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The Hamilton – Jakobi method for the classical mechanics in Grassmann algebra.

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Метод Гамільтона – Якобі для класичної механіки в алгебрі Грасмана.

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Анотація. В роботі розглядається класична механіка в алгебрі Грасмана. Формулюється метод Гамільтона – Якобі, приведено доведення теореми Якобі у випадку вироджених теорій в алгебрі Грасмана. Розглядаються приклади використання методу Гамільтона – Якобі. Приведено розв'язок класичної системи, що описується суперсиметричним Лагранжіаном.

The Hamilton – Jakobi method for the classical mechanics in Grassmann algebra .

Tabunshchyk K.V.

Abstract. The classical mechanics in Grassmann algebra is investigated in the present work. The Hamilton – Jakobi method is formulated and the Jakobi theorem is proved for the degenerated theories in Grassmann algebra. Examples of using of the Hamilton – Jakobi method is considered. The solution of the classical system characterized by the SUSY Lagrangian is given.

1. Introduction.

It is well known that in the quantum field theory the elementary particles are considered as bosons or fermions. However, the quantization of variables which correspond to bosons and fermions is different. The first ones are quantized by commutator, whereas the fermion variables are treating by anticommutator. Early in the development of the quantum field theory, the variables were treated as the classical variables, so that they were considered as commutating functions of the coordinates and the time. In that approach for the Fermi statistic case it was not possible to establish a correspondence between classical and quantum equations of motion. That is why it was clear that the quantum theory for fermions one must build up by quantization of the classical field theory, taking into account that the variables of this theory anticommutate. In other words, this variables should be the generators of Grassmann algebra.

In the present work we consider the classical mechanics in Grassmann algebra. The Hamilton – Jakobi method is formulated and the Jakobi theorem is proved for the degenerated theory in Grassmann algebra. Examples of using of the Hamilton – Jakobi method is considered.

We also consider a classical counterpart of Witten's model.

We assume that the state of mechanical system is described by n ordinary bosonic degrees of freedom $q_1, q_2, \ldots, q_n$ and m fermionic degrees of freedom $\psi_1, \psi_2, \ldots, \psi_m$ which satisfy the following relations

$$\begin{aligned} &q_i q_j - q_j q_i = 0\,, \\ &q_i \psi_j - \psi_j q_i = 0\,, \\ &\psi_i \psi_j + \psi_j \psi_i = 0\,. \end{aligned} \tag{1.1}$$

Thus the set q are the even Grassmann numbers, ψ are the odd Grassmann numbers.

The Lagrange function for the mechanical system depends on the commuting coordinates q, anticommuting coordinates ψ and on the derivatives of the coordinates with respect to time t

$$L = L(q, \psi, \dot q, \dot\psi, t)\,.$$

We assume in addition that the Lagrangian is even function of Grassmann space and further we use the left derivatives with respect to the Grassmann numbers.

Lagrangian which characterized the classical mechanics in Grassmann algebra is proportional to the first power of odd velocity, therefore we shall consider this system within degeneration theory.

We would like to remind some main aspects of this theory.

Theory is degenerated when the matrix of Hessian M_{ab} is degenerated. This means, that the equations of motion (for example let us consider only even Grassmann variable) which we can write in the form

$$M_{ab}\ddot q^b = \frac{\partial L}{\partial q^a} - \frac{\partial^2 L}{\partial \dot q^a \partial q^c}\dot q^c - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot q^a}\right) = K^a(q, \dot q),$$

$$M_{ab} = \frac{\partial^2 L}{\partial \dot q^a \partial \dot q^b}, \qquad \det\|M\| = 0,$$

could not be solved for $\ddot q$ and therefore there is a possibility to exist of a functional arbitrarily for solution.

In the degenerated theory, in addition, constrains arise, i.e. the variables q, P_q satisfy the system of equations $F_\alpha(q, P_q) = 0$.

The equations of motion can be obtained from the variational principle $\delta S = 0$. Here action is defined as

$$S = \int \left[\dot q^a P_q^a - H(q, P_X, t) + \lambda F\right] dt\,, \tag{1.2}$$

$$H(q, P_X, t) = \left(\frac{\partial L}{\partial \dot q^a}\dot q^a - L\right)_{\dot X = \bar{\dot X}},$$

where $\lambda(t)$ is a certain Lagrange coefficients for the constrains $F = 0$ at the moment of time t.

