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FIELD THEORETIC OPERATORS FOR MULTIFRACTAL
MOMENTS

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Теоретико-польові оператори для мультифрактальних моментів

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Анотація. Опуклість мультифрактального спектру суперечить, на перший погляд, загальним теоретико-польовим аргументам, які стверджують, що степінь польових операторів ϕ^f приводить до ввігнутого спектру показників як функції f . Ця проблема вирішується відповідним вибором операторів для опису мультифрактальних моментів. В Лагранжевій польовій теорії полів двох типів ϕ, ψ , що взаємодіють між собою і мають нуль компонент, оператори $O_{f',f} = \psi^{f'} \phi^f$ із "traceless" симетрією приводять до мультифрактального спектру гармонійної дифузії біля фрактального абсорбера.

Field theoretic operators for multifractal moments

Ch. von Ferber, Yu. Holovatch

Abstract. Convexity of multifractal spectra seems to contradict general field theoretic arguments showing that power of field operators ϕ^f yield a concave spectrum of exponents as function of f . This is resolved by appropriate choice of operators to describe multifractal moments. In a Lagrangian field theory of two mutually interacting species of fields ϕ, ψ , operators $O_{f',f} = \psi^{f'} \phi^f$ with traceless symmetry give rise to multifractal spectra of harmonic diffusion near absorbing fractals when evaluated for zero component fields.

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1. Introduction

The concept of multifractality developed in the last decade has proven to be a powerful tool for analyzing systems with complex statistics which otherwise appear to be intractable [1,2]. It has found direct applications in a wide range of fields including turbulence, chaotic attractors, Laplacian growth phenomena etc [2–8]. Here we generalize the idea of Cates and Witten [9,10] by deriving the multifractal (MF) spectrum in the frames of a field theoretical (FT) formalism and make use of renormalization group (RG) methods. We relate the MF spectrum to the spectrum of scaling dimensions of a family of composite operators of Lagrangian ϕ^4 field theory. This gives an example of power of field operators whose scaling dimensions show the appropriate convexity for a MF spectrum [11], while there is no need to include field gradients for this property. We calculate the MF spectrum to third order of perturbation theory using two complementary approaches: zero mass renormalization with successive ε -expansion (see e.g. [12]) and massive renormalization group approach at fixed dimension [13], reproducing previous results obtained in lower order of perturbation theory for special cases [9,10]. The resulting series are asymptotic. We take this into account and obtain numerical values only by careful resummation.

We address a special case of a growth process controlled by a Laplacian field. The latter may describe a variety of phenomena depending on the interpretation of the field. For diffusion limited aggregation this field is given by the concentration of diffusing particles, in solidification processes it is given by the temperature field, in dielectric breakdown it is the electric potential, in viscous fingers formation it is the pressure [5,15]. In all mentioned phenomena the resulting structure appears to be of fractal nature and is characterized by appropriate fractal dimensions [1]. Its growth and spatial correlations lead to (non-trivial) spectra of multifractal dimensions [2]. In general, the boundary conditions determining the field will be given on the surface of the growing aggregate itself. It is this dynamic coupling that produces the rich structure of the phenomena and seems to make the general dynamical problem intractable.

Here we study the simpler case when the fractal has been already formed and look for the distribution of the Laplacian field and its higher moments near the surface of the fractal [9,10]. We will follow the diffusion picture, considering the aggregate as an absorbing fractal, “the absorber”. The field $\rho(\vec{r})$ gives the concentration of diffusing particles and vanishes on the surface of the absorber. More specifically we con-

sider the Laplacian field $\rho(\vec{r})$ in the vicinity of an absorbing (fractal) path, ‘a polymer’, or a junction of absorbing paths, ‘a core of a polymer star’. In general we assume the ensemble of absorbers to be characterized by either random walk (RW) or self-avoiding walk (SAW) statistics. Multifractal scaling is found for the f -moments $\langle \rho^f(\vec{r}) \rangle$ of the field with respect to these ensembles.

This formulation of the problem allows us to use the polymer picture and theory developed for polymer networks and stars [16–18] and extended for copolymer stars [19,20]. The theory is mapped to a Lagrangian ϕ^4 field theory with several couplings [21–23] and higher order composite operators to describe star vertices.

In section 2 we present the path integral solution of the Laplace equation and relate it to polymer theory. Field theoretical representation and renormalization are discussed in section 3. together with the renormalization group flow and expressions for the exponents. In section 4. we define the multifractal spectrum and give its series expansion in both renormalization group approaches adopted here. Section 5. is devoted to resummation of these asymptotic series and numerical results followed by some conclusions and an outlook in section 6..

