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PENNING TRAP WITH AN INCLINED MAGNETIC FIELD

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Пастка Пеннінга з нахиленим магнітним полем

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Анотація. Розглядається видозмінена пастка Пеннінга де однорідне магнітне поле **B** є нахилене до осі симетрії електродів. Кут нахилу довільний. Знайдене канонічне перетворення до змінних у яких гамільтоніан є сумою трьох незв'язаних гармонічних осциляторів. У просторі параметрів, що контролюють динаміку (утримуючий параметр κ та квадрат синуса кута нахилу ϑ_0), побудована область стабільності. Коли кут ϑ_0 не перевищує 54 градуси заряд локалізований всередині робочої камери пастки. Коли кут нахилу 3-вектора **B** набуде критичного значення спостерігається резонанс: магнетронна частота стає рівною циклотронній а аксіальна їх обох перевищує. При цьому орбіта заряду перестає бути замкнутою. В області стабільності виявлено низку резонантних кривих, що описують обмежені орбіти.
В області релятивістських енергій система перестає бути лінійною. Ми показали що вона неінтегровна в сенсі Ліувілля. Усереднивши за модою з найбільшою частотою отримуємо динамічну систему з двома ступенями вільності. Аналіз перерізів Пуанкаре усереднених систем показує область ефективного утримання заряду всередині пастки.

Penning trap with an inclined magnetic field

Yu. Yaremko, M. Przybylska, A.J. Maciejewski

Abstract. Modified Penning trap with a spatially uniform magnetic field **B** inclined with respect to the axis of rotational symmetry of the electrodes is considered. The inclination angle can be arbitrary. Canonical transformation of phase variables transforming Hamiltonian of considered system into a sum of three uncoupled harmonic oscillators is found. We determine the region of stability in space of two parameters controlling the dynamics: the trapping parameter κ and the squared sine of the inclination angle ϑ_0. If the angle ϑ_0 is smaller than 54 degrees, a charge occupies finite spatial volume within processing chamber. A rigid hierarchy of trapping frequencies is broken if **B** is inclined at the critical angle: the magnetron frequency reaches the corrected cyclotron frequency while the axial frequency exceeds them. Apart from this resonance we reveal the family of resonant curves in the region of stability.
In the relativistic regime the system is not linear. We show that it is not integrable in the Liouville sense. The averaging over the fast variable allows to reduce the system to two degrees of freedom. An analysis of the Poincaré cross-section of the averaged systems shows the regions of effective stability of the trap.

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Пастка Пеннінга з нахиленим магнітним полем

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1. Introduction

Cylindrical Penning trap was used in Harvard experiment [1] where the electron magnetic moment and the fine structure constant have been measured with unprecedent level of accuracy: fourteen decimal places. Elimination of possible systematic errors caused by trap imperfections [2, Sec.3.3–3.5] is one of sources of such extraordinary precision. The other imperfection is the inherent anharmonicity caused by special relativity [3, Sec.VII D]. Recently [4] quasirelativistic corrections to the trap's eigenfrequencies are calculated by means of the perturbation theory developed in Ref. [5]. In Ref. [6] we study the dynamics of a single charge in an ideal Penning trap in the relativistic domain. The special relativity produces the quartic terms in effective potential which are analogous to those caused by octupolar electrostatic perturbations [8] to the quadruple potential. Because of the axial symmetry, the dynamical system has two degrees of freedom and it is simply and convenient to use it for studying non-linear phenomena in classical mechanics. Poincaré sections describing evolution of phase space reveal quasiperiodic and periodic orbits among mainly chaotic motions. If the total energy of a charge is well outside the realm of special relativity, the Poincaré sections show that the orbits are periodic (see [6, Fig.11]). This circumstance illustrates that the relativistic frequency shifts [4] are of true physical meaning.

