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R. Folk*, Yu. Holovatch

Critical fluctuations in normal-to-superconducting transition

*Institut für Theoretische Physik, Johannes Kepler Universität Linz *A-4040 Linz, Austria*

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Критичні флюктуації у фазовому переході з нормального в надпровідний стан

Р. Фольк, Ю. Головач

Анотація. Ми досліджуємо фазовий перехід у надпровідний стан беручи до уваги флюктуації параметра впорядкування та векторного магнітного поля і обсуджуємо питання про рід фазового переходу в такій моделі. Ми застосовуємо теоретико-польовий ренормгруповий підхід і розглядаємо калібрувальну модель надпровідника, узагальнену на випадок $n/2$ компонентного комплексного параметра впорядкування. Попередні ренормгрупові обчислення із безпосереднім ε-розкладом стверджували, що в такій моделі відбувається фазовий перехід першого роду. Ми досліджуємо вирази для ренормгрупових функцій в тривимірному просторі в двопетлевому наближенні. Особлива увага приділяється то́му, що відповідні ряди можуть бути асимптотичними і мати нульовий радіус збіжності. Ми розглядаємо різні шляхи аналітичного продовження рядів і, застосовуючи Паде-аналіз і техніку пересумовування Паде-Бореля робимо висновок про те, що в моделі існує можливість фазового переходу другого роду із критичними показниками, відмінними від показників надплинної рідини. Такий висновок узгоджується із результатами недавніх досліджень, виконаних без застосування теорії збурень.

Critical fluctuations in normal-to-superconducting transition

R. Folk, Yu. Holovatch

Abstract. We study the phase transition to the superconducting state taking into account the fluctuations of the order parameter and of the vector magnetic field and discuss the question of the order of the transition occuring in this model. We use the field-theoretical renormalization group approach and consider the gauge model for a superconductor, generalized to a $n/2$ component complex order parameter. Previuos renormalization group calculations within strict ε-expansion suggested that in such a model a first-order phase transition occurs. We examine expressions for the renormalization group functions in a two-loop approximation in three dimensions. Special attention is being payed to the fact, that the corresponding series might be asymptotic ones and therefore have zero radius of convergence. We review different ways of analytical continuation of the series and applying Padé analysis and Padé-Borel resummation technique conclude that in the model under consideration still exists a possibility for the second-order phase transition with critical exponents differing from those of a superfluid liquid. This is in agreement with conclusions made very recently in other nonperturbative treatments.

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1. Introduction

Successes achieved in our modern understanding of critical phenomena due the application [1] of the renormalization group (RG) approach [2] are by now well known and generally recognized (see e.g. the textbooks [3–5]). Scale invariance at the critical point and universality of certain features of critical phenomena found their reflection and explanation in the frames of RG transformation and led to the theory providing quantitative description of critical behaviour of different thermodynamic quantities of interest. In particular, the critical point (in the context of our lecture we specify it as an equilibrium second order phase transition point) in the RG "language" corresponds to the stable fixed point of the RG transformation, where the system is scale invariant. Asymptotic properties of the system are governed solely by the coordinate of the stable fixed point whereas non-asymptotic ones are defined in the regime of approach to the fixed point. Moreover, sometimes RG approach may give an answer about the order of phase transition occurring in a certain model. This is explored by studying the stability of the fixed points of RG transformation: an absence of a stable fixed point is interpreted as an evidence of fluctuation-induced first-order phase transition in the system under consideration. For the models with lack of exact solutions or rigorous proofs of the existence of second order phase transition (and this is the case for the majority of realistic models in statistical physics) RG provides a tool to check the order of transition.

Namely such kind of problem we are going to discuss in this lecture. It can be formulated as: *what is the order of normal-to-superconducting phase transition?* From the theoretical point of view according to Bardeen-Cooper-Schriefer theory of superconductivity the normal-to-superconducting (NS) phase transition is a classical second order phase transition described by the Landau-Ginsburg Hamiltonian with the complex order parameter, corresponding to the wave function of the Cooper pairs. Taking into account the fluctuations of the order parameter one can find the values of corresponding critical exponents which in this case will coincide with the critical exponents of $O(n)$ symmetrical field theoretical model for the case $n = 2$ (XY model). Consequently this leads to the answer that NS phase transition is described by the same set of critical exponents as the phase transition in normal-to-superfluid liquid. The last have been measured with high accuracy [6] and calculated by different methods [7–14]. However taking into account, that for the NS transition the corresponding "superfluid liquid" is charged, essentially complicates the problem. For the first time this question was considered

by B. I. Halperin, T. C. Lubensky and S. Ma [15] and since then different ways of tackling it were proposed. We briefly discuss some of them in the subsequent section.

From the experimental point of view when the question about the order of phase transition first was discussed for a superconductor it was more or less an academic question, since due to of the large correlation length $\xi_0 \sim 10^3 \mathring{A}$ only very near the phase transition the first order character can be seen, otherwise mean field behaviour is to be expected. This situation changed after the discovery of high-T$_c$ superconductors with correlation lengths within the range of the lattice distances ($\xi_0 \sim 1\mathring{A}$) [16]. Here in several experiments critical effects have been observed [17–24].

Our main result presented in this lecture is that being within the frames of the RG method applied to the original model of the superconductor minimally coupled to the gauge field [15] one still can obtain the answer that in such a model there occurs a *second order phase transition with the critical exponents distinct from those of a superfluid liquid.* To prove this we consider two-loop renormalization group functions for this model and pay special attention to the fact that the loop expansion is the asymptotic one [27–29]. In this way we find several fixed points with new scaling exponents and a rich crossover behavior. Some of our results were previously published in [30–32].

The setup of this lecture is as follows. In the next section we give a brief review of the main stages in studying the problem we are interested in. In section 3 we describe the model of a superconductor, give the results of its study in the mean field approximation as well as obtain the expressions for the renormalization group functions in a two loop approximation and describe the results obtained on their basis without applying any resummation procedure. In the section 4 we discuss several ways of resummation of asymptotic series which are applied in the modern theory of critical phenomena. We give the examples of their application in well-established cases as well. Section 5 is central in our account: it is devoted to a study of the RG functions and of the corresponding flows on the basis of the resummation technique. Here we obtain the result about presence of a stable fixed point and, correspondingly, about the evidence of the second order phase transition in the model under consideration. In sections 6 - 8 we calculate the asymptotic and effective values for the critical exponents and give the expressions for the amplitude ratios. The results are discussed in section 9.

2. Normal-to-superconducting transition: 1st or 2nd order?

The order of a phase transition may have severe consequences for physical quantities. So one knows from the first order liquid gas transition phenomena like overheating and undercooling connected with the metastability at the transition. For the second order phase transition divergencies in physical quantities occur (in the thermodynamic limit of course) leading to a dramatic increase of the specific heat or scattering of light (critical opalescence) near the liquid gas critical point.

Similar dramatic changes are connected with the phase transitions which occurred in the early stage of the universe and the questions discussed here for the superconductor are also relevant there (for a recent review see [33]).

As it was mentioned already in the Introduction the question of the order of the NS phase transition becomes complicated when one accounts that the fluctuations of the order parameter are coupled to the "gauge field" (the vector potential of the fluctuating magnetic field created by the Cooper pairs) whose fluctuations also diverge at long distances. Since posed for the first time more than 20 years ago [15] up to now this problem remains a challenging one in the physics of superconductivity. The theoretical model of Halperin, Lubensky, and Ma describing the relevant critical behaviour was the usual $O(n)$ symmetrical ϕ^4 model with the $n/2$-component complex field ϕ coupled to a gauge field describing the fluctuating magnetic field created by Cooper pairs. The answer obtained in [15] states that because of the coupling to the gauge field in mean field approximation a third order term appears in the free energy of the superconductor and the NS phase transition is of the first order. The mean filed analysis is appropriate for the type-I superconductors [34] where the fluctuations in the order parameter have no significant effect on the thermodynamics of transition. The case of a type-II superconductors is more complicated because there the fluctuations can not be neglected. Studying the problem within Wilson-Fisher recursion relations [35] in the first order of ε it was found [15] that a stable fixed point (necessary, but not sufficient for a second order phase transition) exists only for the order parameter components number $n > 365.9$, exceeding to a great extent the superconductor case $n = 2$. The crossover near the first order phase transition was studied [26] and the expression for the crossover function of the specific heat was given within one loop order perturbation theory. The kinetics of fluctuations arising from vortex pairs in a superconductor was studied by means of numerical simulations [36]. The

result lead to the conclusion about a nucleation process typical for the first order phase transition, confirmimg the mean field and RG results of [15]. Note, that the mean field analysis applied to the Ginzburg-Landau free energy of a superconductor [37] including a Chern-Simons term leads to quantitatively different behaviour: for different values of the topological mass in system occurs either a fluctuation-induced first order phase transition or only the second-order transition exists. This result is also confirmed in the frames of one-loop RG calculations [37].

The occurrence of the first order phase transition was found also in massless scalar electrodynamics [38,39] and confirmed in linear order in ε for the n-component Abelian Higgs models by explicit construction of the coexistence curve and the equation of state [40]. CP^{N-1} non-linear sigma model, being related to the model of superconductor in the limit of infinite charge by means of $2+\varepsilon$ expansion was shown [41] to posses behaviour similar to those, observed in [15] as well.

The results for type-II superconductors obtained in ε expansion [15, 26,40] appear to be stable against account of the influence of different physical factors such as a possibility of another (non-magnetic) ordering, presence of disorder and crystal anisotropy when the study is performed by means of strict ε-expansion. Scaling behaviour of a superconducting system with another (non-magnetic) ordering studied in ε-expansion provided one more example of a system where weak first order phase transition occurs [45]. The analysis of influence of quenched impurities on the critical behaviour of superconductors with account of magnetic field fluctuations resulted [46,47] in the answer about appearance of a new stable fixed point for $1 < n < 366$. However it was shown [47] that it describes critical behaviour in the range of space dimensionalities $d_c(n) < d < 4$ with $d_c(2) = 3.8$ and results in a first order phase transition. RG flow for the model of superconductor with quenched impurities was found [48] to exhibit a stable focus surrounded by an unstable limit cycle. The second order phase transition behaviour was found to show up inside the limit cycle. Introducing random fields with short and long range correlations has not lead to a second order behaviour in the region of (d, n) near $(3, 2)$ as well [49]. Note however, that studies of the influence of quenched and annealed gauge fields on the spontaneous symmetry breaking performed in terms of Helmholtz free energy [42] resulted in an answer that in the first nontrivial, or one-loop approximation in the annealed model spontaneous symmetry breaking occurs through a first order transition for $d = 2, 3$ whereas the quenched model displays a continuous phase transition. A more complicated account of fluctuations in the annealed model changes the nature of the transition to a continuous one, whereas spontaneous symmetry breaking is then absent in the model with quenched disorder [42]. The combined influence of crystal anisotropy, magnetic fluctuations, and quenched randomness on the critical behaviour of unconventional superconductors [50] studied by means of the RG analysis within the ε-expansion [51] resulted in the conclusion that only fluctuation-induced first order transitions should occur in unconventional superconductors in the vicinity of the critical point.

However the mean field results where questioned by an already mentioned above calculation of Lovesey [42], which showed that taking into account the gauge field fluctuations in the calculation of free energy leads back to a second order phase transition. A further indication of a second order phase transition came several years later when this problem was studied on the lattice by means of MC calculations and duality arguments [43]. The results confirmed scenarios of the NS transition differing from those obtained in [15]. Namely, the NS transition was found to be of the second order asymptotically equivalent to that of a superfluid with the reversed temperature axis. Subsequent MC simulations [44] performed in different regions of couplings lead to the result that the NS transition is strongly first order deep in the type-I region and becomes more weakly first order moving in the direction of the type-II region. Beyond a certain point the data of [44] suggest a second-order transition. The corresponding $O(n)$ nonlinear σ-model coupled to an Abelian gauge field studied near two dimensions by $2+\varepsilon$ expansion [53] did not show a first order phase transition either.