Here we used the notation

$$X^i = q^i\,, \quad i = 1, \ldots, R\,, \qquad x^j = q^{j+R}\,, \quad j = 1, \ldots, n-R\,,$$

$$\det\left\|\frac{\partial^2 L}{\partial \dot X \partial \dot X}\right\| \neq 0, \qquad \operatorname{rank}\left\|\frac{\partial^2 L}{\partial \dot q^a \partial \dot q^b}\right\| = R\,.$$

And P_X is the canonical momenta conjugate to the variables X. $\bar{\dot X}(q, P_X, \dot x)$ is the solution of the next equation $P_X = \dfrac{\partial L}{\partial \dot X}$ relative to the variables $\dot X$.

Lagrange coefficients may be found from the equations of motion and from the time–independence condition

2. The Hamilton – Jakobi equation. Jakobi theorem.

Let us consider the action as a function of the top limits of integration when the equations of motion is satisfied

$$\begin{aligned} \delta S = \delta \int \left[\dot q P_q + \dot\psi P_\psi - H(q, \psi, P_X, P_\Psi, t) + \lambda F\right] dt = \\ = \delta q \cdot P_q + \delta\psi \cdot P_\psi\,. \end{aligned} \tag{2.1}$$

Therefore we obtained that

$$\frac{\partial S}{\partial q} = P_q\,;\ \frac{\partial S}{\partial \psi} = P_\psi\,. \tag{2.2}$$

From the definition of action (in the case of the second–class constrains theory) we have

$$\begin{aligned} \frac{dS}{dt} &= \dot q\cdot P_q + \dot\psi\cdot P_\psi - \tilde H\big(q,\psi,P_q,P_\psi,\lambda(q,\psi,P_q,P_\psi),t\big) = \\ &= \dot q\cdot\frac{\partial S}{\partial q} + \dot\psi\cdot\frac{\partial S}{\partial\psi} - \tilde H\big(q,\psi,P_q,P_\psi,\lambda(q,\psi,P_q,P_\psi),t\big) \Rightarrow \\ &\Rightarrow \frac{\partial S}{\partial t} = -\tilde H\left(q,\psi,\frac{\partial S}{\partial q},\frac{\partial S}{\partial\psi},\lambda\big(q,\psi,\frac{\partial S}{\partial q},\frac{\partial S}{\partial\psi}\big),t\right). \end{aligned} \tag{2.3}$$

where we took into account that equation of motion is satisfied and found the Lagrange coefficients as a function of q,ψ,P_q,P_ψ from the equations of motion.

$$\tilde H\big(q,\psi,P_q,P_\psi,\lambda(q,\psi,P_q,P_\psi),t\big) = \big(H(q,\psi,P_X,P_\Psi,t)-\lambda F\big)\Big|_{\lambda(q,\psi,P_q,P_\psi)}$$

In the case of the first–class constrains theory we can not find all Lagrange coefficients that is why our equation have the next form

$$\frac{\partial S}{\partial t} = -\tilde H\left(q,\psi,\frac{\partial S}{\partial q},\frac{\partial S}{\partial\psi},\lambda\big(q,\psi,\frac{\partial S}{\partial q},\frac{\partial S}{\partial\psi}\big),\lambda^\alpha,t\right),$$

where λ^α are those coefficients which can not be found.

However we may put a guage for our theory (i.e. transform our degenerate theory with first–class constrains to the physical equivalent theory which is not degenerated or to the physical equivalent theory with second–class constrains).

As example we can take the strong minimal guage which does not shift the equations of motion (the so – called canonical guage $G^{(c)}$).

Consider now the action as a function of the top and bottom limits of integration when the equations of motion is satisfied.

$$dS = -\tilde H_2 dt^{(2)} + dq^{(2)}P_q^{(2)} + d\psi^{(2)}P_\psi^{(2)} - \big(-\tilde H_1 dt^{(1)} + dq^{(1)}P_q^{(1)} + d\psi^{(1)}P_\psi^{(1)}\big).$$

Let us take $t^{(1)} = t$, $t^{(2)} = t+\tau$, $\tau = const$. Then

$$\begin{aligned} dS = &-\big(\tilde H(t+\tau) - \tilde H(t)\big)\cdot dt + \\ &+ dq(t+\tau)\cdot P_q(t+\tau) + d\psi(t+\tau)\cdot P_\psi(t+\tau) - \\ &- dq(t)\cdot P_q(t) - d\psi(t)\cdot P_\psi(t)\,. \end{aligned} \tag{2.4}$$