2. Path Integral Solution of the Laplace Equation and Polymer Absorber Model

In this section we describe the diffusion of particles in the vicinity of a polymer absorber by a “polymer” formalism. Let us formulate the problem first in terms of diffusion of particles in time. The probability of finding a randomly walking particle at point \vec{r}_1 at time t which started at point \vec{r}_0 at time $t = 0$ is given described by the following normalized path integral:

$$G^0(\vec{r}_0, \vec{r}_1, t) = \langle \delta(\vec{r}^{(1)}(0) - \vec{r}_0) \delta(\vec{r}^{(1)}(t) - \vec{r}_1) \rangle_{\mathcal{H}_0(t)}. \quad (1)$$

Angle brackets in (1) denote the normalized integral

$$\langle \dots \rangle_{\mathcal{H}_0(t)} = \frac{\int (\dots) \exp(-\mathcal{H}_0(t)) d\{r^{(1)}\}}{\int \exp(-\mathcal{H}_0(t)) d\{r^{(1)}\}}, \quad (2)$$

which is performed with the Hamiltonian:

$$\mathcal{H}_0(t) = \int_0^t \left(\frac{d\vec{r}^{(1)}(\tau)}{d\tau} \right)^2. \quad (3)$$

Integration in (1) is performed over all paths $\vec{r}^{(1)}(\tau)$ with $0 \leq \tau \leq t$. Note that we have absorbed the diffusion constant into re-definition of time. $G^0(\vec{r}_0, \vec{r}_1, t)$ obeys the following differential equation:

$$\left(\Delta_{r_1} + \frac{d}{2t} - \frac{\partial}{\partial t}\right)G^0(r_0, r_1, t) = 0 \quad (4)$$

with d the dimension of space. For finite d the random walker will visit any site after some finite time and we may assume a steady state limit for $G^0(r_0, r_1, t)$ for $t = \infty$. In this case $G^0(\vec{r}_0, \vec{r}_1, t)$ will become independent of \vec{r}_0 and its limit and defines a field:

$$\rho(\vec{r}_1) = \lim_{t \rightarrow \infty} \frac{1}{V} \int d\vec{r}_0 G^0(r_0, r_1, t), \quad (5)$$

here V is the system volume. The field $\rho(\vec{r})$ obeys the Laplace equation:

$$\Delta\rho(\vec{r}) = 0. \quad (6)$$

We introduce boundary conditions in such a way that the field $\rho(\vec{r})$ equals to some constant ρ_∞ at $r = \infty$ and vanishes on the absorber. The absorber itself we describe by a path $\vec{r}^{(2)}(s)$, $0 \leq s \leq S_2$.

Let us explain the solution of the Laplace equation (5) in the presence of an absorbing path $\vec{r}^{(2)}(s)$. The boundary conditions are implemented by an avoidance interaction u_{12} punishing any coincidence of the path $r^{(1)}$ of the RW and the path $r^{(2)}$ of the absorber. The correlation function of a random walk in the presence of an absorbing path $\vec{r}^{(2)}(s)$ may then be written as

$$G(\vec{r}_0, \vec{r}_1, S_1) = \langle \delta(\vec{r}^{(1)}(0) - \vec{r}_0) \delta(\vec{r}^{(1)}(S_1) - \vec{r}_1) \exp\left\{-u_{12} \int_0^{S_1} ds_1 \int_0^{S_2} ds_2 \delta(\vec{r}^{(1)}(s_1) - \vec{r}^{(2)}(s_2))\right\} \rangle_{\mathcal{H}_0(S_1)}, \quad (7)$$

where we have adopted the notation $t = S_1$.

We are interested in ensemble moments $\langle \rho^{f_2}(\vec{r}_1) \rangle$ of the field in the vicinity of the absorber, assuming an ensemble of RW or SAW absorbers. For the RW ensemble the average is performed with respect to the Hamiltonian $\mathcal{H}_0(S_2)$, for the SAW ensemble an additional interaction has to be included.

Here we choose a more general formulation, which allows us to describe the moments of the field in the vicinity of the core of an absorbing polymer star, or near the junction of f_1 absorbing paths.

The calculation of the f_2 moment of (7) near the junction of f_1 absorbing paths will include the average over f_2 random walks ending at

r_1 and the ensemble average over the configurations of the f_1 absorbing paths with junction at r_1 .