By mapping the model of $3d$ superconductor on a disorder field theory it was predicted [58] the existence of a tricritical point where the second order phase transition changes into a first. The position of tricritical point was located slightly in the type-I regime at the value of Ginzburg parameter [34] $k < 0.8/\sqrt{2}$. Starting from a dual formulation of the Landau-Ginzburg theory by means of the RG arguments it was shown that the critical exponents of the NS transition coincide with those of a superfluid transition with reversed temperature axis [59]. However, while the correlation length critical exponent of the normal-to-superconducting transition was predicted to coincide with the ordinary 3D XY model, the divergency of the renormalized penetration depth was characterised by the mean field value $\nu = 1/2$ [59].

The influence of the critical fluctuations on the order of NS transition was reconsidered on the basis of the ideas of field theoretical RG in [30]. Here the two-loop flow equations [30] for the static parameters and the ζ-functions [52] were obtained and it was indicated that a stable fixed point might be possible and, as a consequence, a second order phase transition might appear. An attractive feature of the flow found in [30] was that it discriminated between type-I and type-II superconductors, depending on the initial (background) values of the couplings. For small values of the ratio (coupling to the gauge field)/(fourth order coupling) (appropriate for type-II superconductors) the flow comes very near to the fixed point of the uncharged model but ends in the new superconducting fixed point. For large values of the ratio (type-I superconductors) the flow runs away. For values of the ratio in between the critical behavior might be influenced by a second (unstable) superconducting fixed point with scaling exponents quite different from the uncharged model.

Qualitative similar to [30] flow picture was obtained in [54] by approximately solving the model of charged superconductor with the help of nonperturbative flow equations: a method which appeared to give very encouraging results for critical scalar field theories [55,56]. Depending on the relative strength of a ratio (coupling to the gauge field)/(fourth order coupling) a first or a second order phase transition was found. The approximate description of the tricritical behaviour was given as well as the estimates of the correlation length critical exponent ν and the pair correlation function critical exponent η governing a second order phase transition were reported. Depending on three different assumptions for the stable fixed point value of the coupling to the gauge field in 2 successive truncations of the potential the following values were obtained: (η,ν) = [(-0.13, 0.50); (-0.20, 0.47)], [(-0.13, 0.53); (-0.17, 0.58)], [(-0.13, 0.59); (-0.15, 0.62)], indicating that independent of truncation the critical exponents belong to the physical region $\eta > 2-d$ and $\nu > 0$ clearly pointing towards a second order phase transition.

In the context of baryogenesis the problem of the order of the NS phase transition was considered in two loop order in [57] and the effective potential was calculated. The ε-expansion was applied to the electroweak phase transition in order to estimate various parameters of it in leading and next-to-leading orders in ε, including the scalar correlation length, latent heat, surface tension, free energy difference, bubble nucleation rate, and baryon nonconservation rate. Of course, the result was a first order phase transition since only run away flows are found in strict ε-expansion perturbation theory. Note that in the electroweak scenario of baryogenesis there exist so-called Sakharov requirement which is met when the transition is strongly of first order rather than second order.

The problem of the NS transition was also studied by means of analytical method which does not rely on the expansions in ε or $1/n$. Using a non-perturbative method of solving approximate Dyson equation for arbitrary d and n [60] it was found [61] that NS phase transition is governed by a "charged" fixed point. The value of the pair correlation function critical exponent η at $d=3, n=2$ appeared to be $\eta(3,2)=-0.38$. It is interesting to note that although the result for η appears to be well behaved function of d and n it breaks down at critical value $n_c \simeq 18$ when expanded in ε. From here the conclusion was drawn that the results of ε-expansion obtained in [15] and, in particular, the absence of stable fixed point solution at $n < n_c \simeq 365.9$ are to be interpreted as the breakdown of the ε-expansion rather then the fluctuation-induced first-order phase transition. On the other hand, near $d=4$ the results of [61] are in a good agreement with the ε-expansion data [15] for high n ($n > 366$).

Recently the same problem studied by the RG technique in fixed dimension $d=3$ in one-loop approximation showed the evidence of an attractive charged fixed point distinct from that of a neutral superfluid leading, in particular, to the correlation length critical exponents values $\nu \simeq 0.53$ and $\eta \simeq -0.70$ [62]. However considered in the form of continuum dual theory [63] the magnetic penetration depth was shown to diverge with the XY exponent, contradicting to the results mentioned above [30,54,61,62]. To investigate this controversy the MC simulations of the 3D isotropic lattice superconductor in zero external magnetic field were performed resulting in the conclusion about single diverging length scale consistent with the universality of the ordinary 3D XY model [65].

Further applications of the model containing coupling to the gauge field have been suggested in the context of the quantum Hall effect [66].

Let us give a brief account of the experimental data relevant for our study. As it was mentioned in the Introduction, the effects of thermodynamic fluctuations are generally small in conventional low-T_c superconductors because of their low transition temperatures and large coherence length. On contrary, the high transition temperatures and small coherence lengths of high-T_c superconductors lead to the relevance of critical fluctuations there. Though critical fluctuations in high-T_c superconductors were observed in a series of experiments (see e.g. [17–24]) their interpretation survived some changes. Deviations from the mean field (i.e. first order) behaviour were accounted for either by $3d$ Gaussian fluctuations (providing, in particular, values for the specific heat critical exponent α and the correlation length critical exponent ν: $\alpha=\nu=1/2$) [17,19] or by a nontrivial XY behaviour characterizing uncharged superfluid (with $\nu \simeq 2/3$ and logarithmic divergencies in α) [18,20,22,24]. Measurements of the heat capacity [22,24], magnetization and electrical conductivity [22] of single-crystal samples of $YBa_2Cu_3O_{7-x}$ in a magnetic fields near T_c supported the existence of a critical regime governed

by the XY-like critical exponents [18,20,22,24,25], a similar conclusion followed from a crossover analysis of the zero-field heat capacity on a comparable sample [21]. However, the maximum applied magnetic field for which the 3D XY scaling holds is different for different materials [67].

To conclude this brief review it is worth mentioning one more physical interpretation of a charged field coupled to a gauge vector potential. Namely it is the nematic-smectic-A transition in liquid crystals [68–73]. Here the nematic phase is an orientationally ordered but translationally disordered phase with rodlike molecules aligned with their long axes parallel to the director and the smectic-A phase contains layers of molecules with their long axes perpendicular to the layer. It has been proposed [68,69] that this transition is described by a model similar to those describing the NS transition in the charged case [15]: now the smectic order parameter (being a complex field $\Psi(r)$ that specifies the amplitude and phase of the density modulation induced by layering) is coupled to the director fluctuations. On contrary to the NS type transition, the nematic-smectic A transition is characterized by the critical region of the experimentally accessible range. Indeed for certain materials it was shown [72] that both the latent heat data obtained through adiabatic scanning calorimetry as well as independent interface velocity measurements can be fit near the Landau tricritical point by a crossover function consistent with a mean field free energy density that has a cubic term [15], implying that the nematic-smectic-A transition is weakly first order. However many liquid crystals appear to exhibit a continuous nematic-smectic-A transition (see [73] and references therein). High-resolution heat-capacity and x-ray studies of the nematic-smectic-A transition performed during past twenty years (see [73] for a comprehensive review) show complex systematic trends to crossover from three-dimensional XY to tricritical behaviour and the anisotropic behaviour arise due to coupling between the smectic order parameter and director fluctuations.

3. The model and its "naive" analysis

As is well known now the influence of the order parameter fluctuations on the NS transition can be described by the Landau-Ginsburg free energy functional:

$$F[\phi] = \int \mathrm{d}^3x \{ \frac{t_0}{2}|\phi_0|^2 + \frac{1}{2}|(\nabla\phi_0|^2 + \frac{u_0}{4!}|\phi_0|^4 \}, \qquad (1)$$

t_0 being temperature-dependent, u_0 is a coupling constant and the complex order parameter ϕ_0 is connected with the wave function of Cooper pairs. The Cooper pairs are charged and therefore create fluctuating magnetic field which leads to the appearance of additional terms in the free energy functional. Note that it is not the case for a normal-to-superfluid transition in neutral (uncharged) fluid, which is well described by (1) without any modifications. Describing the fluctuating magnetic field $\mathbf{B}$ by the vector potential $\mathbf{A}$ ($\mathbf{B} = \mathrm{rot}\mathbf{A}$) and adding to (1) the minimal coupling between the fluctuating vector potential and the order parameter one gets the free energy functional $F[\Psi, \mathbf{A}]$ originally considered in [15] for a generalized superconductor in d dimensions with the d-dimensional vector potential $\mathbf{A}$ and the order parameter Ψ consisting of $n/2$ complex components. Now one can describe the fluctuation effects by an Abelian Higgs model with the gauge invariant Hamiltonian [15]:

$$H = \int \mathrm{d}^d x \{ \frac{t_0}{2}|\Psi_0|^2 + \frac{1}{2}|(\nabla - i e_0 \mathbf{A}_0)\Psi_0|^2 + \frac{u_0}{4!}|\Psi_0|^4 + \frac{1}{2}(\nabla \times \mathbf{A}_0)^2 \}, \tag{2}$$

depending on the bare parameters t_0, e_0, u_0. The parameter t_0 changes its sign at some temperature, the rest of the parameters being considered as temperature-independent. For the coupling constant $e_0 = 0$ no magnetic fluctuations are induced and the model reduces to the usual field theory (1) describing a second-order phase transition and corresponding in the particular case $n = 2$ to the superfluid transition in ^4He.

The mean field results for the critical behaviour of the model with a free energy functional $F[\Psi, \mathbf{A}]$ which corresponds to the Hamiltonian (2) were reported already in the original paper of Halperin, Lubensky and Ma [15]. In the frames of the mean field theory one gets that the systems characterized by the free energy functionals $F[\phi]$ (1) and $F[\Psi, \mathbf{A}]$ ($n = 2$) possess qualitatively different critical behaviour. Neglecting fluctuations in the order parameter (in accordance with the Ginzburg criterion this may be done for a good type-I superconductor) one gets that depending on the sign of t_0 the free energy (1) is minimized by the value of the order parameter $\phi = 0$ for $t_0 > 0$ or by a non-zero value, when $t_0 < 0$, and the appearance of the non-zero order parameter is continuous: in the system under consideration the second order phase transition occurs.

However, being applied to the free energy functional $F[\Psi, \mathbf{A}]$ the mean field theory predicts qualitatively different behaviour. Defining the effective free energy $F_{\mathrm{eff}}[\Psi]$ as a function of single variable Ψ by taking the trace over the configurations of the vector potential $\mathbf{A}$ one finds [15] that the expression for $F[\Psi]$ will contain a term which has a negative sign and is proportional to $|\Psi|^3$. Such a term inevitably leads to a first order transition, $F_{\mathrm{eff}}[\Psi]$ develops a minimum at a finite value of Ψ when the coefficient of the quadratic term is still slightly positive.

As we already mentioned, the above reasoning is appropriate for a type-I superconductor. The case of type-II superconductors is considerably more complicated. Here, the fluctuations in Ψ cannot be neglected and one should choose appropriate technique to study the problem. Originally the critical behaviour of the model (2) at presence of order parameter fluctuations was studied in [15] by means of Wilson-Fisher recursion relations [35] in the first order of $\varepsilon = 4 - d$, resulting, in particular, in the answer that the second order phase transition is absent for $n = 2$ in the region of couplings appropriate for the type-II superconductor either. Below we will reproduce these results of the ε-expansion and proceed further in studying the problem.

