}{P_{(a+1)+}} +$$
$$\left. \mathcal{F}_1''(0) \frac{a \left[(a-3)/2 - (a+1)\eta_a \right]}{P_{a+}} \right), \tag{4.7}$$

where $\mathcal{F}_1'(0) = d\mathcal{F}_1(0)/d\Delta_1$, $P_{a+} = P_+ \prod_{j=a}^{N} (1/2 + \eta_j)$. Moreover, changing every classical functions Z_k by ${}^{\mathcal{F}}Z_k$ we reduce the quantization procedure to the quantization of the classical problem with the new set of "classical observables" ${}^{\mathcal{F}}Z_k$ via the original Weyl rule and we can immediately use the results of Ref. [19]. Using equality (4.7), expressions (4.4), for integrals of motion λ_j, we obtain

$${}^{\mathcal{F}_1}\lambda_j^2 = \frac{m_j^2}{1/2 - \eta_{j-1}} + \sum_{k=1}^{j-1} \frac{m_k^2}{1/2 - \eta_{k-1}} \prod_{i=k}^{j-1} (1/2 + \eta_i)^{-1} +$$
$$+ \sum_{k=1}^{j-2} \frac{\omega^2}{\prod_{i=k+1}^{j-1} (1/2+\eta_i)} \left[(1/4-\eta_k^2) q_k^2 - \right.$$
$$\left. 2\mathcal{F}_1'(0) q_k \left(\frac{1-k}{2} + (k+1)\eta_k \right) + \mathcal{F}_1''(0) \left(\frac{k(k-1)}{1/2+\eta_k} - k(k+1) \right) \right] +$$

$$+\omega^2\left[(1/4-\eta_{j-1}^2)q_{j-1}^2-2\mathcal{F}_1'(0)q_{j-1}\left(\frac{2-j}{2}+j\eta_{j-1}\right)+\right.$$

$$\left.+\mathcal{F}_1''(0)\left(\frac{(j-1)(j-2)}{1/2+\eta_{j-1}}-j(j-1)\right)\right], \tag{4.8}$$

where ${}^{\mathcal{F}_1}\lambda_N^2={}^{\mathcal{F}_1}M^2$. Recurrence relations (4.5) are transformed into

$${}^{\mathcal{F}_1}\lambda_j^2+\omega^2 j(j-1)\mathcal{F}_1''(0)=$$

$$\frac{{}^{\mathcal{F}_1}\lambda_{j-1}^2+\omega^2(j-1)(j-2)\mathcal{F}_1''(0)}{1/2+\eta_{j-1}}+\frac{m_j^2}{1/2-\eta_{j-1}}+$$

$$+\omega^2\Big((1/4-\eta_{j-1}^2)q_{j-1}^2-2\mathcal{F}_1'(0)[(2-j)/2+j\eta_{j-1}]q_{j-1}\Big). \tag{4.9}$$

The separation of motion of the system as a whole by means of canonical variables (2.10) leads on the quantum level to the decomposition of the Hilbert space $\mathcal{H}_N^F$ into the tensor product $\mathcal{H}_N^F=h_{int}\otimes\mathcal{H}_{ext}$, where "inner" and "external" spaces are realized, correspondingly, by functions $\psi(\eta)$, and $f(P_+)$ with the inner products

$$(f_1,f)=\frac{1}{2}\int_0^\infty\frac{dP_+}{P_+}f_1^*(P_+)f(P_+), \tag{4.10}$$

$$(\psi_1,\psi)=\int_{-1/2}^{1/2}\left(\prod_{k=1}^{N-1}\frac{d\eta_k}{1/2-2\eta_k^2}\right)\psi_1^*(\eta)\psi(\eta). \tag{4.11}$$