Thus we obtain

$$\begin{aligned} &-\big(\tilde H(t+\tau) - \tilde H(\tau)\big) = \frac{\partial S}{\partial t}\,; \\ &P_q(t+\tau) = \frac{\partial S}{\partial q(t+\tau)}\,; \qquad P_\psi(t+\tau) = \frac{\partial S}{\partial \psi(t+\tau)}\,; \\ &P_q(t) = -\frac{\partial S}{\partial q(t)}\,; \qquad P_\psi(t) = -\frac{\partial S}{\partial \psi(t)}\,. \end{aligned} \tag{2.5}$$

These formulae (2.5) we may treat as a canonical transformation between the old variables at time t and new variables at time $t+\tau$. The action is the creation function. Therefore the motion for our system we may treat as a canonical transformation.

Jakobi theorem.

Let us consider a full solution of the Hamilton – Jakobi equation $S = S_r(q,\psi,\alpha,\beta,t)$ where α is a set of even Grassmann constant, β is a set of odd Grassmann constant.

Doing the canonical transformation from old variables q, ψ, P_q, P_ψ to the new ones and taking S_r as creation function.

Let us put $\alpha = P_Q$, $\beta = P_\nu$ as new canonical momenta; Q, ν as new coordinates. Then

$$\begin{aligned} &H' = H + \frac{\partial S_r}{\partial t}\,; \\ &P_q = \frac{\partial S_r}{\partial q}\,; \qquad P_\psi = \frac{\partial S_r}{\partial \psi}\,; \\ &Q = \frac{\partial S_r}{\partial P_Q}\,; \qquad \nu = -\frac{\partial S_r}{\partial P_\nu}\,. \end{aligned} \tag{2.6}$$

Since S_r is the solution of Hamiltom – Jakobi equation, we obtain

$$\begin{aligned} &H' = 0 \ \Rightarrow \\ &\Rightarrow \begin{matrix} \dot P_Q = 0\,, & \dot Q = 0\,, \\ \dot P_\nu = 0\,, & \dot\nu = 0\,, \end{matrix} \quad \Rightarrow \quad \begin{matrix} P_Q = \text{const}\,, & Q = \text{const}\,, \\ P_\nu = \text{const}\,, & \nu = \text{const}\,. \end{matrix} \end{aligned} \tag{2.7}$$

From the last relation we can write

$$\begin{aligned} \frac{\partial S_r}{\partial\alpha} &= \text{ const (which is an even Grassmann number)}\,; \\ \frac{\partial S_r}{\partial\beta} &= \text{ const (which is an odd Grassmann number)}\,. \end{aligned} \tag{2.8}$$

3. Fermi oscillator.

As a simple example, let us consider now the next Lagrangian which describes the so–called free Fermi oscillator

$$L = i \cdot \bar{\psi}\dot{\psi} - \bar{\psi}\psi \,, \tag{3.1}$$

where the dot denotes the derivative with respect to t. Here ψ and $\bar{\psi}$ are odd Grassmann numbers. The overbar denotes the Grassmann variant of complex conjugation. These classical models may be viewed as the classical limits of models with one fermionic degree of freedom.
For our system the canonical momenta has the form:

$$\begin{cases} P_{\bar{\psi}} = \dfrac{\partial L}{\partial \dot{\bar{\psi}}} = 0 \,, \\[2ex] P_{\psi} = \dfrac{\partial L}{\partial \dot{\psi}} = -i \cdot \bar{\psi} \,. \end{cases} \tag{3.2}$$

And thus we have two constrains

$$\begin{cases} F_1 = P_{\bar{\psi}} \,, \\[2ex] F_2 = P_{\psi} + i \cdot \bar{\psi} \,. \end{cases} \tag{3.3}$$

The initial Hamiltonian can be written in the form

$$\begin{aligned} H_\lambda = \dot{\psi} \cdot \frac{\partial L}{\partial \dot{\psi}} + \dot{\bar{\psi}} \cdot \frac{\partial L}{\partial \dot{\bar{\psi}}} - L + \lambda_1 F_1 + \lambda_2 F_2 = \\ = \bar{\psi}\psi + \lambda_1 P_{\bar{\psi}} + \lambda_2 (P_\psi + i \cdot \bar{\psi}) \,. \end{aligned} \tag{3.4}$$

The Lagrange coefficient may be found from the next relation

$$\begin{cases} \dot{F}_1 = \{H_\lambda \,, F_1\} = -\psi + i\lambda_2 = 0 \Rightarrow \quad \lambda_2 = -i \cdot \psi \,, \\[2ex] \dot{F}_2 = \{H_\lambda \,, F_2\} = \bar{\psi} + i\lambda_1 = 0 \Rightarrow \quad \lambda_1 = i \cdot \bar{\psi} \,. \end{cases} \tag{3.5}$$

These relations mean that the constrains are time–independent.