In order to describe long-distance properties of model (2) arising in the vicinity of the phase transition point we apply field-theoretical RG approach. Two-loop results [30] for the RG functions corresponding to (2) are obtained on the basis of dimensional regularization and minimal subtraction scheme [74], defining the renormalized quantities so as to subtract all poles at $\varepsilon = 4 - d = 0$ from the renormalized vertex functions. The renormalized fields, mass and couplings are introduced by:

$$\Psi_0 = Z_\Psi^{1/2}\Psi, \qquad \mathbf{A}_0 = Z_{\mathrm{A}}^{1/2}\mathbf{A}, \qquad t_0 - t_{0c} = Z_t Z_\Psi^{-1} t,$$
$$e_0^2 = Z_e^1 Z_{\mathrm{A}}^{-1} Z_\Psi^{-1} e^2 \qquad \mu^\varepsilon S_d^{-1}, u_0 = Z_u Z_\Psi^{-2} u \mu^\varepsilon S_d^{-1}. \tag{3}$$

with $\varepsilon = 4 - d$. Here μ is an external momentum scale, t_{0c} being a shift, which for the results considered here can be set to zero, and S_d stands for the surface of d-dimensional hypersphere: $S_d = 2^{1-d}\pi^{-d/2}/\Gamma(d/2)$. The Z-factors are determined by the condition that all poles at $\varepsilon = 0$ are removed from the renormalized vertex functions.

The RG equations are written noticing the fact that the bare vertex functions $\Gamma_0^{N,M}$ being calculate with the bare Hamiltonian (2) as a sum of one-particle irreducible (1PI) diagrams [75]:

$$\Gamma_0^{N,M}(\{r\}, \{R\}) = < \Psi_0(r_1) \ldots \Psi_0(r_N) A_0(R_1) \ldots A_0(R_M) >_{1\mathrm{PI}} \tag{4}$$

do not depend on the scale μ and their derivative with respect to μ at fixed bare parameters is equal to zero. So one gets

$$\mu\frac{\partial}{\partial\mu}\Gamma_0^{N,M}|_0 = \mu\frac{\partial}{\partial\mu}Z_\Psi^{N/2}Z_A^{M/2}\Gamma_R^{N,M}|_0 = 0, \tag{5}$$

where the index 0 means differentiation at fixed bare parameters. The following RG equations for the renormalized vertex function $\Gamma_R^{N,M}$ follow

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta_u\frac{\partial}{\partial u} + \beta_f\frac{\partial}{\partial f} + \zeta_\nu t\frac{\partial}{\partial t} - \frac{N}{2}\zeta_\Psi - \frac{M}{2}\zeta_A\Big)\Gamma_R^{N,M}(t,u,f,\mu) = 0, \tag{6}$$

where $f = e^2$, $\zeta_\nu = \zeta_\Psi - \zeta_t$ and the RG functions read

$$\beta_u(u,f) = \mu\frac{\partial u}{\partial\mu}|_0, \qquad \beta_f(u,f) = \mu\frac{\partial f}{\partial\mu}|_0,$$
$$\zeta_\Psi = \mu\frac{\partial\ln Z_\Psi}{\partial\mu}|_0, \qquad \zeta_A = \mu\frac{\partial\ln Z_A}{\partial\mu}|_0, \qquad \zeta_t = \mu\frac{\partial\ln Z_t}{\partial\mu}|_0. \tag{7}$$

Using the method of characteristics the solution of the RG equation may be formally written in the form:

$$\Gamma_R^{N,M}(t,u,f,\mu) = X(l)^{N/2}(X'(l))^{M/2}\Gamma_R^{N,M}(Y(l)t,u(l),f(l),\mu l), \tag{8}$$

where the characteristics are the solutions of the ordinary differential equations:

$$l\frac{d}{dl}\ln X(l) = \zeta_\Psi(u(l),f(l)), \qquad l\frac{d}{dl}\ln X'(l) = \zeta_A(u(l),f(l)),$$
$$l\frac{d}{dl}\ln Y(l) = \zeta_\nu(u(l),f(l)), \tag{9}$$
$$l\frac{d}{dl}u(l) = \beta_u(u(l),f(l)), \qquad l\frac{d}{dl}f(l) = \beta_f(u(l),f(l)) \tag{10}$$

with

$$X(1) = X'(1) = Y(1) = 1, \qquad u(1) = u, \qquad f(1) = f. \tag{11}$$

For small values of l equation (8) is mapping the large scales of length (the critical region) to the noncritical point $l = 1$. In this limit the scale-dependent values of the couplings $u(l)$, $f(l)$ approach the stable fixed point, if it exists.

The fixed points u^*, f^* of differential equations (10) are given by the solutions of the system of equations:

$$\beta_f(u^*, f^*) = 0,$$
$$\beta_u(u^*, f^*) = 0. \tag{12}$$

The stable fixed point is defined as the fixed point where the stability matrix

$$B_{ij} = \frac{\partial\beta_{u_i}}{\partial u_j}, \qquad u_i = \{u,\, f\} \tag{13}$$

possess positive eigenvalues (or the eigenvalues with positive real parts, if complex). The stable fixed point corresponds to the critical point of the system: as we mentioned above, in the limit $l \to 0$ (corresponding to the limit of infinite correlation length) the renormalized couplings reach their values in the stable fixed point.

Now we write down the results for the RG functions obtained in a two-loop approximation [30] following the above described procedure in frames of dimensional regularization and minimal subtractions schemes. From a Ward identity one has $Z_\Psi = Z_e$, and the remaining Z-factors are to be found from the corresponding vertex functions $\Gamma^{2,0}$, $\Gamma^{0,2}$, and $\Gamma^{4,0}$. Since the gauge field is massless, the renormalization has been performed at finite wave vector. The results in two-loop order read:

$$Z_\Psi = 1 + \frac{1}{\varepsilon}\{3e^2 - u^2(n+2)/144 + e^4[(n+18)/4\varepsilon - (11n+18)/48]\}, \tag{14}$$
$$Z_\mathrm{A} = 1 + \frac{1}{\varepsilon}\{-ne^2/6 - ne^4/2\}, \tag{15}$$
$$Z_t = 1 + \frac{1}{\varepsilon}\{(n+2)u/6 + u^2[(n+2)(n+5)/36\varepsilon - (n+2)/24] + ue^2[-(n+2)(1/2\varepsilon - 1/3)] + e^4[(3n+6)/2\varepsilon + (5n+1)/4]\}, \tag{16}$$

$$Z_u = 1+\frac{1}{\varepsilon}\{(n+8)u/6+18e^4/u+u^2[(n+8)^2/36\varepsilon - (5n+22)/36]+ue^2[-(n+8)/2\varepsilon+(n+5)/3]+ e^4[(3n+24)/\varepsilon+(5n+13)/2]+ e^6/u[3(n+18)/\varepsilon-7n/2-45]\}. \quad (17)$$

Following the standard procedure one obtains that the expressions for β-functions in two-loop approximation read:

$$\beta_f = -\varepsilon f+\frac{n}{6}f^2+nf^3, \quad (18)$$

$$\beta_u = -\varepsilon u+\frac{n+8}{6}u^2-\frac{3n+14}{12}u^3-6uf+18f^2 +\frac{2n+10}{3}u^2f+\frac{71n+174}{12}uf^2-(7n+90)f^3. \quad (19)$$

The previous analysis of the equations of type (18), (19) either on one-loop [15] or on two-loop level [30] was based on the direct solutions of the equation for the fixed point. In the present study we want to attract attention to the fact that the series have zero radius of convergence and they are known to be asymptotic at best. Therefore some additional mathematical methods have to be applied in order to obtain reliable information on their basis.

We start by recalling the results of a ε^2-expansion for β-functions [15,30]. In second order in ε one obtains three fixed points: the Gaussian ($u^{*G}=f^{*G}=0$), the "Uncharged" ($u^{*U}\neq 0, f^{*U}=0$) and the "Charged" ($u^{*C}\neq 0, f^{*C}\neq 0$), to be denoted as G, U, C. The expressions for them read:

$$\mathrm{G}: \quad u^{*G}=0, \quad f^{*G}=0, \quad (20)$$

$$\mathrm{U}: \quad u^{*U}=u_1^U\varepsilon+u_2^U\varepsilon^2, \quad f^{*U}=0, \quad (21)$$

$$\mathrm{C}: \quad u^{*C}=u_1^C\varepsilon+u_2^C\varepsilon^2, \quad f^{*C}=f_1^C\varepsilon+f_2^C\varepsilon^2, \quad (22)$$

where

$$u_1^U=\frac{6}{n+8}, \quad u_2^U=\frac{18(3n+14)}{(n+8)^3},$$

$$u_1^C=\frac{3(n+36)+(n^2-360n-2160)^{1/2}}{3n(n+8)},$$

$$u_2^C=\frac{a_2}{a_1}, \quad f_1^C=\frac{6}{n}, \quad f_2^C=-\left(\frac{6}{n}\right)^3 n,$$

with

$$a_1=1+\frac{n+8}{3}u_1^C-\frac{36}{n},$$

$$a_2=\frac{3n+14}{12}\left(u_1^C\right)^3-6nu_1^C\left(\frac{6}{n}\right)^3+36n\left(\frac{6}{n}\right)^4 -\frac{(n+5)4}{n}\left(u_1^C\right)^3-\frac{3(71n+174)}{n^2}u_1^C+\left(\frac{6}{n}\right)^3(7n+90).$$

Almost all physical results concerning phase transition described by the field theory (2) were to some extend based on the information given by (20) - (22). The main of them read:

(i) fixed point U is unstable with respect to the presence of f-symmetry at $d<4$ with the stability exponent

$$\lambda_f(u=u^{*U}, f=f^{*U}=0)=\frac{\partial\beta_f}{\partial f}|_U=-\varepsilon;$$

(ii) fixed point C appears to be complex for $n<n_c=365.9$ [15] already on one-loop level. The stability exponent is given by

$$\lambda_u(u=u^{*C}, f=f^{*C})=\frac{\partial\beta_u}{\partial u}|_C$$

and on the two-loop level it reads:

$$\lambda_u=-\varepsilon s, \quad s=\left[\left(1+\frac{36}{n}\right)^2-\frac{432(n+8)}{n^2}\right]^{1/2}$$

leading to an oscillatory flow in u in one-loop order below n_c with the solution [26,30]:

$$f(l)=\frac{6fl^{-\varepsilon}}{6+n\varepsilon f(l^{-\varepsilon}-1)}, \quad (23)$$

$$u(l)=f(l)\frac{n}{2(n+8)}\left\{s\tan\left[\frac{s}{2}\ln\left(f(l)f^{-1}l^{\varepsilon}\right) +\arctan\left(\frac{2(n+8)}{sn}\frac{u}{f}+\frac{n+36}{ns}\right)\right]-\frac{n+36}{n}\right\}, \quad (24)$$

here f and u are the initial parameters at $l = 1$;

(iii) from the condition of positiveness of the fixed point coordinate f^* ($f = e^2$) follows that at $\varepsilon = 1$ n has to be larger that 36. This questions the applicability of the ε-expansion for $n = 2$ to $d = 3$.

Finally the conclusion follows that for the "superconductor" case $n = 2$ being of most physical interest there does not exist a stable fixed point and therefore the observed phase transition is of the first order.

In what follows below we will study the RG equations in minimal subtraction scheme in the frames of $d = 3$ theory [11–13] putting $\varepsilon = 1$ in expressions for the RG functions and studying the perturbation theory in powers of coupling constants. The last correspond to the number of loops in Feynman diagrams and thus one develops the perturbation theory in succesive number of loops. Direct calculations based on the equations (18), (19) at fixed $d = 3$ do not bring qualitativly new features to the described above analysis. In the one-loop approximation, leaving square terms in (18), (19) one finds that there exists only one nontrivial fixed point $u^* = 6/(n+8)$, $f^* = 0$. The β-functions $\beta_u(u,f)$, $\beta_f(u,f)$ in one-loop approximation at $d = 3, n = 2$ are shown in the Fig. 1. Simultaneous intersection of the surfaces corresponding to both functions with the plane $\beta = 0$ results in the fixed points: these for the $n = 2$ case have coordinates $u^* = f^* = 0$ and $u^* = 0.6$, $f^* = 0$ are seen at the picture. In the two-loop approximation only the Gaussian fixed point survives, as one may see from the Fig. 2.