We are thus lead to consider the partition function of a star of walks which are in part mutually-avoiding. We will give this partition function here for the more general case of f walks of which f_2 random walks describe the field and f_1 walks correspond to the absorber. This situation describes the f_2 th moment of the flux of the field to the core of an absorbing star of f_1 walks. We allow for additional avoidance interactions among these absorbing paths:

$$\mathcal{Z}_{*f} = \frac{1}{\mathcal{Z}_{*f}^0} \left\langle \prod_{i=1}^f \delta(\vec{r}_i^{(i)}(0) - \vec{r}_0) \exp\left\{-\frac{1}{6} \sum_{a,b=1}^f u_{ab} \int_0^{S_a} ds_a \int_0^{S_b} ds_b \delta(\vec{r}^{(a)}(s_a) - \vec{r}^{(b)}(s_b))\right\} \right\rangle_{\mathcal{H}_0^{*f}\{S_a\}}, \quad (8)$$

here

$$\mathcal{H}_0^{*f}\{S_a\} = \sum_{i=1}^f \mathcal{H}_0(S_i)$$

where $\mathcal{H}_0(S_i)$ is given by (2). \mathcal{Z}_{*f}^0 stands for the partition function of star with zero interactions $u_{ab} = 0$. The matrix u_{ab} is given in the following form:

$$u_{ab} = \begin{cases} u_{11}^0 & \text{if } a, b \leq f_1 \\ u_{12}^0 & \text{if } a \leq f_1 < b \leq f_1 + f_2 \\ & \text{or } b \leq f_1 < a \leq f_1 + f_2 \\ 0 & \text{else} \end{cases} \quad (9)$$

This corresponds to the partition function of co-polymer stars consisting of two species of chains [19,20] with f_1 chains of one species and f_2 chains of the other. u_{11}^0 is the interaction between absorbing paths.

3. Field Theory and Renormalization

As is well known, the polymer model may be mapped to the limit of $m = 0$ of $O(m)$ -symmetrical Lagrangian field theory [26]. To describe polymers and interacting random walks at the same time we adopt the formalism developed for multicomponent polymer solutions [23]. Its field theory is described by the following Lagrangian:

$$\mathcal{L}\{\phi_b, \mu_b\} = \frac{1}{2} \sum_{a=1}^f \int d^d r (\mu_a \phi_a^2 + (\nabla \phi_a(r))^2)$$

$$+ \frac{1}{4!} \sum_{a,a'=1}^f u_{a,a'} \int d^d r \phi_a^2(r) \phi_{a'}^2(r). \quad (10)$$

in general m -component theory

$$\phi_a^2 = \sum_{\alpha=1}^m (\phi_a^\alpha)^2. \quad (11)$$

μ_a is a chemical potential conjugated to the Gaussian surfaces S_a in (7). Correlation functions in this theory are defined by averaging with the weight given by (10):

$$\langle (\dots) \rangle_{\mathcal{L}} = \int \mathcal{D}[\phi_a(r)] (\dots) \exp[-\mathcal{L}\{\phi_b, \mu_b\}] |_{m=0}. \quad (12)$$

here functional integration $\int \mathcal{D}[\phi_a(r)]$ is defined in such a way that normalization is already included: $\langle 1 \rangle_{\mathcal{L}} = 1$ if all $u_{a,a'} = 0$. The limit $m = 0$ in (12) can be understood as a certain rule to calculate the diagrams appearing in the perturbation theory expansions and can be easily checked diagrammatically.

The partition function \mathcal{Z}_{*f} defined in (7) is mapped to the field theoretical correlation function $\tilde{\mathcal{Z}}_{*f}$ via a Laplace transform in the Gaussian surfaces S_a to conjugate chemical potentials (“mass variables”) μ_a :

$$\tilde{\mathcal{Z}}_{*f}\{\mu_a\} = \int_0^\infty \prod_b dS_b e^{-\mu_b S_b} \mathcal{Z}_{*f}\{S_a\}, \quad (13)$$

and

$$\tilde{\mathcal{Z}}_{*f}\{\mu_a\} = \left\langle \int dr_a \prod_{a=1}^f \phi_a(r_0) \phi_a(r_a) \right\rangle_{\mathcal{L}} \quad (14)$$

Our interest is in the scaling properties of these functions. Note that by (14) these are governed by the spectrum of scaling dimensions of the composite operators $\prod_{a=1}^f \phi_a$. To extract them we use renormalization group methods [27,28]. Here we use the results of our previous approaches to the problem of co-polymer stars: massless renormalization group scheme with successive ε -expansion (see e.g. [12]) and massive renormalization group approach at fixed dimension [13] compiled in a pseudo- ε expansion [30]. On the basis of correlation functions it is standard to define vertex functions $\Gamma_{u_{ab}}^4$ corresponding to the couplings u_{ab} as well as vertex functions $\Gamma_{\Pi\phi_a}^{*f}$ with insertion of composite operators $\prod_a \phi_a$.