All the operators $\hat\lambda_j$ act only in h_{int}. When we pass from the functions ψ with inner product (4.11) to the functions

$$\varphi(\eta)=\psi(\eta)\prod_{b=1}^{N-1}(1/2-2\eta_b^2)^{-1/2} \tag{4.12}$$

with the inner product

$$(\varphi_1,\varphi)=\int_{-1/2}^{1/2}\varphi^*(\eta)\varphi(\eta)\prod_{a=1}^{N-1}d\eta_a\ , \tag{4.13}$$

then classical functions ${}^{\mathcal{F}_1}Z_k$ depending only on the inner variables are transformed into the following integral operators [19]

$$(\hat Z\varphi)(\eta)=\int_{-1/2}^{1/2}W(\eta,\eta')\varphi(\eta')\prod_{n=1}^{N-1}d\eta_n, \tag{4.14}$$

with the kernel

$$W(\eta,\eta')=\left(\prod_{d=1}^{N-1}[(1/2+\eta_d')(1/2+\eta_d)]^{(d-1)/2}\mathcal{D}_{d+1}^{-1}\right)\times$$

$$\times\int_{-\infty}^{\infty}{}^{\mathcal{F}_1}Z_k\ (\tilde q,\tilde\eta)exp\left(i\sum_{a=1}^{N-1}\tilde q_a(\eta_a-\eta_a')Y_a\right)\prod_{b=1}^{N-1}\frac{d\tilde q_b}{\pi} \tag{4.15}$$

where

$$\mathcal{D}_a=\prod_{j=a}^{N}(1/2+\eta_j)+\prod_{j=a}^{N}(1/2+\eta_j'), \tag{4.16}$$

$$Y_a=4\mathcal{D}_{a+1}^{-2}\prod_{j=a+1}^{N}(1/2+\eta_j)(1/2+\eta_j'). \tag{4.17}$$

The quantities $\tilde q,\tilde\eta$ have the form

$$\tilde q_a=\frac{\mathcal{D}_{a+1}}{2\mathcal{D}_a}\left(\frac{\mathcal{D}_a q_a}{\prod_{j=a+1}^{N-1}(1/2+\eta_j)}+\right.$$

$$\left.+\sum_{\nu=1}^{a-1}(\eta_\nu'-\eta_\nu)q_\nu\prod_{j=\nu+1}^{N-1}\frac{1/2+\eta_j'}{1/2+\eta_j}\right), \tag{4.18}$$

$$\tilde\eta_a=\mathcal{D}_{a+1}^{-1}\left(\eta_a\prod_{j=a+1}^{N}(1/2+\eta_j)+\eta_a'\prod_{j=a+1}^{N}(1/2+\eta_j')\right). \tag{4.19}$$

Substituting the expressions for the integrals ${}^{\mathcal{F}_1}\lambda_j^2$ into kernel (4.15) we obtain expression for the operators $\hat\lambda_j^2$:

$$\hat\lambda_j^2=\sum_{k=1}^{j-1}\frac{m_k^2}{1/2-\eta_{k-1}}\prod_{i=k}^{j-1}(1/2+\eta_i)^{-1}+\frac{m_j^2}{1/2-\eta_{j-1}}-$$

$$\sum_{k=1}^{j-2}\frac{\omega^2}{\prod_{i=k+1}^{j-1}(1/2+\eta_i)}\left[(\frac{1}{4}-\eta_k^2)\frac{\partial^2}{\partial\eta_k^2}-\Big(i(1-k)\mathcal{F}_1'(0)+\right.$$

$$\left.2\Big(i(1+k)\mathcal{F}_1'(0)+1\Big)\eta_k\Big)\frac{\partial}{\partial\eta_k}-i(1+k)\mathcal{F}_1'(0)-\frac{1}{4}\right]-$$

$$-\omega^2\left[(1/4-\eta_{j-1}^2)\frac{\partial^2}{\partial\eta_{j-1}^2}-\left(i(2-j)\mathcal{F}_1'(0)+\right.\right.$$
$$\left.+2\Big(i\mathcal{F}_1'(0)j+1\Big)\eta_{j-1}\right)\frac{\partial}{\partial\eta_{j-1}}-$$
$$\left.-i\mathcal{F}_1'(0)j-\frac{j}{4}+j(j-1)\mathcal{F}_1''(0)\right] \qquad (4.20)$$