Nevertheless one should note that such a straightforward interpretation of the above expansions data was questioned and a way of analyzing the series for β-functions (18),(19) avoiding strict ε-expansion and exploiting the information on the accurate solution for the pure model case at $d = 3$ was proposed [30]. Also from the comparison of ε-expansion data for f^* (giving positive value of f^* only for $n > 36$) with the value of f^* obtained without ε-expansion (remaining positive for all n) the conjecture was made that the lower boundary for n resulting in the negative f^* might be an artifact of the expansion procedure. Let us consider now expressions for the RG functions more carefully, paying attention to their possible asymptotic nature and treating them by some resummation procedure.

4. Resummation

The appropriate resummation technique applied in the theory of critical phenomena to the asymptotic series for the RG functions enables

one to obtain extremely accurate values of the critical exponents [76]. In fact the asymptotic nature of the series for the RG functions has been proved only in the case of the ϕ^4 model containing one coupling of $O(n)$-symmetry (n-vector model) as well as the high-order asymptotics for these series is known [27–29] in analytical form. These results gave the possibility to obtain precise values of the critical exponents for the n-vector model by the resummation of the corresponding series for the renormalization group functions (see e.g. [7,8,11]). For the "charged" model we are considering here up to our knowledge no information similar to those obtained in [27–29] for the "uncharged" case ($f = 0$) is available. In the case of the models containing several couplings of different symmetry the asymptotic nature of the corresponding series for the RG functions is rather a general belief than a proven fact. As one of the examples important in the course of our future analysis we mention here the weakly diluted n-vector model, describing a ferromagnetic ordering in a system of N_1 classical n-component "spins" located in N sites of a lattice ($N_1/N < 1$) and quenched in a certain configuration. Using the replica trick in order to perform the quenched averaging one gets [82] that an effective Hamiltonian of such a model contains two fourth order terms of different symmetry and reads:

$$H = \int \mathrm{d}^d x \Big\{ \frac{1}{2} \sum_{\alpha=1}^{m} \Big[|\nabla \vec{\phi}^\alpha|^2 + m_0^2 |\vec{\phi}^\alpha|^2 \Big] - \frac{v_0}{8} \left(\sum_{\alpha=1}^{m} |\vec{\phi}^\alpha|^2 \right)^2 + \frac{u_0}{4!} \sum_{\alpha=1}^{m} \left(|\vec{\phi}^\alpha|^2 \right)^2 \Big\}, \quad (25)$$

were $\vec{\phi}^\alpha$ is a n-component vector $\vec{\phi}^\alpha = (\phi^{\alpha,1}, \phi^{\alpha,2}, \ldots, \phi^{\alpha,n})$; $u_0 > 0, v_0 > 0$ are bare coupling constants; m_0 is bare mass and in the final results a replica limit $m \to 0$ is to be taken. The RG functions for these model are obtained in the form of double series in renormalized couplings u, v and the asymptotic nature of the series has not been proven for this model up till now [84]. Nevertheless the appropriate resummation technique (applied *as if* these series are the asymptotic ones) enables one to obtain accurate values for critical exponents in three dimensions [77–81] and to describe (in $n = 1$ case) the experimentally observed crossover to a new type of critical behavior caused by weak dilution [85,86]. These results are also confirmed by Monte-Carlo [87,88] and Monte-Carlo RG [89] calculations.

Two main ways of resummation commonly used for the asymptotic series arising in the RG approach are: (**i**) resummation based on the conformal mapping technique and (**ii**) Padé-Borel resummation. The case (**i**) is based on the conformal transformation, which maps a part of the domain of analyticity containing the real positive axis onto a circle centered at the origin and the asymptotic expansion for a certain function is thus re-written in the form of a new series (see [8]). However this resummation is based on the knowledge of subtle details of asymptotics (location of the pole, high-order behavior) which are not available in our case.

In the absence of any knowledge about the singularities of the series the most appropriate method which can be used to perform the analytical continuation is the Padé approximation resulting in Padé-Borel resummation technique (**ii**) (see e.g. [7]). In the following we are going to apply it for the special case of $f = 0$ so let us concentrate on it in detail.

Starting from the Taylor series for the function $f(u)$:

$$f(u) = \sum_{j\geq 0} c_j\ u^j, \tag{26}$$

one constructs the Borel-Leroy transform

$$F(ut) = \sum_{j\geq 0} \frac{c_j}{\Gamma(j+p+1)}\ (ut)^j, \tag{27}$$

with $\Gamma(x)$ being Euler's gamma-function and p - arbitrary positive number [90]. Then one represents (27) in the form of Padé approximant $F^{\text{Padé}}_{[L/M]}(ut)$:

$$F^{\text{Padé}}_{[L/M]}(x) = \frac{\sum_{i=0}^{L} a_i x^i}{\sum_{j=0}^{M} b_j x^j} \tag{28}$$

(in the subsequent analysis, proceeding in two-loop approximation we will use the [1/1] Padé approximant) and the resummed function is given by:

$$f^{\text{Res}}(u) = \int_0^\infty dt\ e^{-t}\ t^p\ F^{\text{Padé}}(ut). \tag{29}$$

The scheme of resummation (27) – (29) of the (asymptotic) series in one variable (26) is easy generalized to the two-variable case when the series is given in a form:

$$f(u,v) = \sum_{j,j\geq 0} c_{i,j}\ u^i\ v^j, \tag{30}$$

with the Borel-Leroy transform

$$F(u,v,t) = \sum_{i,j\geq 0} \frac{c_{i,j}}{\Gamma(i+j+p+1)}\ (ut)^i (vt)^j. \tag{31}$$

Now the procedure postulates to choose an appropriate form of the analytic continuation of the series (31). Two most common ways to proceed are the Borel resummation combined with Chisholm approximants and the Borel resummation of the resolvent series, presented in a form of Padé approximant. In the first way in order to write an analytic continuation of the series (31) one uses the rational approximants of two variables: so-called Canterbury approximants or generalized Chisholm approximants [91,93] which are generalization of Padé approximants in the case of several variables, representing (31) in a form:

$$F^{\text{Chisholm}}(u,v,t) = \frac{\sum_{i,j} a_{i,j} u^i v^j t^{i+j}}{\sum_{i,j} b_{i,j} u^i v^j t^{i+j}}, \tag{32}$$

(sums in the numerator and denominator being limited by the condition of correspondence between known numbers of terms in the initial series and that in the approximant). Again, the resummed function is given by an integral (29):

$$f^{\text{Res}}(u,v) = \int_0^\infty dt\ e^{-t}\ t^p\ F^{\text{Chisholm}}(ut). \tag{33}$$

Proceeding in a second way, one writes for the series of two variables (30) the so-called resolvent series $\mathcal{F}(u,v,\tau)$ [92,93] introducing an auxiliary variable τ, which allows to separate contributions from different orders of the perturbation theory in the variables u, v:

$$\begin{aligned} \mathcal{F}(u,v,\tau) &= \sum_{i,j\geq 0} c_{i,j}\,(u\tau)^i\,(v\tau)^j, \\ f(u,v) &= \mathcal{F}(u,v,\tau=1). \end{aligned} \tag{34}$$

Now the resummation of the series $\mathcal{F}(u,v,\tau)$ is performed with respect to variable τ as for the series in single variable, applying the above described scheme (27) – (29).

Let us illustrate how the resummation procedure works in the case of the effective Hamiltonian (25). In order to allow for a direct comparison with the superconductor case, let us take the β-functions obtained for the model (25) in the minimal subtraction scheme in the two-loop approximation, though the high-order results are available for this model

[81,95] as well as the results [77–80] obtained in the $d = 3$ massive field theoretic approach [94]. The expressions for the β-functions, corresponding to the renormalized couplings u, v in the replica limit $m \to 0$ for the Ising model case (n=1) read:

$$\beta_u = -\varepsilon u + \frac{3}{4}u^2 - 6uv - \frac{17}{12}u^3 + \frac{23}{2}u^2 v - \frac{41}{2}uv^2, \quad (35)$$

$$\beta_v = -\varepsilon v + uv + -4v^2 - \frac{5}{12}u^2 v + \frac{11}{2}uv^2 - \frac{21}{2}v^3. \quad (36)$$

We do not present here the expressions for the other RG functions, as far as for the purpose we are interested here we are going only to study the fixed point equations.

Looking for the solutions of the fixed point equations for functions (35), (36) one gets that in the one-loop approximation in addition to the Gaussian fixed point $u^* = v^* = 0$ there exist two solutions more $u^* = 2/3$, $v^* = 0$ and $u^* = 0$, $v^* = -1/5$ and the solution $u^* \neq 0$, $v^* \neq 0$ is absent [96]. Fixed point with $v^* < 0$ is out of the region of parameters describing the diluted magnet [97] and the pure model fixed point $u^* \neq 0, v^* = 0$ appears to be unstable with respect to the v-coupling (we propose to the reader to do this check looking on the stability matrix $B_{ij}(u, v)$ (13) eigenvalues at the fixed points). Corresponding plot of the functions β_u, β_v in the one-loop approximation is shown in the figure 3. Passing to the two-loop approximation makes the result even worse: only the Gaussian fixed point is present (see Fig. 4). Returning back to the initial problem statement one should conclude that the obtained picture corresponds to the absence of a second order phase transition in a $d = 3$ Ising model with weak dilution as well as without dilution (absence of a fixed point $u^* \neq 0$, $v^* = 0$). Which contradicts of course to the real situation. Let us note as well that the obtained behaviour for the β-functions of model (25) in the one- and two-loop approximations (Figs. 3, 4) resembles those for the superconductor case in the corresponding approximations (Figs. 1, 2).

However, applying the resummation procedure to the series (35), (36) in the two loop approximation one reconstitutes fixed points ($u^* \neq 0$, $v^* = 0$), ($u^* = 0$, $v^* \neq 0$) and obtains a new stable fixed point $u^* \neq 0$, $v^* \neq 0$ which governs a second order phase transition in a weakly diluted Ising model. The obtained picture appears to be stable with respect to successive account of the higher order terms in perturbation theory, when the appropriate resummation technique is being applied. As we already claimed above, this RG results are confirmed by different other theoretical approaches and correspond to the experimentally observed

second order phase transition in the weakly diluted Ising magnet with critical exponents differing from those of the pure case. In the Fig. 5 we show the crossing of the $\beta_u(u, v)$ and $\beta_v(u, v)$ surfaces for the resummed function. The calculations were performed by means of the Padé-Borel resummation technique for the resolvent series (34) of two-loop functions (35), (36) as described above [98]. The Gaussian ($u^* = v^* = 0$) and pure ($u^* = 1.3146$, $v^* = 0$) fixed points can be seen at the back side of the cube. The cross-section of u and v plains in the picture is chosen to pass through the stable fixed point ($u^* = 1.6330$, $v^* = 0.0835$) corresponding to a new critical behaviour.

The example we considered above is a typical situation happening in a $d = 3$ RG theory: being considered without appropriate resummation technique, RG analysis might give not only quantitatively not precise numbers for a critical exponents but also a qualitatively wrong answer about absence of a stable fixed point for a certain model, resulting in absence of a second order phase transition. Now with this information in hand let us pass to the analysis of a model of superconductor described in the two-loop approximation by the RG functions (35), (36).