Explicit expressions may be found in [19,20]. We define renormalization and introduce renormalized couplings g_{ab} by:

$$u_{ab} = \mu^\varepsilon Z_{\phi_a} Z_{\phi_b} Z_{ab} g_{ab}. \quad (15)$$

The renormalizing Z -factors are power series in g_{ab} according to the following conditions:

$$Z_{\phi_a}(g_{aa}) \frac{\partial}{\partial k^2} \Gamma_{aa}^{(2)}(u_{aa}(g_{aa})) = 1 \quad (16)$$

$$Z_{ab}(g_{ab}) \Gamma_{aabb}^{(4)}(u_{ab}(g_{ab})) = \mu^\varepsilon g_{ab} \quad (17)$$

μ is a scale parameter (equal to the mass at which the massive scheme is evaluated and giving the scale of external momenta in the massless scheme).

In order to renormalize the star vertex functions we introduce renormalization factors $Z_{\Pi\phi_a}^{*f}$ by:

$$\left(\prod_{a=1}^f Z_{\phi_a}^{1/2} \right) Z_{\Pi\phi_a}^{*f} \Gamma_{\Pi\phi_a}^{*f}(u_{ab}(g_{ab})) = \mu^{\delta_{\Pi\phi_a}}, \quad (18)$$

where $\delta_{\Pi\phi_a}$ is the engineering dimension of the composite operator

$$\delta_{\Pi\phi_a} = f \left(\frac{\varepsilon}{2} - 1 \right) + 4 - \varepsilon \quad (19)$$

The dependence of the renormalized couplings g_{ab} and of renormalizing Z -factors on the scale parameter μ is expressed by the following relations:

$$\mu \frac{d}{d\mu} g_{ab} = \beta_{ab}(g_{a'b'}), \quad (20)$$

$$\mu \frac{d}{d\mu} \ln Z_{\Pi\phi_a}^{*f}(g_{ab}) = \eta_{\Pi\phi_a}(g_{ab}). \quad (21)$$

We are going to look on the situation of having two sets of walks of different species. In this case only three different couplings remain. We will refer to them as g_{11} , g_{22} , $g_{12} = g_{21}$. The corresponding functions β_{11} , β_{22} , β_{12} define a flow in the space of couplings. This renormalization group flow was discussed in [22,23]. Its fixed points are determined by a set of equations:

$$\beta_{ab}(g_{ab}^*) = 0, \quad a, b = 1, 2. \quad (22)$$

In the space of the three couplings one finds [23] 8 fixed points corresponding to absence or presence of inter- and intra- species interaction.

Table 1: Fixed points for the interactions of a system of polymers of two species.

	G_0	U_0	U'_0	S_0	G	U	U'	S
g_{11}	0	g^*	0	g^*	0	g^*	0	g^*
g_{22}	0	0	g^*	g^*	0	0	g^*	g^*
g_{12}	0	0	0	0	g_G^*	g_U^*	g_U^*	g^*

They are given in the table 1 where g^* corresponds to the fixed point of the theory containing only 1 species, whereas g_G^* corresponds to the case of having only inter-species interactions, g_U^* describes a set of random walks interacting with another set of self-avoiding walks.

The phenomenon we address in this article corresponds to the case of non-vanishing interaction between the two species of walks, while one set has no self-interaction. Thus we consider the two fixed points labeled G and U . The first corresponds to a set of random walks interacting with random walks of another species and thus describes absorption on random walk absorbers, the second corresponds to a set of random walks interacting with another set of self-avoiding walks and thus describes absorption on SAW (polymer) absorbers.

Having f_1 walks of the first species and f_2 walks of second species we define the following exponents in the fixed points G, U :

$$\eta_{f_1 f_2}^G = \eta_{\Pi\phi_a}(g_{11} = g_{22} = 0, g_{12} = g_G^*), \quad (23)$$

$$\eta_{f_1 f_2}^U = \eta_{\Pi\phi_a}(g_{11} = g^*, g_{22} = 0, g_{12} = g_U^*), \quad (24)$$

which govern the scaling properties of the partition sum (8).

The scaling may be formulated in terms of the size R of the walks: We have to normalize the partition function by the number of configurations of the absorber given by $\mathcal{Z}_{*f_1 0}$. For large R the resulting quantity scales like

$$\mathcal{Z}_{*f_1 f_2} / \mathcal{Z}_{*f_1 0} \sim R^{-\lambda_{f_1 f_2}} \quad \text{for } R = S^\nu \rightarrow \infty \quad (25)$$

Here $\lambda_{f_1 f_2} = \eta_{f_1 f_2} - \eta_{f_1 0}$, ν is the correlation length critical exponent of the walks: $\nu = 1/2$ for random walks and $\nu \simeq 0.588$ for self-avoiding walks at $d = 3$. For the fixed point G we have $\eta_{f_1 0}^G = 0$ and $\nu = 1/2$ for all walks.