and boundary conditions

$$\lim_{\eta_j\to\pm1/2}(1/4-\eta_j^2)\frac{\partial\varphi_j}{\partial\eta_j}=\lim_{\eta_j\to\pm1/2}\varphi_j(\eta_j)=0, \qquad (4.21)$$

which ensure the hermiticity of (4.20). The operators $\hat{\lambda}_j^2$ can be determined by means of the following recursive relations

$$\hat{\lambda}_j^2+\omega^2\left(j(j-1)\mathcal{F}_1''(0)-\frac{j-1}{4}\right)=$$
$$\frac{\hat{\lambda}_{j-1}^2+\omega^2\left((j-1)(j-2)\mathcal{F}_1''(0)-\frac{j-2}{4}\right)}{1/2+\eta_{j-1}}+$$
$$+\frac{m_j}{1/2-\eta_j}-\omega^2\left[(1/4-\eta_{j-1}^2)\frac{\partial^2}{\partial\eta_{j-1}^2}-\left(i(2-j)\mathcal{F}_1'(0)+\right.\right.$$
$$\left.\left.+2\Big(i\mathcal{F}_1'(0)j+1\Big)\eta_{j-1}\right)\frac{\partial}{\partial\eta_{j-1}}-i\mathcal{F}_1'(0)j-\frac{1}{4}\right]. \qquad (4.22)$$

Putting $j=N$ we have the expression of the total mass operator $\hat{M}^2$. Operators (4.20) mutually commute

$$[\hat{\lambda}_j,\hat{\lambda}_k]=0 \qquad (4.23)$$

and therefore they have a common set of eigenfunctions. Thus, we see that the Weyl-type quantization rules $W_{\mathcal{F}_1}$ preserve additional (concerning to the Poincaré-invariance) symmetries which are responsible for the integrability of the system.

Let $\varphi(\eta)$ be an eigenfunction of $\hat{M}^2=\hat{\lambda}_N^2$. Putting $\varphi(\eta)=\prod_{i=1}^{N-1}\varphi_i(\eta_i)$ reduces the eigenvalue problem for the operators $\hat{\lambda}_j$ to the system of $N-1$ differential equation of the hypergeometric type

$$\left\{\frac{\lambda_{j-1}^2+\omega^2\left((j-1)(j-2)\mathcal{F}_1''(0)-(j-2)/4\right)}{1/2+\eta_{j-1}}+\right.$$

$$+\frac{m_j}{1/2-\eta_j}-\omega^2\left[(\frac{1}{4}-\eta_{j-1}^2)\frac{\partial^2}{\partial\eta_{j-1}^2}-\right.$$
$$-\left(i(2-j)\mathcal{F}_1'(0)+2\Big(i\mathcal{F}_1'(0)j+1\Big)\eta_{j-1}\right)\frac{\partial}{\partial\eta_{j-1}}-$$
$$\left.\left.-i\mathcal{F}_1'(0)j-\frac{j}{4}+j(j-1)\mathcal{F}_1''(0)-\frac{\lambda_j^2}{\omega^2}\right]\right\}\varphi_i(\eta_i)=0, \qquad (4.24)$$

where $\prod_{i=1}^{j-1}\varphi_i(\eta_i)$ is an eigenfunction and λ_j^2 is an eigenvalue of the operator $\hat{\lambda}_j^2$ and λ_{j-1}^2 is an eigenvalue of $\hat{\lambda}_{j-1}^2$. Solving system (4.24) and taking into account boundary conditions (4.21) we find the functions

$$\varphi_{n_j}(\eta_j)=C_{n_j}(1/2-\eta_j)^{a_j}(1/2+\eta_j)^{b_j}P_{n_j}^{(2a_j,2b_j)}(2\eta_j), \qquad (4.25)$$

and the eigenvalues of the quantum integrals $\hat{\lambda}_j$

$$\lambda_j^2=\omega^2\left(a_0+\sum_{k=1}^{j}(a_k+n_k+1/2)\right)^2+$$
$$\omega^2\left[(j-1)\left(\frac{1}{4}-j\mathcal{F}_1''(0)\right)+(j\mathcal{F}_1'(0))^2\right]. \qquad (4.26)$$