Therefore we have a system which consists of the second–class constrains. Really, the next matrix is not degenerated:

$$\det \| \{F \,, F\} \| \neq 0 \,.$$

Then the equations of motion in the Hamiltonian representation can be written as

$$\begin{cases} \dot{\psi} = \{H_f \,, \psi\} \,, \; F_1 = 0, \\[2ex] \dot{\bar{\psi}} = \{H_f \,, \bar{\psi}\} \,, \; F_2 = 0. \end{cases} \tag{3.6}$$

where the Hamiltonian

$$H_f = \tilde{H} = H_\lambda \Big|_{\lambda(\psi,\bar{\psi})} = iP_\psi \cdot \psi - iP_{\bar{\psi}} \cdot \bar{\psi} \,, \tag{3.7}$$

is obtained by the substitution the Lagrange coefficient (3.5) in the initial Hamiltonian (3.4).

The classical equations of motion (which can be derived also from the Lagrange equations), reads

$$\begin{cases} \dot{\psi} = -i \cdot \psi \;\Rightarrow \quad \psi = \psi_o \cdot \exp(-it) \,, \\[2ex] \dot{\bar{\psi}} = i \cdot \bar{\psi} \;\Rightarrow \quad \bar{\psi} = \bar{\psi}_o \cdot \exp(it) \,, \end{cases} \tag{3.8}$$

where ψ_o, $\bar{\psi}_o$ are the odd Grassmann constants of integration.

Let us make an ansatz for the action

$$S(t,\psi,\bar{\psi}) = S_o(t) + \psi \cdot S_1(t) + \bar{\psi} \cdot S_2(t) + \psi\bar{\psi} \cdot S_3(t) \,, \tag{3.9}$$

and after the substitution our ansatz (3.9) in to the Hamilton – Jakobi equation

$$\frac{\partial S}{\partial t} + i \frac{\partial S}{\partial \psi} \cdot \psi - i \frac{\partial S}{\partial \bar{\psi}} \cdot \bar{\psi} = 0 \,,$$

we obtained the next system of equations

$$\begin{cases} \dfrac{\partial S_o}{\partial t} = 0 \,, \\[2ex] \dfrac{\partial S_1}{\partial t} = iS_1 \,, \\[2ex] \dfrac{\partial S_2}{\partial t} = -iS_2 \,, \\[2ex] \dfrac{\partial S_3}{\partial t} = 0 \,. \end{cases} \tag{3.10}$$

The solution of this system can be written as

$$S(t,\psi,\bar{\psi}) = c_1 + \psi \cdot \theta_1 e^{it} + \bar{\psi} \cdot \theta_2 e^{-it} + \psi\bar{\psi} \cdot c_2 \,. \tag{3.11}$$

Here c_1, c_2 are even Grassmann numbers and θ_1, θ_2 are odd Grassmann numbers.

Thus we have the action and now we can use the Jakobi theorem

$$\frac{\partial S}{\partial \theta_1} = -\psi \cdot e^{it} = -\psi_o = const \;\Rightarrow\; \psi = \psi_o \cdot \exp(-it)\,,$$

$$\frac{\partial S}{\partial \theta_2} = -\bar{\psi} \cdot e^{it} = -\bar{\psi}_o = const \;\Rightarrow\; \bar{\psi} = \bar{\psi}_o \cdot \exp(-it)\,,$$

$$\frac{\partial S}{\partial c_2} = -\bar{\psi}\psi = const\,.$$

The canonical momenta we shall find from the relation

$$P_\psi = \frac{\partial S}{\partial \psi} \;\;,\;\; P_{\bar{\psi}} = \frac{\partial S}{\partial \bar{\psi}} \;\;.$$

4. Supersymmetric classical mechanics.

Classical supersymmetric model forms a subclass of pseudoclassical mechanics, the notion originally introduced by Casalbuoni. Pseudoclassical mechanics deals with the classical systems which are described in term of Grassmann variables rather than the usual Cartesian variables.