The exponent $\lambda_{2,n}$ corresponds to the n th moment of the flux onto the center segment of an absorbing linear chain. Considering the absorber to be either a random walk or a self-avoiding walk let us define the

exponents:

$$\lambda_{RW}(n) \equiv \lambda_{2,n}^G = -\eta_{2n}^G, \quad (26)$$

$$\lambda_{SAW}(n) \equiv \lambda_{2,n}^U = -\eta_{2n}^U + \eta_{20} \quad (27)$$

Previously [19,20] we obtained the expressions for the exponents $\eta_{f_1 f_2}^G, \eta_{f_1 f_2}^U$ in terms of ε -expansion and pseudo- ε expansion series in massless and massive renormalization group schemes. Whereas the first corresponds to collecting perturbation theory terms of the same powers of $\varepsilon = 4 - d$, in the pseudo- ε expansion series [30] at each power of the pseudo- ε parameter (τ) one collects contributions from the dimension-dependent loop integrals of the same order. In the final results $\tau = 1$. Based on the expressions for the exponents $\eta_{f_1 f_2}^G, \eta_{f_1 f_2}^U$ [19,20] we find:

$$\lambda_{RW}(\varepsilon) = \varepsilon n - \frac{n(n-1)\varepsilon^2}{4} + \frac{n(n-1)(-1+n+3\zeta(3))\varepsilon^3}{8} \quad (28)$$

$$\lambda_{SAW}(\varepsilon) = \frac{3\varepsilon n}{4} + \left(\frac{7n}{128} - \frac{9n^2}{64}\right)\varepsilon^2 + \left(-\frac{149n}{2048} - \frac{21n^2}{1024} + \frac{27n^3}{512} - \frac{69n\zeta(3)}{512} + \frac{135n^2\zeta(3)}{512}\right)\varepsilon^3 \quad (29)$$

$$\lambda_{RW}(\tau) = \tau\varepsilon n + \left(\frac{\varepsilon n^2}{2} - \varepsilon n^2 i_1 - \frac{\varepsilon n}{2} + \varepsilon n i_1\right)\tau^2 + \lambda_{RW}^{3loop}\tau^3. \quad (30)$$

$$\lambda_{SAW}(\tau) = \frac{3\tau\varepsilon n}{4} + \left(\frac{\varepsilon n i_1}{4} + \frac{9\varepsilon n^2}{32} - \frac{9\varepsilon n^2 i_1}{16} + \frac{\varepsilon n i_2}{16} - \frac{\varepsilon n}{8}\right)\tau^2 + \lambda_{SAW}^{3loop}\tau^3. \quad (31)$$

Here $\zeta(3) \simeq 1.202$ is the Riemann zeta function, i_j are the loop integrals dependent on the space dimension d : at $d = 3$ $i_1 = 2/3$, $i_2 = -2/27$. The expressions for the three-loop terms $\lambda_{RW}^{3loop}, \lambda_{SAW}^{3loop}$ in (30),(31) are given elsewhere [31].

4. Multifractal Spectrum

A widely used description for the MF spectrum is obtained from a Legendre transform, the spectral function $f(\alpha)$, of the analytically continued

spectrum $\lambda(n) - n\lambda(1)$ by

$$f(\alpha) = (\lambda(n) - n\lambda(1)) + n\alpha \quad \text{with} \quad \alpha = -\frac{d}{dn}(\lambda(n) - n\lambda(1)) \quad (32)$$

For standard moments of a MF measure not including an ensemble average $f(\alpha)$ is called the spectral function. Here this notion is kept. The spectral function is widely used to characterize the multifractal nature of many processes [2]. In the standard approach the function $f(\alpha)$ defined for a multifractal measure on a set X gives for every α the fractal dimension of the subset of X for which the measure at scale ℓ is characterized by ℓ^α with the Hölder exponent α , in the limit $\ell \rightarrow 0$. Due to this interpretation as a property of a (multifractal) measure, strict convexity conditions hold. The standard $f(\alpha)$ has the shape of a cap. While simple power of field operators ϕ^f will not generate such a spectrum [11], the operators constructed here accord to this condition.