5. Fixed points and flows in three dimensions

We will proceed here by considering the flow equations (10) directly at $d = 3$. Let us look for the solutions of the fixed point equations at $d = 3$ paying attention to the possible asymptotic nature of the corresponding series (18),(19). Consider first the equation for the uncharged fixed point U. Substituting value $f^* = 0$ into (19) one obtains the following expression for the function $\beta_u^U \equiv \beta_u(u, f^* = 0)$:

$$\beta_u^U = -u + \frac{n+8}{6}u^2 - \frac{3n+14}{12}u^3. \quad (37)$$

Solving this polynomial for the fixed point one obtains for the non-trivial $u^* > 0$:

$$u^{*U} = \frac{n+8}{3n+14} + \frac{\sqrt{n^2 - 20n - 104}}{3n+14} \quad (38)$$

and immediately the "condition of the existence of non-trivial solution u^{*U}" qualitatively very similar to those, appearing in the frames of the ε-expansion technique (see [15,30] and formula (21) of the present article as well) follows : the solution exists only for certain values of $n > n_c = 24.3$! From Fig. 6 one can see that the function β_u^U (37) does not intersect the u-axis at any non-zero value of u for $n = 2$. In the $O(n)$-symmetric

ϕ^4-theory at $d = 3$ this situation is well-known (see e.g. [94,99]): the β-function calculated directly at $d = 3$ does not possess a stable zero for the realistic values of n, nevertheless in three-loop order the presence of the stable fixed point is restored. To avoid this artifact appearing in the two-loop calculation one can either resume the series for β-function or construct the appropriate Padé approximant [100] in order to perform the analytical continuation of (37) out of the domain of convergence (which is equal to zero for the series in the right-hand side of (37). Let us try both ways. Representing (37) in the form of [1/1] Padé approximant:

$$\beta_u^{U,Padé} = u\frac{-1 + A_u u}{1 + B_u u} \qquad (39)$$

one obtains:

$$A_u = \frac{n^2 + 7n + 22}{6(n+8)}, \qquad B_u = \frac{3n + 14}{2(n+8)}, \qquad (40)$$

and, solving the equation for the fixed point:

$$\beta_u^{U,Padé}(u^{*P,Padé}) = 0 \qquad (41)$$

one obtains:

$$u^{*U,Padé} = \frac{6(n+8)}{n^2 + 7n + 22}. \qquad (42)$$

So we obtained a qualitatively different situation. The behavior of the function $\beta_u^{U,Padé}(u)$ for $n = 2$ is shown in Fig. 6 by the dashed curve. If one is interested in more accurate values of u^* some resummation has to be applied. Choosing the Padé-Borel resummation technique [101] and following scheme (26)-(29) one obtains for the resummed function $\beta_u^{U,Res}$ [98]:

$$\beta_u^{U,Res} = u[2(1 - A_u/B_u)(1 - E(\frac{2}{uB_u})) \; - \; 1], \qquad (43)$$

the coefficients A_u, B_u are given by (40), $E(x) = xe^x E_1(x)$, where the function

$$E_1(x) \; = \; e^{-x}\int_0^\infty dte^{-t}(x+t)^{-1}$$

is connected with the exponential integral by the relation [102]:

$$E_1(x \pm i0) = -Ei(-x) \mp i\pi.$$

The behavior of the function $\beta_u^{U,Res}(u)$ is shown in Fig. 6 by the solid curve. And the fixed point coordinate $u^{*U,Res}$ is obtained solving the non-linear equation:

$$\beta_u^{U,Res}(u^{*U,Res}) = 0. \qquad (44)$$

The coordinates of the fixed point u^{*U} obtained on the basis of Padé approximation and Padé-Borel resummation ($u^{*U,Padé}$, $u^{*U,Res}$) for different n are given in Table 1.

We conclude from this analysis: in $d = 3$ theory Padé approximants (as an analytical continuation of β-functions) qualitatively may change the picture and lead to the values of fixed points comparable to those obtained by the Padé-Borel resummation technique.

Consider now the equation for the charged fixed point C applying the above considerations to β_f for which the expression at $d = 3$ reads (18):

$$\beta_f = -f + \frac{n}{6}f^2 + nf^3. \qquad (45)$$

The behavior of β_f as a function of f is shown in Fig. 7 by asterisks. Note however that in this case the function β_f even without any resummation possess a non-trivial zero f^{*M} (its value $f^{*C,Dir}$ is given in the 2nd row of Table 2). Representing (45) in the form of [1/1] Padé approximant:

$$\beta_f^{Padé} = f\frac{-1 + A_f f}{1 + B_f f} \qquad (46)$$

one has for A_f, B_f:

$$A_f = \frac{n + 36}{6}, \quad B_f = -6, \qquad (47)$$

and, solving the equation for the fixed point coordinate $f^{*C,Padé}$:

$$\beta_f^{Padé}(f^{*C,Padé}) = 0 \qquad (48)$$

one obtains:

$$f^{*C,Padé} = \frac{6}{n + 36}. \qquad (49)$$

The function $\beta_f^{Padé}(f)$ is shown in Fig. 7 by the dashed line, the coordinate $f^{*C,Padé}$ is given in the 3rd row of Table 2. But now the series (45) is not alternating and this results in the presence of a pole (at $f = \frac{1}{6}$) in the approximant (46). Therefore (46) correctly represents the function $\beta_f(f)$ only for $f < 1/6$. Let us note however that for all the positive n a fixed point exists and its coordinate $f^{*M,Padé}$ lies within the limits $0 < f^{*C,Padé} < 1/6$, where no pole in (46) exists. Comparing this result with those obtained for the uncharged fixed point one can note that the representation of β_f in the form of the Padé approximant does not qualitatively change the picture (a solution for $\beta_f(f) = 0$ exists at $d = 3$ even without an analytical continuation) but results in a decrease of the fixed point coordinate. Contrary to the ε-expansion values (22) there does not exist any border line values of n for the positivity of f^{*C}. Unfortunately we can not check this result by means of Padé-Borel resummation technique: the above mentioned presence of a pole in the denominator of the Padé approximant makes the corresponding integral representation problematic [103]. In order to find the u-coordinate of the fixed point C, u^{*C}, we have to deal with a function of two variables, $\beta_u(u,f)$, represented by a rather short series (19). Another problem arises due to the fact that function $\beta_u(u,f)$ contains generating terms (i.e. $\beta_u(u=0,f)\neq 0$). In order to perform some kind of the analytic continuation of the function of two variables one can use the Chisholm approximants (32) [91,93]. But the presence of generating terms makes this choice rather ambiguous. The most reliable way in such a case is the representation of $\beta_u(u,f)$ in the form of a resolvent series $B(u,f,\tau)$ (34) [92,93] introducing an auxiliary variable τ, which allows to separate contributions from different orders of the perturbation theory in the coupling constants. The series for $B(u,f,\tau)$ then reads:

$$B(u,f,\tau)\equiv\beta_u(u\tau,f\tau)=\sum_{j\geq 0} b_j\ \tau^j, \tag{50}$$

with obvious notations for the coefficients b_j. Now one considers (50) as a series in the *single* variable τ. This series can be represented in the form of Padé approximant $B^{Padé}(u,f,\tau)$ as the analytical continuation of the function $B(u,f,\tau)$ for the general value of τ. In particular at $\tau=1$ the equality holds $B(u,f,\tau=1)=\beta_u(u,f)$ and the approximant

$$B^{Padé}(u,f,\tau=1)\equiv\beta_u^{Padé}(u,f)$$

represents the initial function $\beta_u(u,f)$. In our case the expression for $B(u,f,\tau)$ reads:

$$B(u,f,\tau)=\tau(b_1+b_2\tau+b_3\tau^2), \tag{51}$$

where:

$$b_1=-u,\qquad b_2=\frac{n+8}{6}u^2-6uf+18f^2,$$

$$b_3=-\frac{3n+14}{12}u^3+\frac{2n+10}{3}u^2f+\frac{71n+174}{12}uf^2-(7n+90)f^3. \tag{52}$$

Representing the expression in brackets in the right-hand side of (51) in form of a [1/1] Padé approximant we have:

$$B^{Padé}(u,f,\tau)=\tau\ b_1\frac{1+A_{u,f}\tau}{1+B_{u,f}\tau}, \tag{53}$$

where

$$A_{u,f}=\frac{b_2}{b_1}-\frac{b_3}{b_2},\qquad B_{u,f}=\frac{-b_3}{b_2}. \tag{54}$$

Let us note here that the function $B(u,f,t)$ obtained in this way as the approximant for the function of two variables $\beta_u(u,f)$ obeys certain projection properties in the single-variable case: substituting $f=0$ or $u=0$ into (53) one obtains the [1/1] Padé approximant for $\beta_u^U(u)$ or the [0/1] Padé approximant for $\beta_u(u=0,f)$. Finally the expression for $\beta_u(u,f)$ approximated in such a way reads:

$$\beta_u^{Padé}(u,f)=\ b_1\frac{1+A_{u,f}}{1+B_{u,f}}. \tag{55}$$

Substituting into the equation for the fixed point $\beta_u(u^{*C},f^{*C})=0$ the value for the coordinate $f^{*C}=f^{*C,Padé}$ (49) one obtains the non-linear equation for $u^{*C,Padé}$:

$$\beta_u^{Padé}(u,f=f^{*C,Padé})=0. \tag{56}$$

Solving (56) with respect to u one obtains the values $u^{*C,Padé}$ given in Table 3. The intersection of the function $\beta_u^{Padé}(u,f)$ (55) with the plane $f=f^{*C,Padé}$ is shown for $n=2$ in Fig. 8. The first fixed point (C1) given in the 2nd row of Table 3 turns out to be unstable, while the fixed point C2 is stable also for the case $n=2$ we are mainly interested in.

The resulting picture of β-functions surfaces is shown in the Fig. 9 . The Gaussian and uncharged fixed points may be seen at the back side of the picture, whereas the intersection of the u- and f-planes was chosen in the picture to cross the stable fixed point C2. Unstable fixed point C1 is seen as well.

The crossover to the asymptotic critical behavior is described by the solutions of the flow equations (10) with the initial values of $u(\ell_0)$ and $f(\ell_0)$ at $\ell=\ell_0$ [104]. Substituting for the β-functions entering the right-hand side of (10) their analytical continuation in form of the Padé approximants (46), (55) we get the following system of differential equations:

$$l\frac{df}{dl} = f\frac{-1+A_f f}{1+B_f f}, \qquad (57)$$

$$l\frac{du}{dl} = -u\frac{1+A_{u,f}}{1+B_{u,f}}, \qquad (58)$$

where A_f, B_f and $A_{u,f}$, $B_{u,f}$ are given by (47) and (54) correspondingly.

Solving equations (57), (58) numerically one gets the flow diagram shown in Fig. 10 for the case of $n = 2$. The space of couplings is divided into several parts by separatrices (thick lines in Fig. 10) connecting the fixed points. Besides the Gaussian (G) there exist three fixed points, one corresponding to the uncharged (U) and two other corresponding to the charged (C1, C2) cases. The fixed points G, C1 and U are unstable (solid circles in Fig. 10) and the fixed point C2 is the stable one (shown as a solid box in Fig. 10). Several different flow lines are shown in Fig. 10. They can be compared with the corresponding flow picture obtained by a direct solution of the flow equations for the two-loop β-functions expressed by the third-order polynomials in couplings u, f (18), (19) (see Fig.2a in [30]). There one can see that no stable fixed point existed and even the fixed point U was absent. Comparing Fig. 10 and Fig. 2b from [30] one can see how an analytical continuation of the β-functions (10), (18), done only partly in [30] and performed here in the form of Padé approximants restores the presence of the fixed point U (unstable) and leads to the appearance of a new stable fixed point C2 for the charged model. The coordinates of the fixed points U, C1, C2 are given in the corresponding rows of Tables 1, 2, 3 and for $n = 2$ they are equal to:

$$\mathrm{U}: u^* = 1.500, f^* = 0,$$
$$\mathrm{C1}: u^* = 0.181, f^* = 0.158,$$
$$\mathrm{C2}: u^* = 2.457, f^* = 0.158.$$

6. Critical exponents

The values of critical exponents can be determined by the fixed point values of the ζ-functions defined on the basis of renormalizing Z-factors (15) - (17) by:

$$\zeta_i = \mu\partial \ln Z_i/\partial\mu, \qquad (59)$$

where the derivative is taken at fixed unrenormalized couplings. The expressions for the ζ-functions related to the order parameter and the temperature field renormalization in two-loop approximation read [30]:

$$\zeta_\psi = -3f + \frac{(n+2)}{72}u^2 + \frac{(11n+18)}{24}f^2, \qquad (60)$$

$$\zeta_t = \frac{-(n+2)}{6}u + \frac{(n+2)}{12}u^2 - \frac{2(n+2)}{3}uf - \frac{(5n+1)}{2}f^2, \qquad (61)$$

$$\zeta_A = \frac{n}{6}f + nf^2. \qquad (62)$$

If there exists a stable fixed point, the critical exponent ν of the correlation length, the critical exponent γ of the order parameter susceptibility and the critical exponent α of the specific heat are given by:

$$\nu = (2-\zeta_\nu^*)^{-1}, \qquad (63)$$

$$\gamma = (2-\zeta_\nu^*)^{-1}(2-\zeta_\psi^*), \qquad (64)$$

$$\alpha = (2-\zeta_\nu^*)^{-1}(\varepsilon - 2\zeta_\nu^*), \qquad (65)$$

$$\eta = \zeta_\psi^*, \qquad (66)$$

here $\zeta_\nu = \zeta_\psi - \zeta_t$. The exponents (63) - (66) are related by the familiar scaling laws. From the analysis given above it follows that the charged fixed point $C2$ is the stable one and this results in the values for exponents (63)–(65) different from the values of the uncharged fixed point U, i.e. they are not given by the ^4He values as it is sometimes stated (see e.g. [16,59,25]).