The other well-studied imperfection is a misalignment of the magnetic field 3-vector **B** [2, Sec.3.4.3], see also [7]. Since the misalignment is experimentally inevitable, it should be made negligible by means of a careful design. But even tiny inclination of **B** from vertical direction yields shifts of the trap's eigenfrequencies. Brown and Gabrielse in [3] calculated these eigenfrequencies solving the characteristic polynomial under assumption that the inclination angle is very small. Independently Kretzschmar in [9] determined theses eigenvalues using perturbation theory. In this paper we consider the misalignment not as an undesirable perturbation, but we incline the magnetic field intentionally. We suppose that the angle at which the magnetic field slopes to the axis of symmetry of quadruple potential changes from 0 to $\pi/2$. Our objective is to study of the particle's dynamics at regimes which are sensitive to very small disturbances such as relativistic effects which can not be eliminated by means of improvement of the geometry of electrodes and so on.

The paper is organized as follows. In Sec. 2 we formulate non-relativistic Hamiltonian description of dynamics of a charge in the processing chamber of a modified Penning trap where **B** is not parallel to the axis of rotational symmetry of electrodes. In Sec. 2.1 we find the eigenfrequencies of the system as functions of the trapping parameter $\kappa = 2\omega_z^2/\omega_c^2$ and the squared sine $\sigma = \sin^2\vartheta_0$ of the misalignment angle ϑ_0. In Sec. 2.2 we define the region of stability in (κ, σ) plane where condition of stability of the equation of motion is satisfied. A charged particle is trapped if the magnetic field slopes to a fixed vertical direction at the acute angle smaller than specific critical value $\vartheta_c(\kappa)$. If the trapping parameter κ is much smaller than 1, the critical angle is asymptotically equal to 54 degrees. If the angle of inclination reaches $\vartheta_c(\kappa)$, the motion becomes unstable. In Sec. 3 we perform the normalization of Hamiltonian governing the dynamics in the Penning trap with inclined magnetic field. In new canonical variables the Hamiltonian is the sum of three uncoupled harmonic oscillators. The main result of Sec. 4 are the Hamilton equations of motion for relativistic particle in the trap with inclined magnetic field. Their non-integrability is proved in Sec. 5. In Sec. 6 we normalize quadratic part of the relativistic Hamiltonian and we reobtain three characteristic frequencies. Under the usual operating conditions of a Penning trap [1–3] one of frequencies is much greater than the others and we make averaging over corresponding fast angle variable. Obtained averaged Hamiltonian with two degrees of freedom is analyzed by means of the Poincaré sections.

2. Dynamics

In an ideal Penning trap [2,3] a strong homogeneous magnetic field $\mathbf{B}$ is perfectly aligned along the axis of symmetry of quadruple potential

$$e\Phi(\mathbf{r}) = \frac{m\omega_z^2}{2}\left(-\frac{x^2+y^2}{2} + z^2\right), \tag{2.1}$$

i.e., $\mathbf{B} = (0, 0, B)$. In this paper we assume the magnetic field is directed arbitrarily:

$$e\mathbf{B} = m\omega_c\left(\cos\phi_0 \sin\vartheta_0, \sin\phi_0 \sin\vartheta_0, \cos\vartheta_0\right). \tag{2.2}$$

Here ϑ_0 is the angle between vector $\mathbf{B}$ and the axis of symmetry of potential (2.1), i.e., $\mathbf{e}_z = (0, 0, 1)$; ϕ_0 is the angle between the x axis and the projection of $\mathbf{B}$ onto (x, y) plane. The positively defined cyclotron frequency

$$\omega_c = \frac{eB}{m}, \qquad B = |\mathbf{B}|, \tag{2.3}$$

characterizes the intensity of magnetic trapping of a charge e with rest mass m.