The model which we consider now is characterized by the following Lagrangian

$$L = \frac{\dot{q}^2}{2} - \frac{1}{2}V^2(q) - \frac{i}{2}(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi}) - U(q)\cdot\bar{\psi}\psi\,. \tag{4.1}$$

In the above q denotes a bosonic degree of freedom and hence is an even Grassmann number. In contrast to this ψ and $\bar{\psi}$ denote fermionic degrees of freedom and, therefore, are odd Grassmann numbers, The real–valued function V is the so–called superpotential then $U(q) = V'(q)$. Starting from (4.1) we can write the canonical momenta in form

$$\begin{cases} P_q = \dfrac{\partial L}{\partial \dot{q}} = \dot{q}\,, \\[2ex] P_{\bar{\psi}} = \dfrac{\partial L}{\partial \dot{\bar{\psi}}} = -\dfrac{i}{2}\cdot\psi\,, \\[2ex] P_\psi = \dfrac{\partial L}{\partial \dot{\psi}} = -\dfrac{i}{2}\cdot\bar{\psi}\,. \end{cases} \tag{4.2}$$

The momenta conjugate to the fermionic variables do not depend on $\dot{\psi}$ and $\dot{\bar{\psi}}$. Hence, the system is subject to the second–class constrains,

$$\begin{cases} F_1 = P_\psi + \frac{i}{2}\cdot\bar{\psi}\,. \\[2ex] F_2 = P_{\bar{\psi}} + \frac{i}{2}\cdot\psi\,, \end{cases} \tag{4.3}$$

which have a non–vanishing Poisson bracket $\{F_1, F_2\} \neq 0$.

The initial Hamiltonian can be written in the form

$$H_\lambda = \dot{q}\frac{\partial L}{\partial \dot{q}} + \dot{\psi}\cdot\frac{\partial L}{\partial \dot{\psi}} + \dot{\bar{\psi}}\cdot\frac{\partial L}{\partial \dot{\bar{\psi}}} - L + \lambda_1 F_1 + \lambda_2 F_2 = \tag{4.4}$$

$$= \frac{P_q^2}{2} + \frac{1}{2}V^2(q) + U(q)\cdot\bar{\psi}\psi + \lambda_1(P_\psi + \frac{i}{2}\cdot\bar{\psi}) + \lambda_2(P_{\bar{\psi}} + \frac{i}{2}\cdot\psi)\,.$$

The Lagrange coefficient may be found from the next relation

$$\begin{cases} \dot{F}_1 = \{H_\lambda\,,\,F_1\} = 0 \;\Rightarrow\; \lambda_2 = -iU(q)\cdot\psi\,, \\[2ex] \dot{F}_2 = \{H_\lambda\,,\,F_2\} = 0 \;\Rightarrow\; \lambda_1 = iU(q)\cdot\bar{\psi}\,. \end{cases} \tag{4.5}$$

These relations mean that the constrains are time–independent.

Therefore we have a system which consists of the second–class constrains.

$$\det \|\{F\,,\,F\}\| \neq 0\,.$$

We can, finally, written the Hamiltonian of our system in the form

$$H_f = \tilde{H} = H_\lambda\Big|_{\lambda(q,\psi,\bar{\psi})} = \frac{P_q^2}{2} + \frac{1}{2}V^2(q) - iU(q)P_{\bar{\psi}}\cdot\bar{\psi} + iU(q)P_\psi\cdot\psi\,. \tag{4.6}$$

The classical equations of motion, which can be derived from the Hamiltonian (4.6), read

$$\begin{cases} \dot{\psi} = \{H_f\,,\,\psi\} = -iU(q)\cdot\psi\,,\; F_1 = 0, \\[2ex] \dot{\bar{\psi}} = \{H_f\,,\,\bar{\psi}\} = iU(q)\cdot\bar{\psi}\,,\; F_2 = 0. \\[2ex] \ddot{q} + V(q)V'(q) + U'(q)\cdot\bar{\psi}\psi = 0, \end{cases} \tag{4.7}$$

where the prime and the dot denote the derivative with respect to x and t, respectively. The first – order differential equation for the fermionic degree of freedom can be presented in the next form (see.[5])

$$q(t) = x_{qc}(t) + q_o(t)\cdot\bar{\psi}_o\psi_o\,, \tag{4.8}$$

where $q_o(t)$ and $x_{qc}(t)$ are real – valued functions of time. Only for the special initial condition ($\psi = \bar{\psi} = 0$) we have that $q_o(t)$ and $x_{qc}(t)$ coincide.
The energy conservation law have the form

$$\frac{1}{2}\dot{q} + \frac{1}{2}V^2(q) + U(q)\cdot\bar{\psi}_o\psi_o = E + F\cdot\bar{\psi}_o\psi_o \quad , \quad (E, F \in \mathbb{R})$$