Recently an interesting consequence of the existence of a stable charged fixed point (C_2) has been observed [62]. According to the renormalization of the charge (3) the β_f-function reads

$$\beta_f = f\ (\varepsilon - \zeta_A(f,u)) \qquad (67)$$

Thus at a fixed point with f^* nonzero the value of the gauge field ζ-function is **exactly** given by $\zeta_A^* = \varepsilon$. That means that the penetration depth λ and the correlation length ξ are proportional and the temperature dependence follows a power law with the exponent ν [62]. At the fixed point with $f^* = 0$ this is not the case, there we have $\zeta_A^* = 0$ (each loop contribution to the ζ_A-function contains at least one f-factor). Then the penetration depth behaves as $\lambda \sim \xi^{\frac{2-\varepsilon}{2}}$ and one would have two different critical length scales.

Trying to obtain the numerical values of the critical exponents on the basis of the values of fixed point $C2$ coordinates $f^{*C,Padé}$, $u^{*C2,Padé}$ given in Tables 1, 2 in order to be self-consistent let us perform the same type of analytical continuation for the series for ζ-functions, as those, which have been applied to the β-functions (18), (19). So, introducing the auxiliary variable τ let us represent functions (63) - (65) in the form of resolvent series in τ and then we will chose the [1/1] Padé approximants for these series, which at $\tau = 1$ will give us the analytical continuation of the series requested. Obtained in such a way expression for the critical exponent $\phi\,(\phi \equiv \{\nu, \gamma, \alpha)\}$ reads:

$$\phi = a_\phi^{(0)}\,\frac{1 + A_\phi}{1 + B_\phi}. \tag{68}$$

The expressions for the coefficients A_ϕ, B_ϕ in (68) read:

$$A_\phi = a_\phi^{(1)} \;+\; B_\phi; \quad B_\phi = -a_\phi^{(2)}/a_\phi^{(1)}, \tag{69}$$

and $a_\phi^{(i)}$ are to be determined from the resolvent series in τ:

$$\phi = \sum_{i\geq 0} a_\phi^{(i)}\;\tau^i|_{\tau=1}. \tag{70}$$

Substituting (60) and (61) into (63) - (65) and representing (63) - (65) in the form of (70) one finds:

$$\begin{aligned}
&a_\nu^{(0)} = 1/2,\\
&a_\nu^{(1)} = (n+2)/12\;u - 3/2\;f,\\
&a_\nu^{(2)} = (n^2 - n - 6)/144\;u^2 + (71n + 138)/48\;f^2 +\\
&(n+2)/12\;uf,
\end{aligned} \tag{71}$$

$$\begin{aligned}
&a_\gamma^{(0)} = 1,\\
&a_\gamma^{(1)} = (n+2)/12\;u,\\
&a_\gamma^{(2)} = (n^2 - 2n - 8)/144\;u^2 + (5n+1)/4\;f^2 +\\
&5(n+2)/24\;uf,
\end{aligned} \tag{72}$$

$$\begin{aligned}
&a_\alpha^{(0)} = 1,\\
&a_\alpha^{(1)} = -3(n+2)/12\;u \;+\; 9/2\;f,\\
&a_\alpha^{(2)} = (-3n^2 + 3n + 18)/144\;u^2 - (71n + 138)/16\;f^2 -\\
&(n+2)/4\;uf.
\end{aligned} \tag{73}$$

Now considering the case $n = 2$ and substituting coordinates of the fixed point $C2$ ($f^{*C,Padé} = .158$), $u^{*C2,Padé} = 2.457$ (see Tables 1, 2) into (71) - (73) one obtains for the critical exponents (63) - (66) [105]:

$$\begin{aligned}
&\nu = 0.86, \quad \gamma = 1.88,\\
&\alpha = -1.14, \quad \eta = -0.19.
\end{aligned} \tag{74}$$

The application of the Padé approximants for the analytical continuation of the functions may result in the appearance of poles in these functions. If the pole is located in the region of expansion parameters which is unphysical (e.g. negative coupling u or f) this does not complicate the analysis. This was the case for the β-functions in the region of couplings less than the fixed point values. For the ζ-functions however considering the non-asymptotic behavior (and thus being far from the stable fixed point) one passes through a region of couplings where the Padé approximation for the ζ-functions becomes ambiguous resulting in the appearance of a pole. Therefore studying the crossover behavior in the next subsection we will still keep the polynomial representation for ζ-functions instead of the Padé approximants. Then for the asymptotic values of critical exponents one gets:

$$\begin{aligned}
&\nu = 0.77, \quad \gamma = 1.62,\\
&\alpha = -0.31, \quad \eta = -0.10.
\end{aligned} \tag{75}$$

Comparing the values (75) and (76) show a numerical difference of 15% in ν and γ and a considerable increase of α. However there is no qualitative change (e.g. the sign of the specific heat exponent remains the same). This should be compared with the values, given by other authors: $\nu = 0.53$ and $\eta = -0.70$ [62] and $\eta = -0.38$ [61].

Since for the conventional superconductors the experimentally accessible regime lies in the precritical region further away from T_c, let us discuss here several non-asymptotic quantities such as effective exponents and amplitude ratios.

7. Amplitude ratio for the specific heat

One of the most interesting measurable quantity is the specific heat. Asymptotically near a second order phase transition it follows a power law

$$C_0^\pm = \frac{A^\pm}{\alpha}|t|^{-\alpha} + \text{const.} \tag{76}$$

where $\pm$ indicates the specific heat C and its non universal amplitude A above and below T_c. The amplitude ratio A^+/A^- found from the ratio $C^+(t^+)_0/C^-(t^-)_0$ after subtracting the non singular background value constitutes at T_c a universal quantity depending only on dimension and the number of components of the order parameter.

The calculation of this ratio can be extended to the non asymptotic region [106,107] resulting in a temperature dependent measurable quantity, which also tests the description of non asymptotic behavior by a certain flow in the interaction space of the Hamiltonians, as it was discussed for the effective exponents. The starting point in the calculation is the renormalization group equation for the specific heat $C^\pm$

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_u\frac{\partial}{\partial u}+\beta_f\frac{\partial}{\partial f}+\zeta_\nu\left(2+t\frac{\partial}{\partial t}\right)\right]C^\pm(t,u,f,\mu)= \mu^{-\varepsilon}B(u,f) \tag{77}$$

where the inhomogeneity B comes from the additive renormalization. The formal solution reads

$$C^\pm(t,u,f,\mu)=\mu^{-\varepsilon}\exp\left[-\int_1^l(\varepsilon-2\zeta_n u(x)\frac{dx}{x}\right] \left\{F^\pm(l)-\int_1^l\frac{dy}{y}B(y)\exp\left[-\int_l^y(\varepsilon-2\zeta_\nu(x))\frac{dx}{x}\right]\right\} \tag{78}$$

The amplitude ratio is most easily calculated by choosing the same value of flow parameter above and below T_c, which means for the temperatures $t^+=-2t^-$ [108]. We then recover the asymptotic expression found in [107]

$$\frac{A^+}{A^-}=2^\alpha\frac{B\nu+F^+\alpha}{B\nu+F^-\alpha} \tag{79}$$

where the functions B and $F^\pm$ are taken at the fixed point. We use for this functions the lowest order result known from the Ψ^4 theory neglecting the coupling to the gauge field; $B=2n$, $F^+=-n$ and $F^-=12/u^*-4$. Then we have for $n=2$

$$\frac{A^+}{A^-}=2^\alpha\frac{2\nu-\alpha}{2\nu-2\alpha+6\alpha/u^*} \tag{80}$$

In the Table 4 we collect the values obtained at the different fixed points. It is interesting to note the reasonable estimate for this ratio presented in [109] for the superfluid phase transition. The authors found a value of $A^+/A^-=1.067$, which is surprisingly near the value obtained in the stable charged fixed point using both calculation schemes although the exponents are very different. However we did not take into account changes in the scaling functions due to the coupling f.

In the comparison with experiments [18] the amplitude ratio of the Gaussian n-vector model without coupling to the gauge field $A^+/A^-=n/2^{3/2}$ has been used since the dimension of the order parameter was unclear. Later on this expression for the amplitude ratio was calculated for other than isotropic symmetry. This leads to a dependence of the ratio on the higher order couplings [110].

8. Effective exponents

Effective exponents are usually defined by the logarithmic temperature derivatives of the corresponding correlation functions (see e.g. [26,111]). These can be found from the solutions of the renormalization group equation for the renormalized vertex functions. These effective exponents contain two contributions, one from the corresponding ζ-functions now taken at the values of $u(\ell)$, $f(\ell)$ of the flow curve considered ("exponent part"), and one from the change of the corresponding scaling function ("amplitude part"). For the analysis we give below we neglect the last contributions since we expect them to be smaller than the differences for the fixed point values of the exponents coming from the different treatments discussed before. Thus we have:

$$\nu = (2-\zeta_\nu(\ell))^{-1}, \tag{81}$$

$$\gamma = (2-\zeta_\nu(\ell))^{-1}(2-\zeta_\psi(\ell)), \tag{82}$$

$$\alpha = (2-\zeta_\nu(\ell))^{-1}(\varepsilon-2\zeta_\nu(\ell)). \tag{83}$$

The flow parameter ℓ can be related to the relative temperature distance T_c by the matching condition $t(\ell)=(\xi_0^{-1}\ell)^2$, with ξ_0 the amplitude of the correlation length.

We have computed these effective exponents, see Fig. 11 - Fig. 13, along the flow lines shown in Fig. 10 by inserting [112] values of the couplings $u(\ell)$ and $f(\ell)$ into Eqs. (81)-(83). For the separatrix 1 we started with initial conditions leading to a flow, which did not stick in the fixed point C1 but slightly missed it although the flow curve did not differ from the separatrix within the thickness of the lines shown in Fig. 10. For the curve number 4 we started somewhat further away from the Gaussian fixed point G leading to the initial values of the effective exponents between their Gaussian values and their values for the uncharged fixed point U. Note that the values of the effective exponent γ for the uncharged fixed point U and the charged fixed point C1 are the same within the accuracy given by the scale of the figure.

Note that when coupling f to the gauge field fluctuations is small (i.e. for extreme type-II superconductors) the RG flow pass very close to the uncharged fixed point U and the effective exponents, within some region of temperatures, coincide with those of the uncharged superfluid liquid. In this region the effective Hamiltonian (2) may be considered as that of a superconductor in a constant magnetic field neglecting magnetic field fluctuations. Recently for such a model it was shown that near the zero-field critical point the singular part of the free energy scales as $F_{sing} \simeq |t|^{2-\alpha}\mathcal{F}(B|t|^{-2\nu})$with ν being the coherence length exponent [113].