Behavior of a charged particle moving in this electromagnetic field is governed by the standard Lorentz force equation. The system of the second order differential equations can be put into Lagrangian framework

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - e\Phi(\mathbf{r}) + \frac{e}{2}\left([\mathbf{B}\times\mathbf{r}]\,\dot{\mathbf{r}}\right). \tag{2.4}$$

Here, $\mathbf{r} = (x, y, z)$ is the radius vector of the charged particle in the rectangular coordinate system with the origin $(0, 0, 0)$ at the saddle point of electrostatic potential (2.1). The magnetic field is included as the contraction $A_i v^i$ where the magnetic vector potential is introduced as

$$\mathbf{A} = \frac{1}{2}\left[\mathbf{B}\times\mathbf{r}\right], \tag{2.5}$$

and $v^i = \dot{r}^i$. We use a natural gauge which yields a vanishing vector potential for a vanishing magnetic field.

Following Ref. [9], we rotate the coordinate frame till new z axis be aligned the magnetic field 3-vector (2.2):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\vartheta_0 & 0 & \sin\vartheta_0 \\ 0 & 1 & 0 \\ -\sin\vartheta_0 & 0 & \cos\vartheta_0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}. \tag{2.6}$$

So, in new coordinate frame the magnetic field 3-vector looks like that in an ideal Penning trap, i.e. either $\mathbf{B}' = (0, 0, B)$ for positive charge, or $\mathbf{B}' = (0, 0, -B)$ for negative one. The "primed" vector potential simplifies

$$e\mathbf{A}' = \frac{1}{2}m\omega_c\left(-y', x', 0\right), \tag{2.7}$$

and Lagrangian (2.4) takes the form

$$L = \frac{m}{2}\left(\dot{x'}^2 + \dot{y'}^2 + \dot{z'}^2\right) - e\Phi(\mathbf{r}') + \frac{m}{2}\omega_c\left(x'\dot{y'} - y'\dot{x'}\right).$$

In new coordinates the quadruple potential (2.1) is more complicated:

$$e\Phi(x', y', z') = \frac{m\omega_z^2}{2}\left[-\frac{\alpha}{2}x'^2 - \frac{1}{2}y'^2 + \beta z'^2 - \gamma x' z'\right], \tag{2.8}$$

$$\alpha = 1 - 3\sin^2\vartheta_0, \; \beta = 1 - \frac{3}{2}\sin^2\vartheta_0, \; \gamma = 3\sin\vartheta_0\cos\vartheta_0. \tag{2.9}$$

Constants α, β and γ do not depend on angle ϕ_0.

It is straightforward to perform the Legendre transformation to derive the Hamiltonian $H(\mathbf{r}', \mathbf{p}')$. To simplify it we introduce dimensionless canonical variables

$$\mathbf{q} = \sqrt{m\omega_c}\mathbf{r}', \qquad \mathbf{p} = \frac{\mathbf{p}'}{\sqrt{m\omega_c}}. \tag{2.10}$$

In terms of these variables the Hamiltonian takes the form $H = \omega_c H_2$, where

$$\begin{aligned} H_2 &= \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{1}{2}\left(p_1 q_2 - p_2 q_1\right) & (2.11)\\ &+ \frac{\kappa}{4}\left(-\frac{\alpha}{2}q_1^2 - \frac{1}{2}q_2^2 + \beta q_3^2 - \gamma q_1 q_3\right) + \frac{1}{8}\left(q_1^2 + q_2^2\right), & (2.12)\end{aligned}$$

and $\kappa = 2\omega_z^2/\omega_c^2$ is the trapping parameter [2, 3, 10]. If $\alpha = \beta = 1$ and $\gamma = 0$ we restore Hamiltonian of an ideal Penning trap.