Using the perturbation expansions for the λ exponents given to third loop order both in ε and τ expansion in massless and massive renormalization (28) - (31) and the relations for $\lambda(n)$ and the spectral function some algebra leads to the corresponding expansions for α_n and $f(\alpha_n)$:

$$\alpha_{RW}(\varepsilon) = 2 + \left(-\frac{n}{2} + 1/4\right)\varepsilon^2 + \left(-\frac{n}{2} - \frac{3\zeta(3)}{8} + \frac{3n^2}{8} + \frac{3n\zeta(3)}{4} + 1/8\right)\varepsilon^3. \quad (33)$$

$$f_{RW}(\varepsilon) = 2 - \frac{\varepsilon^2 n^2}{4} + \left(\frac{n^3}{4} + \frac{3n^2\zeta(3)}{8} - \frac{n^2}{4}\right)\varepsilon^3 \quad (34)$$

$$\alpha_{RW}(\tau) = 2 + \left(-\frac{\varepsilon}{2} + \varepsilon i_1 + \varepsilon n - 2\varepsilon n i_1\right)\tau^2 + \alpha_{RW}^{3loop}\tau^3 \quad (35)$$

$$f_{RW}(\tau) = 2 + \left(\frac{\varepsilon n^2}{2} - \varepsilon n^2 i_1\right)\tau^2 + f_{RW}^{3loop}\tau^3 \quad (36)$$

$$\alpha_{SAW}(\varepsilon) = 2 - 1/4\varepsilon + \left(\frac{7}{128} - \frac{9n}{32}\right)\varepsilon^2 \left(-\frac{149}{2048} - \frac{69\zeta(3)}{512} - \frac{21n}{512} + \frac{81n^2}{512} + \frac{135n\zeta(3)}{256}\right)\varepsilon^3 \quad (37)$$

$$f_{SAW}(\varepsilon) = 2 - 1/4\varepsilon + \left(-\frac{9n^2}{64} - \frac{11}{128}\right)\varepsilon^2 \left(\frac{27n^3}{256} + \frac{135n^2\zeta(3)}{512} - \frac{21n^2}{1024} - \frac{83}{2048} + \frac{33\zeta(3)}{256}\right)\varepsilon^3 \quad (38)$$

$$\alpha_{SAW}(\tau) = 2 - \frac{\varepsilon}{4}\tau + \left(-\frac{\varepsilon}{8} - \frac{9\varepsilon n i_1}{8} + \frac{9n\varepsilon}{16} + \frac{\varepsilon i_1}{4} + \frac{\varepsilon i_2}{16}\right)\tau^2 + \alpha_{SAW}^{3loop}\tau^3 \quad (39)$$

$$f_{SAW}(\tau) = 2 - \frac{\varepsilon}{4}\tau + \left(-\frac{9\varepsilon n^2 i_1}{16} + \frac{9\varepsilon n^2}{32} + \frac{5\varepsilon}{32} + \frac{\varepsilon i_2}{16} - \frac{5\varepsilon i_1}{16}\right)\tau^2 + f_{SAW}^{3loop}\tau^3 \quad (40)$$

Here $\zeta(3) \simeq 1.202$ is the Riemann zeta function, i_j are the loop integrals dependent on the space dimension d : at $d = 3$ $i_1 = 2/3$, $i_2 = -2/27$.

Again $\zeta(3)$ is the Riemann zeta function, i_1 and i_2 are the two-loop integrals and the explicit form of the three-loop contributions in (35), (36), (39), (40) is given elsewhere [31].

5. Resummation

As is well known, the series of type (28)-(31), (33) - (40), as they occur in field theory appear to be of asymptotic nature with zero radius of convergence. However, knowledge of the asymptotic behavior of the series as derived from the renormalization group theory allows us to evaluate these asymptotic series (see e.g. [29]). To this end several procedures are available depending on the additional information known for the series to be resummed. We assume our series to be of ε -expansion character together with additional information for the case of the Lagrangian [23] (10) we consider here. So we expect the following behavior of the k th order perturbation theory term A_k for any of given above quantities:

$$A_k \sim k! k^b (-a)^k \quad (41)$$

the constant a for the ε -expansion of Lagrangian ϕ^4 field theory with one coupling was derived in [34,35]: $a = 3/8$. For the unsymmetric fixed point, where two different couplings are present we use the value $a = 27/64$ [23]. We assume as well that the same properties hold also for the pseudo- ε expansion in terms of τ . With the above information in hand one can make use of the Borel summation technique improved by the conformal mapping procedure which up to now served a powerful tool in the field theory (see [29] for example).