The first term from (4.8) can be found in quadratures from the equation

$$\dot{x}_{qc}^2 = 2E - V^2(x_{qc})\,. \tag{4.9}$$

The equation for the second term from (4.8)

$$\dot{q}_o(t) = \frac{1}{\dot{x}_{qc}(t)}\left[F - U(x_{qc}(t)) - V(x_{qc}(t))V'(x_{qc}(t))q_o(t))\right], \tag{4.10}$$

may be solvable by quadratures

$$q_o(t) = \frac{\dot{x}_{qc}(t)}{\dot{x}_{qc}(0)}\left[q_o(0) - \int\limits_0^t d\tau\frac{F - U(x_{qc}(\tau))}{2E - V^2(x_{qc}(\tau))}\right]. \tag{4.11}$$

Let us demonstrate the described above procedure (the Hamilton – Jakobi method) for the case of SUSY – system.

Starting from the obtained above Hamiltonian (4.6), the equation of Hamilton – Jakobi is written in the form

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}V^2(q) - iU(q)\frac{\partial S}{\partial\bar{\psi}}\cdot\bar{\psi} + iU(q)\frac{\partial S}{\partial\psi}\cdot\psi = 0 \tag{4.12}$$

Let us make an ansatz for the action

$$S(t,q,\psi,\bar{\psi}) = S_o(t,q) + \psi\bar{\psi}\cdot S_1(t,q) + \psi\cdot S_2(t,q) + \bar{\psi}\cdot S_3(t,q)\,, \tag{4.13}$$

where S_o and S_1 are even Grassmann functions and S_2 and S_3 are odd Grassmann functions.

After the substitution ansatz (4.13) in the (4.12) and decomposition on equation this one Grassmann parity we obtained the next system of

equations

$$\begin{cases} \dfrac{\partial S_o}{\partial t} + \frac{1}{2}\left(\dfrac{\partial S_o}{\partial q}\right)^2 + \frac{1}{2}V^2(q) = 0\,, \\ \dfrac{\partial S_2}{\partial t} + \dfrac{\partial S_o}{\partial q}\cdot\dfrac{\partial S_2}{\partial q} - iU(q)S_2 = 0\,, \\ \dfrac{\partial S_3}{\partial t} + \dfrac{\partial S_o}{\partial q}\cdot\dfrac{\partial S_3}{\partial q} + iU(q)S_3 = 0\,, \\ \dfrac{\partial S_1}{\partial t} + \dfrac{\partial S_o}{\partial q}\cdot\dfrac{\partial S_1}{\partial q} = 0\,, \\ \dfrac{\partial S_2}{\partial q}\cdot\dfrac{\partial S_3}{\partial q}\cdot\psi\bar{\psi} = 0 \end{cases} \tag{4.14}$$

The first equation can be integrated by using the method of decomposition of variable.

Thus we obtain

$$S_o = \int\sqrt{2E - V^2(q)}\,dq\ - Et\,, \tag{4.15}$$

where E is constant of integration.

Starting form the expression (4.15) for the fourth equation we obtained the following solution

$$S_1 = \int\frac{A\ dq}{\sqrt{2E - V^2(q)}}\ - At\,, \tag{4.16}$$

here, $(A, E \in \mathbb{R})$.

The solutions for second and third equation may be written as

$$S_2 = \phi_1\left(\int\frac{dq}{\sqrt{2E - V^2(q)}}\ - t\right)\exp\left(i\int\frac{U(q)\,dq}{\sqrt{2E - V^2(q)}}\right)\ ; \tag{4.17}$$

$$S_3 = \phi_2\left(\int\frac{dq}{\sqrt{2E - V^2(q)}}\ - t\right)\exp\left(-i\int\frac{U(q)\,dq}{\sqrt{2E - V^2(q)}}\right)\ ;$$

where ϕ_1 and ϕ_2 is any odd Grassmann functions. For our account it is sufficiently to take that $\phi_1 = const$, $\phi_2 = const$.