After rescaling of time $\tau = \omega_c t$ we obtain the "dimensionless" Hamiltonian $H_2 = H/\omega_c$

$$H_2 = \frac{1}{2}\mathbf{x}^T \mathbb{H}\mathbf{x}, \tag{2.13}$$

where $\mathbf{x}^T = (q_1, q_2, q_3, p_1, p_2, p_3)$ and $\mathbb{H}$ is the symmetric 6×6 matrix of coefficients of quadratic form (2.11). The corresponding Hamilton's equations are linear

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbb{A}\mathbf{x}, \quad \text{where matrix} \tag{2.14}$$

$$\mathbb{A} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 1 & 0 & 0\\ -\frac{1}{2} & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ -\frac{1}{4}(1-\alpha\kappa) & 0 & \frac{1}{4}\gamma\kappa & 0 & \frac{1}{2} & 0\\ 0 & -\frac{1}{4}(1-\kappa) & 0 & -\frac{1}{2} & 0 & 0\\ \frac{1}{4}\gamma\kappa & 0 & -\frac{1}{2}\beta\kappa & 0 & 0 & 0 \end{bmatrix} \tag{2.15}$$

is the product $\mathbb{J}\mathbb{H}$ of six dimensional symplectic unit matrix $\mathbb{J}$ and Hamiltonian matrix $\mathbb{H}$.

2.1. Frequencies

Characteristic polynomial $\det[\mathbb{A} - \lambda\mathbb{I}]$ of matrix (2.15) takes the form

$$p(\lambda) = \lambda^6 + \lambda^4 + \frac{1}{4}\kappa\left(2\beta - \frac{3}{4}\kappa\right)\lambda^2 - \frac{1}{32}\kappa^3. \tag{2.16}$$

The equilibrium $x = 0$ of the system of linear equation (2.14) is stable if and only if all eigenvalues of this polynomial are pure imaginary. Substituting $\lambda = \mathrm{i}\Omega$ where $\Omega \in \mathbb{R}$ and introducing the variable $w = \Omega^2 = -\lambda^2$ we obtain

$$w^3 - w^2 + \frac{1}{4}\kappa\left(2\beta - \frac{3}{4}\kappa\right)w - \frac{1}{32}\kappa^3 = 0, \tag{2.17}$$

see e.g. [3, eqs. (2.75)-(2.79)]. According to the Vieta formulae, the sum of roots $w_1 + w_2 + w_3 = 1$ while the product of roots $w_1 w_2 w_3 = \kappa^3/32$. Restoring dimensions, we arrive at the so-called "invariance theorem" [3, Sec. II D]

$$\Omega_1^2 + \Omega_2^2 + \Omega_3^2 = \omega_c^2, \tag{2.18}$$

which allows to derive the cyclotron frequency from measurable eigenfrequencies of an imperfect trap where $\mathbf{B}$ is not parallel to the axis of symmetry of electrodes.

Standard substitution $w = z + 1/3$ transforms polynomial (2.17) into reduced cubic $z^3 + a_1 z + a_0 = 0$ where polynomial coefficients

$$\begin{aligned} a_1 &= -\frac{1}{3} + \frac{1}{4}\kappa\left(2\beta - \frac{3}{4}\kappa\right),\\ a_0 &= -\frac{2}{27} + \frac{1}{12}\kappa\left(2\beta - \frac{3}{4}\kappa\right) - \frac{1}{32}\kappa^3, \end{aligned} \tag{2.19}$$

are negative real numbers. We are interested in three distinct real and positive roots which satisfy the condition of stability and yield periodic orbits. They can be obtained using the cosine and arccosine functions

$$z_k = A\cos\left(\psi + \frac{2\pi}{3}k\right), \tag{2.20}$$

for $k = 0, 1, 2$.The amplitude is

$$A = \sqrt{\left(\frac{2}{3} - \frac{\kappa}{2}\right)^2 + \kappa\sin^2\vartheta_0}\,. \tag{2.21}$$

The argument of cosine function is $\psi = \frac{1}{3}\arccos\Psi(\kappa, \sigma)$ where

$$\Psi(\kappa, \sigma) = \frac{1}{A^3}\left(A^2 - \frac{4}{27} + \frac{\kappa^3}{8}\right). \tag{2.22}$$

Both the amplitude (2.21) and phase (2.22) depend on the trapping parameter κ and $\sigma = \sin^2\vartheta_0$.