Here

$$a_j=\sqrt{\frac{m_{j+1}^2}{\omega^2}-(\mathcal{F}_1'(0))^2},$$
$$b_j=a_0+\sum_{k=1}^{j-1}(a_k+n_k+1/2), \qquad (4.27)$$

n_k are integers: $n_k=0,1,2,...$, and $P_{n_j}^{(2a_j,2b_j)}(2\eta_j)$ are Jacobi polynomials. The constants C_{n_j} are determined by the equality:

$$|C_{n_j}|^2=\frac{n_j!(2n_j+1+2a_j+2b_j)\Gamma(n_j+1+2a_j+2b_j)}{\Gamma(n_j+1+2a_j)\Gamma(n_j+1+2b_j)}. \qquad (4.28)$$

Thus, the mass spectrum of the system is

$$M_n^2=\left[\sum_{a=1}^{N}\sqrt{m_a^2-(\omega\mathcal{F}_1'(0))^2}+\omega\sum_{b=1}^{N-1}(n_b+1/2)\right]^2+$$
$$+\omega^2\left[(N-1)\left(\frac{1}{4}-N\mathcal{F}_1''(0)\right)+(N\mathcal{F}_1'(0))^2\right]. \qquad (4.29)$$

The discrete spectrum exists only for real a_j. This leads to the inequality

$$\omega|\mathcal{F}_1'(0)| \le \min\{m_a\}, a=\overline{1,N}, \tag{4.30}$$

which gives additional restriction for the type of quantization rules $W_{\mathcal{F}_1}$. In the two-particle case mass spectrum (4.29) coincides with the result of Ref.[8] where the ambiguity in the quantization procedure of the two-particle oscillator-like interaction (4.1) have been considered within framework of purely algebraic method and two classes of ordering rules without specifying any representation.

We see that the mass spectrum depends essentially on the choice of quantization rule. In the case $\mathcal{F}_1 = 1$ we come to the spectrum of the system with the interaction (4.1) which has been obtained by the original Weyl quantization in Ref.[19]. In this work the generalization of the pure oscillator-like interaction has been considered too. This new interaction function contains also the terms which are linear in the coordinates:

$$V \to \tilde{V} = V + \alpha \sum\sum_{a<b} r_{ab}(p_a - p_b). \tag{4.31}$$

The original Weyl quantization gives the following result (see Ref.[19]):

$$M_n^2 = \left[\sum_{a=1}^{N}\sqrt{m_a^2 - \frac{\alpha^2}{4\omega^2}} + \omega\sum_{b=1}^{N-1}(n_b+1/2)\right]^2 + \\ + \frac{N-1}{4}\omega^2 + \frac{\alpha^2 N^2}{4\omega^2}. \tag{4.32}$$

Comparing the equalities (4.29), (4.32) we see that the quantizations $W_{\mathcal{F}_1}$, $\mathcal{F}_1'(0) \neq 0$, $\mathcal{F}_1''(0) = 0$ of the classical system with the pure oscillator-like interaction (4.1) gives the terms in the expression for mass spectrum (4.29) which we can treat as a presence of the linear interaction with

$$\alpha = -2\omega^2\mathcal{F}_1'(0). \tag{4.33}$$

Then such a quantum system is equivalent to those which is obtained from the classical system with the interaction (4.31) by means of the original Weyl quantization. Thus, the use of different quantization rules may lead to essentially different quantum results. Moreover different quantizations may lead to quantum systems with physically different interactions!