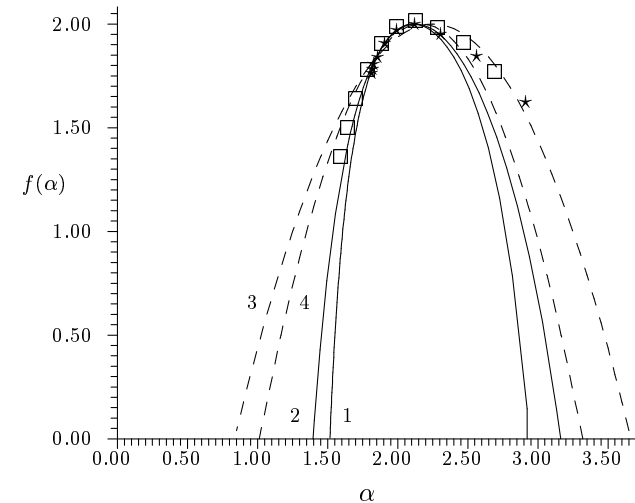
Table 2 contains the results for the exponents $\lambda_{RW}(n)$ and $\lambda_{SAW}(n)$ obtained in ε and in pseudo- ε expansion techniques from the corresponding values of exponents η_{f_1, f_2} [19] with the application of the resummation procedure as described above.

Table 2: Exponents $\lambda_{RW}(n)$ and $\lambda_{SAW}(n)$ obtained in ε and in pseudo- ε expansion techniques.

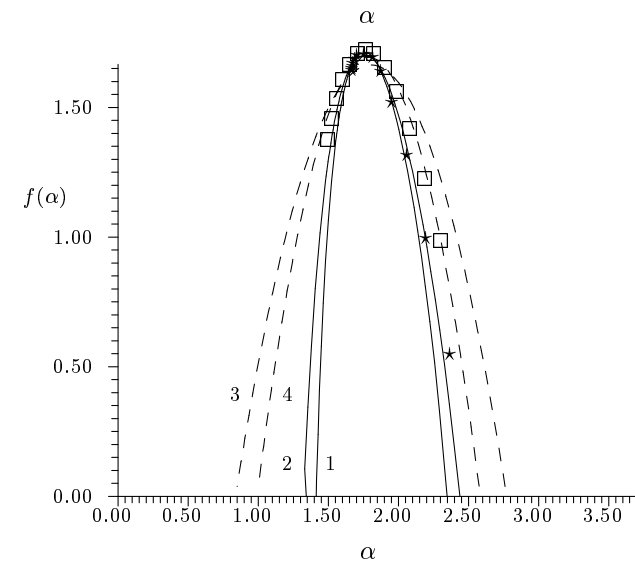
n	$\lambda_{RW}(\varepsilon)$	$\lambda_{RW}(\tau)$	$\lambda_{SAW}(\varepsilon)$	$\lambda_{SAW}(\tau)$
1	0.99	0.99	0.71	0.71
2	1.77	1.81	1.31	1.33
3	2.45	2.53	1.86	1.92
4	3.01	3.17	2.34	2.44
5	3.51	3.75	2.78	2.94
6	3.95	4.28	3.19	3.41

We have calculated the spectral function as a Legendre transform from the series of exponents received from the scaling of the moments of measure defined by diffusion. These moments were calculated as averages over all configurations of the absorber instead of performing a site average. Thus the interpretation of $f(\alpha)$ itself does not directly correspond to the picture developed above for the standard MF. All the same deriving $f(\alpha)$ in the same way from the spectrum of scaling exponents of moments of a measure the above discussed properties of a MF spectral function hold also for $f(\alpha)$ in this case. In particular $\max f(\alpha) = \tau_0$ gives the fractal dimension of the absorber and f is convex $f''(\alpha) < 0$.

Our numerical results for the spectral function are presented in Figs. 1a,1b. They were obtained from the series for α_n and $f(\alpha_n)$ as functions of n . We show the results of the resummation procedure described above applied to the series in both RG approaches. For comparison we also show the curve for direct summation of the ε and τ series to the 2nd order. In addition we have performed an analytical continuation of our series in form of [2/1] Padé approximants for the ε^3 and τ^3 series. It is obvious that direct summation of ε^3 and τ^3 series fails to converge and gives comparable values for $\alpha_n, f(\alpha_n)$ only for small values of n , i.e. near the maximum of $f(\alpha)$ at $n = 0$. The symmetry of the Padé approximant holds only in the region shown and may be an artifact of the method. On the left wing, where it coincides with the resummed results the Padé approximant gives a continuation which is compatible with the estimation for the minimal α value $\alpha_{\min} = d - 2$. The Padé result is $\alpha_{\min}(\varepsilon) = 1.333$, $\alpha_{\min}(\tau) = 1.017$ for the RW absorber and $\alpha_{\min}(\varepsilon) = 1.250$, $\alpha_{\min}(\tau) = 1.013$ for the SAW absorber, which is calculated here