Therefore, the roots of polynomial (2.17) are

$$w_k(\kappa,\sigma) = 1/3 + z_{k-1}(\kappa,\sigma). \tag{2.23}$$

In the specific case $\sigma = 0$ (ideal Penning trap) the roots become the corrected cyclotron frequency $\omega_+ = 1/2\left(1+\sqrt{1-\kappa}\right)$, magnetron frequency $\omega_- = 1/2\left(1-\sqrt{1-\kappa}\right)$, and axial frequency $\omega_z = \sqrt{\kappa/2}$ [3]. The argument $\Psi(\kappa,0)$ of arccosine function (2.22) decreases from $\Psi(0,0)=1$ to its minimal value $\Psi(8/9,0) = -1$ and then increases to 1 when the parameter κ rises from 8/9 to 1. For $\kappa > 8/9$ instead of phase ψ we have to take $\psi' = 2\pi/3 - \psi$. Therefore, we have the following correspondence in intervals $\kappa \in]0, 8/9]$ and $\kappa \in [8/9, 1[$:

- $\kappa \in]0, 8/9]$: $\omega_+ = \sqrt{w_1},\ \omega_- = \sqrt{w_2},\ \omega_z = \sqrt{w_3}\,$;
- $\kappa \in [8/9, 1[$: $\omega_+ = \sqrt{w_3},\ \omega_- = \sqrt{w_2},\ \omega_z = \sqrt{w_1}$.

At point $(8/9, 0)$ the resonance $(2,1,2)$ is observed [11]: $\omega_+ = 2/3$, $\omega_- = 1/3$, $\omega_z = 2/3$.

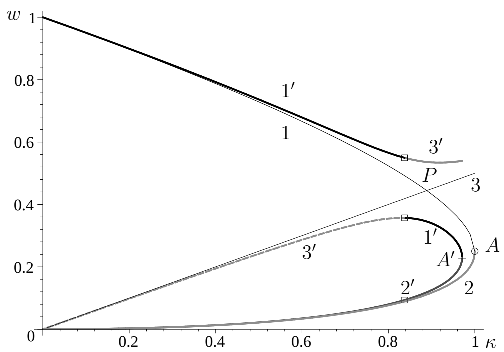


Figure 1: Curves defined by Eq. (2.23) represent changes of squared frequencies when κ increases from 0 to 1 for fixed σ. Primed numbers denote the graphs of spectral curves for inclined magnetic field. Continuous curves describe the spectrum of an ideal Penning trap. Point P indicates the resonance $(2,1,2)$. The corrected cyclotron frequency decreases (curves 1 and $1'$) while the others, magnetron frequency (curves 2 and $2'$) and axial frequency (curves 3 and $3'$) increase.

If $\sigma \neq 0$ the minimal value of argument of arccosine function $\Psi(\kappa,\sigma)$ is larger than -1. To find the extreme curve on the plane (κ,σ) we equate to zero the partial derivatives of this function. The equation $\partial\Psi/\partial\sigma = 0$

defines the curve at which the function exceeds 1. The equation $\partial\Psi/\partial\kappa = 0$ gives the curve at which the argument of arccosine function takes minimal value which is smaller than 1. The curve can be expressed in the form either

$$\kappa_{\min}(\sigma) = \frac{10/9-\sigma}{1-\sigma} - \sqrt{\left(\frac{10/9-\sigma}{1-\sigma}\right)^2 - \frac{8}{3}\frac{(2/3-\sigma)^2}{1-\sigma}} \tag{2.24}$$

or

$$\sigma_{\min}(\kappa) = \frac{3}{16}\left[(1-\kappa)^2 + \frac{23}{9}\right. \\ \left. - \sqrt{\left[(1-\kappa)^2+\frac{23}{9}\right]^2 - \frac{32}{3}\left(\frac{8}{9}-\kappa\right)\left(\frac{4}{3}-\kappa\right)}\right]. \tag{2.25}$$

At this minimum the squared cyclotron frequency $\omega_+^2(\kappa,\sigma)$ and the squared axial frequency $\omega_z^2(\kappa,\sigma)$ interrupt (see Fig. 1).