In the nonrelativistic case all the ambiguities in the mass spectrum (4.29) vanish and we obtain well known energy spectrum of nonrelativistic system with the oscillator interaction. But the first relativistic correction to the nonrelativistic energy depends on the type of quantization:

$$E \approx \hbar\omega\sum_{b=1}^{N-1}(n_b+1/2) + \\ + \frac{\hbar^2\omega^2}{2c^2}\left\{\frac{1}{m}\left(\sum_{b=1}^{N-1}(n_b+1/2)\right)^2 - \left(\mathcal{F}_1'(0)\right)^2\sum_{a=1}^{N}\frac{1}{m_a} + \right. \\ \left. + \frac{1}{m}\left[(N-1)\left(\frac{1}{4} - N\mathcal{F}_1''(0)\right) + \left(N\mathcal{F}_1'(0)\right)^2\right]\right\}. \tag{4.34}$$

Here we renewed constants $\hbar$, c.

Let us note that for the quantization of the oscillator-like interaction we have used only quantizations preserving the commutability of the diagram (3.24). Using the quantization rules $W_{\mathcal{F}}$ (3.16), which preserve only the commutation relations of the Poincaré algebra $\mathfrak{p}(1,1)$, we could obtain more ambiguous results for the mass spectrum.

IV. Conclusions

We have considered the problem of construction of a unitary representation of the group $\mathcal{P}(1,1)$ by means of quantization of the classical canonical realization of the Poincaré algebra corresponding to N-particle relativistic system with an interaction in the two-dimensional space-time $\mathbb{M}_2$ in the front form of dynamics.

The Lie algebra of the Poincré group $\mathcal{P}(1,1)$ has three generators: P_+, P_-, K. Two of them, namely, P_+ , P_- belong to the comutative ideal $\mathfrak{h}$. These two generators determine the square of classical total mass function $M^2 = P_+P_-$ and the Hamiltonian (evolution generator) $H = \frac{1}{2}(P_+ + M^2/\hat{P}_+)$. The structure of the Lie algebra of $\mathcal{P}(1,1)$ permits to reduce the quantization problem to the quantization of classical generator P_-. This generator is the only one which contains an interaction. On the quantum level, operators belonging to the ideal $\mathfrak{h}$ generate the evolution and the mass spectrum of the system via Eqs (3.38), (3.40).

For the construction of a unitary realization of the group $\mathcal{P}(1,1)$ we have applied only Weyl-type quantization rules (3.2). It has been demonstrated that the requirement of preservation of the Lie algebra $\mathfrak{p}(1,1)$ restricts the set of quantization rules but does not by itself remove the ambiguity of the quantization procedure.

In the classical case the square of total mass function M^2 is the invariant of the group $\mathcal{P}(1,1)$. Thus, to obtain in the quantum case the algebraic structure which is most closely related to the classical one, the quantum Kasimir operator $\hat{M}^2 = \hat{P}_+\hat{P}_-$ must be the quantization result of the classical expression $M^2 = P_+P_-$. This additional requirement imposes additional restriction on the family of the Weyl-type quantization rules. But it does not destroy the ambiguity of the quantization either.

We have also demonstrated that the Weyl-type quantization rules are split into equivalence classes. Quantization rules from the same equivalence class lead to the same realization of the ideal $\mathfrak{h}$ and therefore give the same mass spectrum and the evolution of quantized system. They lead to equivalent unitary representations of the group $\mathcal{P}(1,1)$. The quantizations which belong to different classes lead to non-equivalent unitary representation of $\mathcal{P}(1,1)$ and give different mass spectra. We have demonstrated this fact by the example of the N-particle system with the oscillator-like interaction. As it has been shown, in the expressions of the mass spectrum (4.29) there appear the terms which one can treat as the presence of the additional linear interaction. Thus, the choice of different quantizations may change the type of interaction and lead to quantum systems with physically different interactions. This unexpected result means that if we start with the classical description of a mechanical system then quantization rule seems to be the essential part of the definition of the corresponding quantum system.

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