The last equation is only drive to some conditions on function ϕ_1 and function ϕ_2, which is satisfies then the functions is constant.

Thus we can present the action in the next form

$$S = \int \sqrt{2E - V^2(q)}\, dq \; - Et +$$
$$+ \int \frac{A\; dq}{\sqrt{2E - V^2(q)}} \cdot \psi\bar{\psi} - At \cdot \psi\bar{\psi} + \tag{4.18}$$
$$+\psi \cdot \phi_1 \left(\int \frac{dq}{\sqrt{2E - V^2(q)}} - t \right) \exp\left(i \int \frac{U(q)\, dq}{\sqrt{2E - V^2(q)}} \right) +$$
$$+\bar{\psi} \cdot \phi_2 \left(\int \frac{dq}{\sqrt{2E - V^2(q)}} - t \right) \exp\left(-i \int \frac{U(q)\, dq}{\sqrt{2E - V^2(q)}} \right) .$$

Then we could used the Jakobi theorem

$$\frac{\partial S}{\partial \phi_1} = const \;, \; \frac{\partial S}{\partial \phi_2} = const \,. \tag{4.19}$$

The solution of this equation (4.19) gives

$$\psi = \psi_o \cdot \exp(-i \textstyle\int U(q(\tau))\, d\tau) \,,$$
$$\bar{\psi} = \bar{\psi}_o \cdot \exp(i \textstyle\int U(q(\tau))\, d\tau) \,. \tag{4.20}$$

For the bosonic degree of freedom we can take the next series

$$q(t) = x_{qc}(t) + q_o(t) \cdot \bar{\psi}\psi \,. \tag{4.21}$$

But, from (4.20) we obtain the relation

$$\bar{\psi}\psi = \bar{\psi}_o \psi_o \,.$$

Thus we have:

$$q(t) = x_{qc}(t) + q_o(t) \cdot \bar{\psi}_o \psi_o \,. \tag{4.22}$$

Then, from the Jakobi theorem we obtain

$$-\frac{\partial S}{\partial A} = \int \frac{dq}{\sqrt{2E - V^2(q)}} \cdot \bar{\psi}_o \psi_o - t \cdot \; \bar{\psi}_o \psi_o = const \,. \tag{4.23}$$

From (4.22) and (4.23) we can write

$$\int \frac{dx_{qc}}{\sqrt{2E - V^2(x_{qc})}} - t = const \,. \tag{4.24}$$

This relation may be rewritten in form (4.9)

$$\dot{x}_{qc}^2 = 2E - V^2(x_{qc}) \,.$$

Then if we calculate the derivative $\dfrac{\partial S}{\partial E} = const$; substitution in this relation our result and taken into account next expansions:

$$U(q) = U(x_{qc}) + U'(x_{qc}) q_o \cdot \bar{\psi}_o \psi_o \,,$$
$$V^2(q) = V^2(x_{qc}) + 2V'(x_{qc}) V(x_{qc}) q_o \cdot \bar{\psi}_o \psi_o \,,$$
$$f(V^2(q)) = f(V^2(x_{qc})) + f'(V^2(x_{qc})) 2V'(x_{qc}) V(x_{qc}) q_o \cdot \bar{\psi}_o \psi_o \,, \tag{4.25}$$

and using relation (4.24) we obtain

$$\int \frac{dq_o}{\sqrt{2E - V^2(x_{qc})}} = \tag{4.26}$$
$$= \int \frac{\left[A - U(x_{qc}(\tau)) - V(x_{qc}(\tau)) V'(x_{qc}(\tau)) q_0(\tau)\right]}{2E - V^2(x_{qc}(\tau))} d\tau + const \,. \tag{4.27}$$

From the last relation we have presented $q_o(t)$ in form

$$q_o(t) = \frac{\dot{x}_{qc}(t)}{\dot{x}_{qc}(0)} \left[q_o(0) - \int\limits_0^t d\tau \frac{F - U(x_{qc}(\tau))}{2E - V^2(x_{qc}(\tau))} \right] . \tag{4.28}$$

Thus we obtain the result, from the Hamilton – Jakobi method, which coincides with the result obtained from the Hamilton (or Lagrangian) equation of motion.

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