(Fig.1a)



(Fig.1b)

Figure 1: Spectral function $f(\alpha)$ for absorption on (a) RW and (b) SAW . Solid curves: 1 - [2/1] Padé approximant for ε^3 results, 2- [2/1] Padé approximant for pseudo- ε^3 results; dashed curves: 3 - ε^2 results without resummation, 4 - pseudo- ε^2 results without resummation; stars - resummed ε^3 results; boxes - resummed pseudo- ε^3 results.

only from 3rd order perturbation theory.

Note that though the results obtained for α_n and $f(\alpha_n)$ for a specific value of n differ in both approaches, the same curve $f(\alpha)$ is described with better coincidence for the left wing of the curves, corresponding to positive n . The resummation techniques we apply have proven to be a powerful tool already in the field theoretical approach to critical phenomena and have lead to high precision values for critical exponents. We hope that the application of these methods to calculations of MF phenomena allows to improve the reliability and comparability of the results.

We currently work on the generalization of our present approach to find the series of spectral functions obtained for absorption at the core of a polymer with any number of arms.

As can be extrapolated from the Padé approximant and as was shown also on the basis of high order approximations [9], $f(\alpha)$ as it is defined here, will become negative near α_{\min} and α_{\max} . For this reason the identification of $f(\alpha)$ as the fractal dimension of some identifiable subset is not possible here. Also the extrapolation of the resummed data seems to indicate such a behavior. Note, however, that the perturbative approach, even in combination with resummation and analytical continuation is still not capable to give reliable results for high values of the expansion parameters. In particular this method is only good near the maximum of $f(\alpha)$.

The possible negative values of the spectral function were discussed already in [9] and a physical interpretation of $f(\alpha)$ was given as a histogram of the measure μ plotted in logarithmic variables. In this interpretation negative $f(\alpha)$ indicate that the number of sites with a certain logarithmic measure $\alpha \sim \ln \mu$ decreases as the size of the absorber R increases. Thus for large R this number can only be defined by an ensemble average, as performed here.

6. Conclusions

We have studied the characteristics of harmonic diffusion in the presence of a fractal absorber. We related the description of diffusing particles near absorbing paths to interacting walks. Following the model proposed by Cates and Witten [9] we used the polymer formalism to describe both the absorber and the random walks of diffusing particles. The ensemble of absorbers we considered to have random walk or polymer chain (self avoiding walk) statistics. The flux of diffusing particles onto such kind of the absorber and its higher moments generate a multifractal measure [1].

The MF properties of this measure we described by the spectral function formalism [2].

We performed our calculations in the frames of a field theoretical approach, relating our study to the study of scaling properties of composite field operators and defining their spectrum. Namely we show that the “copolymer star” [19,20] - star operators $\prod_a^{f_1} \prod_b^{f_2} \psi_a \phi_b$ in a field theory with interactions u_ψ and $u_{\psi\phi}$ generate a spectrum of scaling dimensions which transforms to a MF spectrum with the appropriate convexity property. To calculate this spectrum we have used the massless renormalization group scheme and massive renormalization group approach at fixed dimension.

We give the explicit expressions for different kinds of exponents describing our problem (formulas (28)-(31), (33), (35), (37), (39) as well as for the spectral function (34), (38), (36), (40) in terms of power series in $\varepsilon = 4 - d$ and in pseudo- ε expansion. All calculations were performed in the third order of perturbation theory. In particular, in the second order in ε we recover previously obtained results [9].

Special attention was payed to the fact that the series are asymptotic and have zero radius of convergence. We have used Padé approximants to obtain analytic continuation of the series under consideration for non-zero value if the expansion parameter. In addition we applied resummation techniques well approved in field theoretic calculations in order to obtain reliable information for the spectral function $f(\alpha)$, Hölder exponent α and exponents λ governing scaling behavior of averaged density moments of diffusing particles (see figs. 1a, 1b and table 2). While standard in field theoretical studies of critical phenomena, the resummation technique as to our knowledge was not applied in the theory of multifractals. We hope that our attempt will attract attention for this possibility in the context of other problems arising in the theory of multifractal measures as well as that the presumed accuracy of our results might evoke comparable efforts by numerical simulation. Further studies devoted to the set of spectra associated with diffusion near the core of absorbing stars with higher numbers of arms. We hope to gain more insight on the unsymmetric behavior of the spectral function and find the envelope of the family of spectra.

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