9. Conclusions

Does the above account give the definite conclusion about the order of a phase transition occurring in a model of superconductor minimally coupled to the gauge field? First of all one should keep in mind that such kind of answer may be given in the frames of the analytic theory only by obtaining exact result or a rigorous proof. Here, the problem was treated by the perturbation theory approach and the account of the influence of fluctuations on the order of phase transition was studied within the field theoretical RG technique. We show that remaining inside this approach one may get an answer about the second order phase transition occurring in the above mentioned model.

The main point which is discussed in this context is whether the equations for β-functions possess a stable fixed point or not. The absence of the stable fixed point is often interpreted as a change of the order of phase transition (caused by the presence of the magnetic field fluctuations) and evidence of the fluctuation-induced first-order phase transition. However this change of the order of the phase transition (being of the second-order in the absence of the coupling to the gauge) is confirmed only by perturbation theory calculations in low orders ([15], see [30] and the references therein as well).

Applying a simple Padé analysis to the series under discussion [114] we have shown how one can recover a stable fixed point in the RG equations. In the case of one coupling such an approach gives a qualitatively correct picture of the phase transition and restores the presence of a stable fixed point ([94], see formulas (38), (42) of this article as well). The same situation happens here in the case of two couplings: at $n = 2$ the "uncharged" fixed point U (having coordinates $f^{*U,Padé} = .158$, $u^{*U,Padé} = 2.457$) appears to be stable, which leads to a new set of critical exponents. However we note however that the pair correlation function critical exponent η calculated by familiar scaling relations on the basis of sets of values (75) or (76) remains negative, which agrees with the result of [61,62,54]. Being calculated only in a two-loop approximation with the application of Padé analysis, these values for the critical exponents are to be considered as preliminary ones. The main point we claim here is that within the framework of the renormalization group analysis for the superconductor model there still exists the possibility of a second-order phase transition characterized by a set of critical exponents differing from those of ^4He.

Another important task could be to calculate the nonasymptotic specific heat in order to compare with experiments within the region of crossover to the background.

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98. The results are given for the value of parameter $p = 0$ in the Borel-Leroy image.
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100. The last possibility was chosen by G. Parisi [94] in order to restore the presence of stable solution for the fixed point in two-loop approximation.
101. The series in (37) appears to be an alternating one and this scheme can be applied without any difficulties.
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103. In this case the principal value of the integral (29) could be taken, but generally speaking it is preferable to avoid such situations (see [7] as well).
104. We take $\ell_0 = 1$.
105. The value of η has been found by the scaling law: $\eta = 2 - \gamma/\nu$.
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112. In fact we have solved the flow equations (57), (58) starting near the unstable fixed points, for the initial value of the flow parameter we took $\ell = 1$. Using different initial values ($u(1)$ and $f(1)$ on the seperatrix) would amount to rescale the flow parameter.
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114. Because they are double series in two coupling constants we made use of representing them in the form of resolvent series which enabled us then to pass to Padé analysis.

n	1	2	3	4	5	6	7	8
$u^{*U,Padé}$	1.800	1.500	1.269	1.091	0.951	0.840	0.750	0.676
$u^{*U,Res}$	1.315	1.142	1.002	0.888	0.794	0.717	0.652	0.597

Table 1. Fixed point U coordinate u^{*U} as a function of n. $u^{*U,Padé}$: obtained on the basis of [1/1] Padé approximant; $u^{*U,Res}$: obtained by Padé-Borel resummation.

n	1	2	3	4	5	6	7	8
$f^{*C,Dir}$	0.920	0.629	0.500	0.424	0.372	0.333	0.304	0.280
$f^{*C,Padé}$	0.162	0.158	0.154	0.150	0.146	0.143	0.140	0.136
$f^{*C,\varepsilon}$	6.000	3.000	2.000	1.500	1.200	1.000	0.857	0.750
f^{*C,ε^2}	-210.000	-51.000	-22.000	-12.000	-7.440	-5.000	-3.551	-2.625

Table 2. Fixed point C coordinate f^{*C} as a function of n. $f^{*C,Dir}$: obtained by direct solution of the equation for fixed point; $f^{*C,Padé}$: obtained on the basis of [1/1] - Padé approximant; $f^{*C,\varepsilon}$: ε-expansion result with the linear accuracy in ε; f^{*C,ε^2}: ε-expansion result with the square accuracy in ε.

n	1	2	3	4	5	6	7	8
C1	0.184	0.181	0.179	0.177	0.175	0.175	0.176	0.179
C2	3.309	2.457	1.781	1.150	0.473	0.369	0.305	0.256

Table 3. Fixed point C coordinates $u^{*C,Padé}$ obtained on the basis of [1/1] Padé approximant for the "resolvent" series as a function of n. C1 : unstable fixed point; C2 : stable fixed point.

F.P.	A^+/A^-	α	ν	u^*
U	1.81	-0.33	0.72	1.500
C2	1.07	-1.14	0.86	2.457
U	0.78	0.15	0.62	1.500
C2	1.06	-0.31	0.77	2.457

Table 4. Asymptotic values for the specific heat amplitude ratio at various fixed points. The exponents in the first two lines correspond to the procedure leading to (75). The third and the fourth lines correspond to the procedure leading to (76).

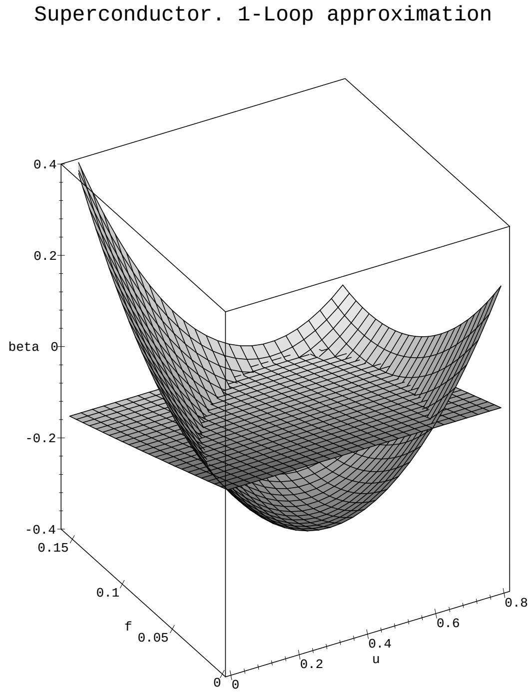

Figure 1. β-functions of the model of superconductor $\beta_u(u,f)$, $\beta_f(u,f)$ in one-loop approximation for $d = 3$, $n = 2$. The fixed points have coordinates $(u^* = f^* = 0)$, $(u^* = 0.6,\ f^* = 0)$.

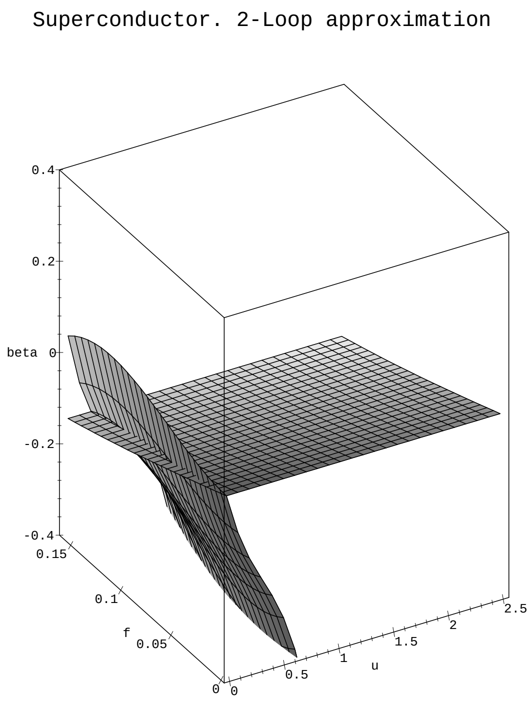

Figure 2. β-functions of the model of superconductor $\beta_u(u,f)$, $\beta_f(u,f)$ in two-loop approximation for $d=3$, $n=2$. Only the Gaussian fixed point $u^*=f^*=0$ survives.

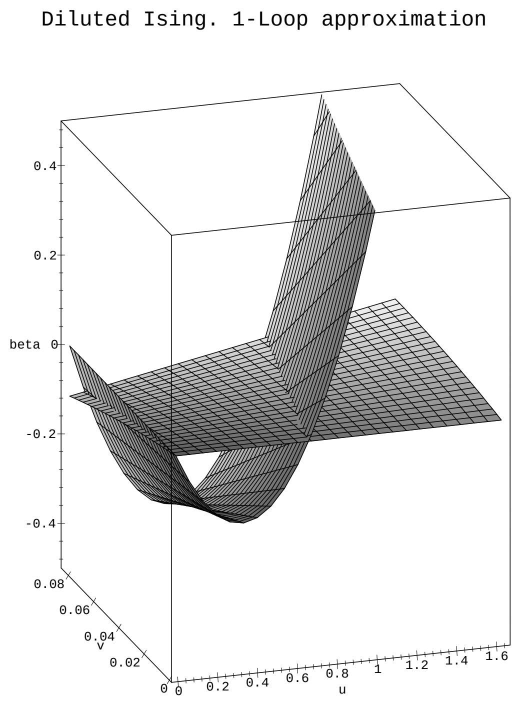

Figure 3. β-functions of the diluted Ising model $\beta_u(u,v)$, $\beta_v(u,v)$ in one-loop approximation for $d=3$. The fixed points have coordinates $(u^*=v^*=0)$, $(u^*=0.667,\, v^*=0)$.

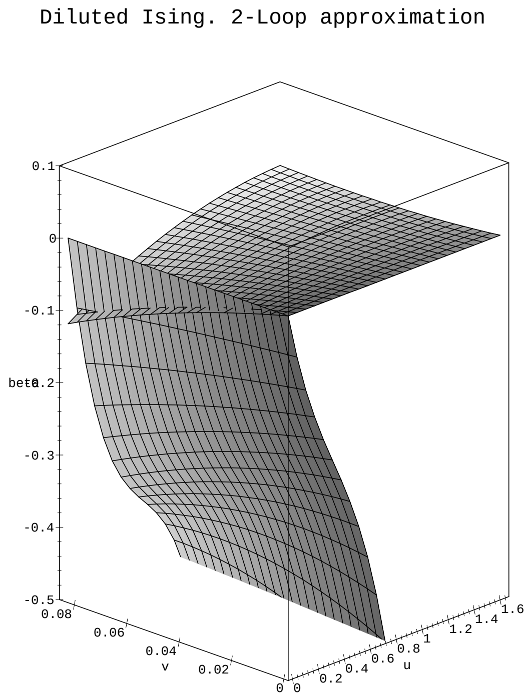

Figure 4. β-functions of the diluted Ising model $\beta_u(u,v)$, $\beta_v(u,v)$ in two-loop approximation for $d=3$. Only the Gaussian fixed point $u^*=v^*=0$ survives.

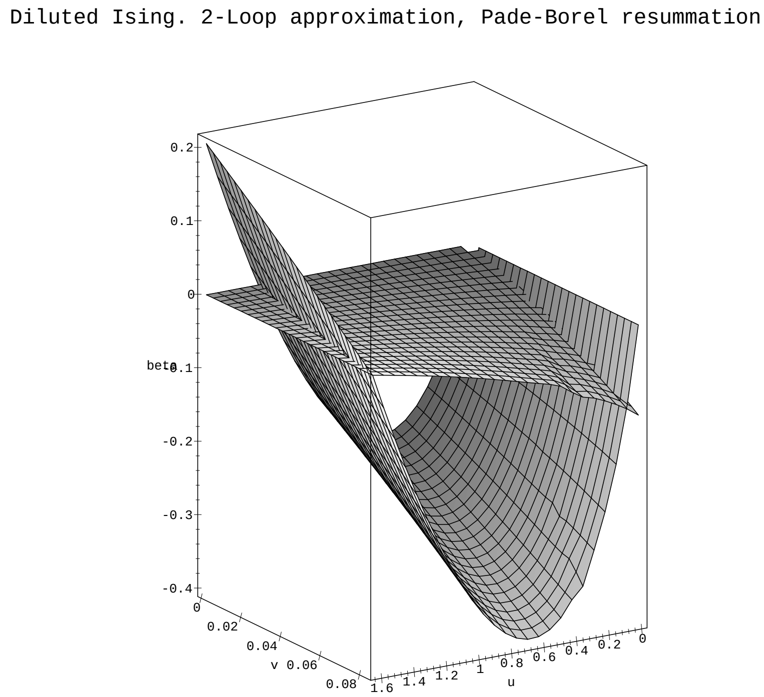

Figure 5. β-functions of the diluted Ising model $\beta_u(u,v)$, $\beta_v(u,v)$ in two-loop approximation for $d=3$ obtained by applying Padé-Borel resummation technique. Resummations restores the presence of the fixed point $u^*\neq 0$, $v^*=0$ and results in the appearance of a new stable fixed point $u^*\neq 0$, $v^*\neq 0$.

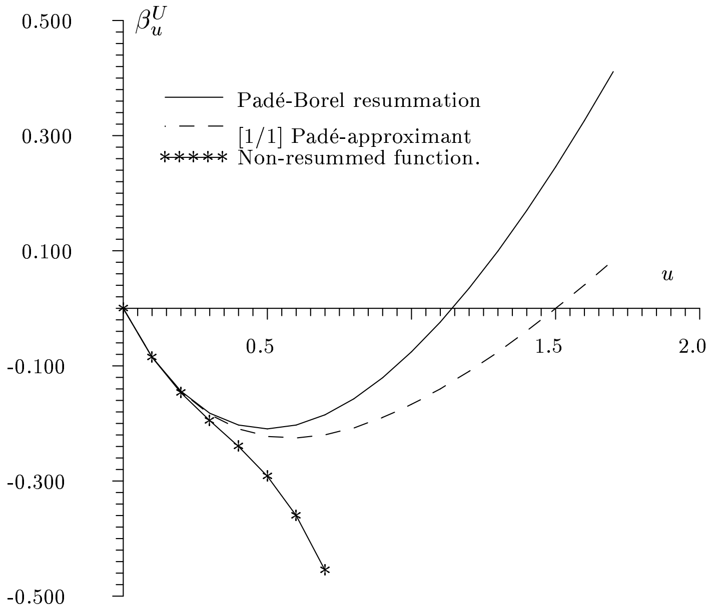

Figure 6. β_u-function of the uncharged model β_u^U at $d = 3$, $n = 2$.

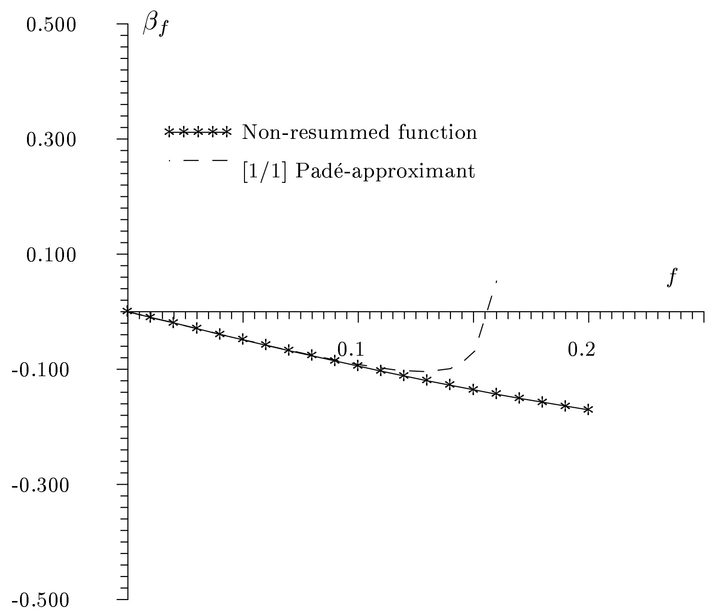

Figure 7. β_f-function at $d = 3, n = 2$.

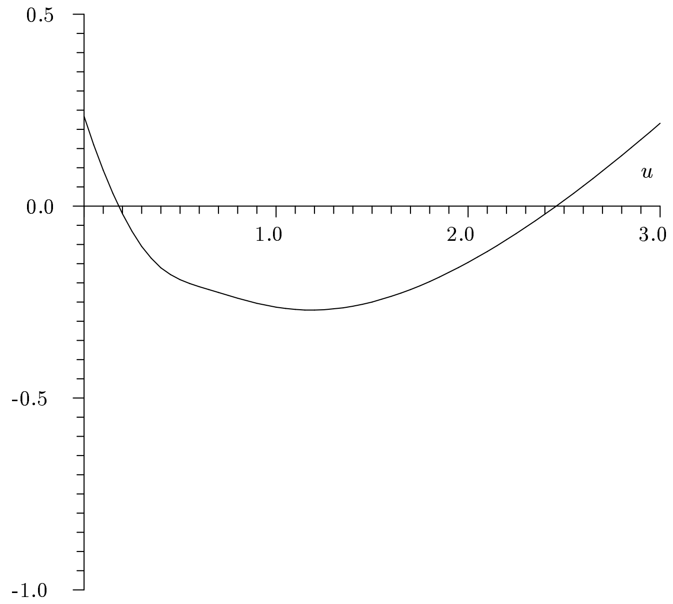

Figure 8. Intersection of the function $\beta_u^{Padé}(u,f)$ at $d=3$, $n=2$ with the plain $f=f^{*C,Padé}$ in two-loop approximation.

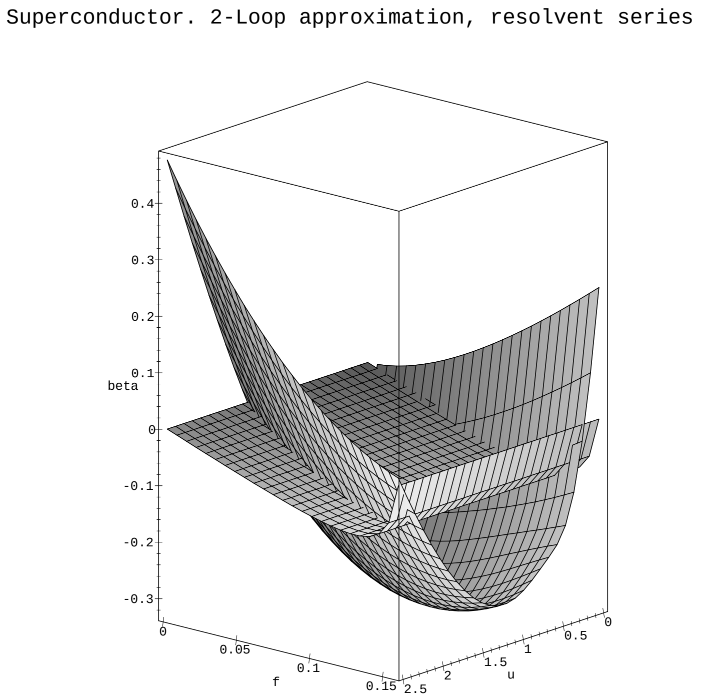

Figure 9. β-functions of the model of superconductor $\beta_u(u,f)$, $\beta_f(u,f)$ in two-loop approximation for $d=3$, $n=2$ obtained by Padé analysis for the resolvent series. The stable "charged" fixed point C2 with coordinates $u^*=2.457$, $f^*=0.158$ as well as the unstable fixed point C1 $u^*=0.181$, $f^*=0.158$ are seen on the front side of the cube. Gaussian ($u^*=f^*=0$) and "uncharged" ($u^*=1.500$, $f^*=0$) fixed points are located on the back side of the cube.

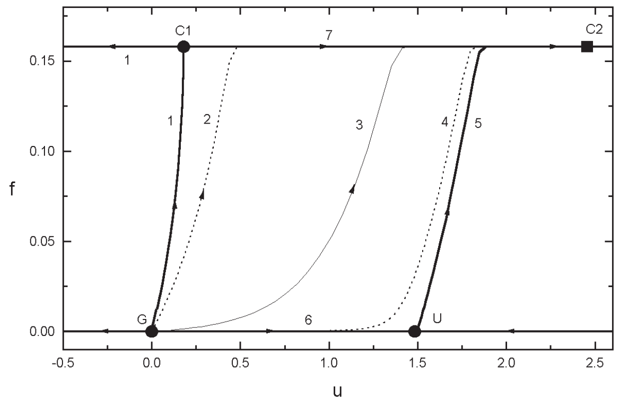

Figure 10. Flow lines for the case $n = 2$, $d = 3$ given by equations (57), (58). Fixed points G, U, C1 are unstable, fixed point C2 (shown by a box) is a stable one (for further description see text).

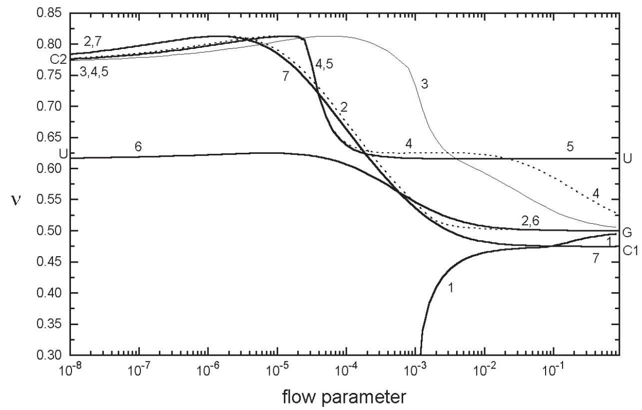

Figure 11. Effective exponent ν for the flows shown in Fig.10 (for further description see text).

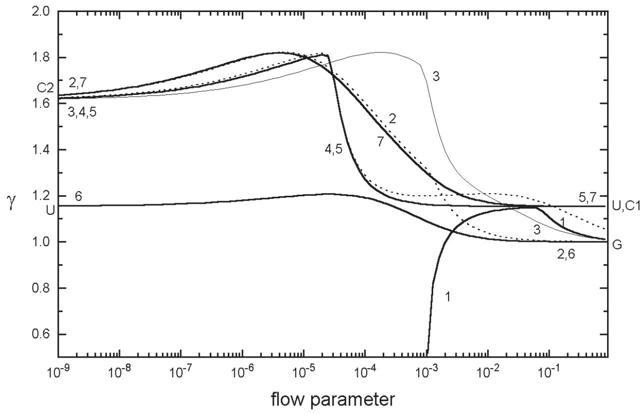

Figure 12. Effective exponent γ for the flows shown in Fig.10 (for further description see text).

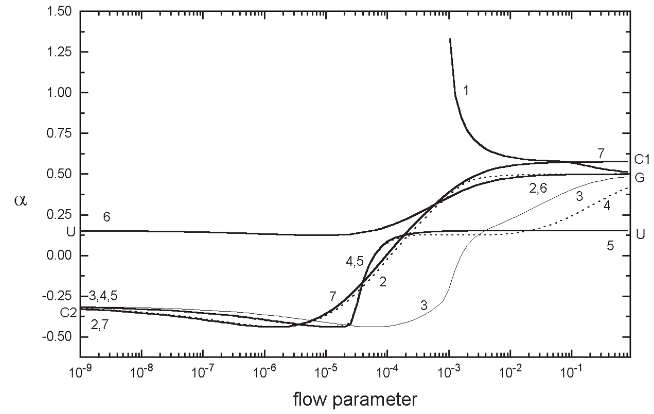

Figure 13. Effective exponent α for the flows shown in Fig.10 (for further description see text).

Рейнгард Фольк
Юрій Головач

Критичні флюктуації у фазовому переході з нормального в надпровідний стан

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