2.2. Region of stability

In original variables the electromagnetic field is described by the quadruple potential (2.1) and magnetic field 3-vector (2.2), so it depends on four parameters $(\omega_c, \omega_z, \phi_0, \vartheta_0)$. After rotation (2.6) and rescaling of variables (2.10), we obtain the dimensionless equations of motion (2.14) depending on two parameters κ and σ. Let us define the stability region in (κ,σ) space where trap confines a charge.

The region of stability is defined by the condition of stability of the system of linear equations (2.14). To fulfill the condition, the roots w_1, w_2, and w_3 of the spectral polynomial (2.17) should be distinct, real and positive. Thus the Cardano's discriminant should be positive:

$$D = -\left(\frac{a_0^2}{4}+\frac{a_1^3}{27}\right) \\ = \left[A^3 - \left(A^2 - \frac{4}{27}+\frac{\kappa^3}{8}\right)\right]\left[A^3 + A^2 - \frac{4}{27}+\frac{\kappa^3}{8}\right] > 0. \tag{2.26}$$

Vieta's formula gives the second condition:

$$Q = w_1w_2 + w_1w_3 + w_2w_3 = \frac{\kappa}{4}\left(2 - 3\sigma - \frac{3}{4}\kappa\right) > 0. \tag{2.27}$$

Region in (κ,σ) space where both the conditions $D > 0$ and $Q > 0$ are fulfilled is

$$S = \{(\kappa,\sigma) \in \mathbb{R}^2 \,|\, 0 < \kappa < 1,\ 0 < \sigma < \sigma_c(\kappa)\}. \tag{2.28}$$

The upper limiting curve is the appropriate root of algebraic equation $D = 0$:

$$\sigma_c(\kappa) = \kappa^{-1}\left[A_c^2 - \left(\frac{2}{3} - \frac{\kappa}{2}\right)^2\right], \quad \text{where} \tag{2.29}$$

$$A_c = \frac{1}{3} + \frac{2}{3}\cos\left(\phi + \frac{4\pi}{3}\right), \ \phi = \frac{1}{3}\arccos\left(\frac{27}{16}\kappa^3 - 1\right). \tag{2.30}$$

It divide (κ, σ) space into regions of stability and instability. For a point inside the deformed triangle with boundary $\sigma_c(\kappa)$ and segment $[0, 2/3]$ (see Fig. 2), the motion of the charge is bounded.

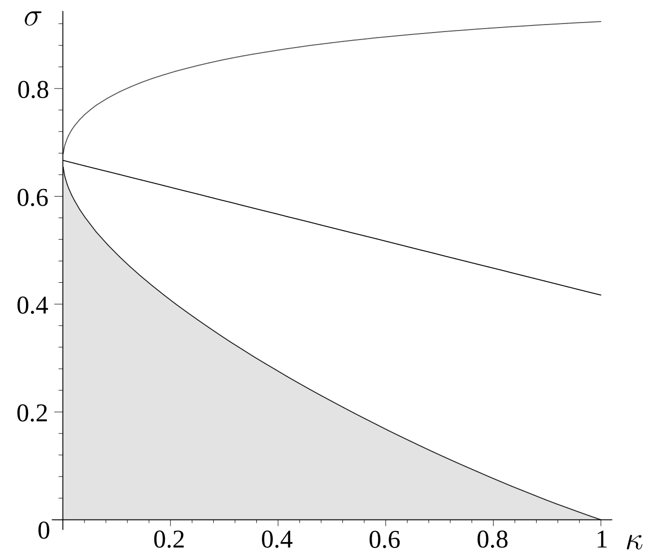


Figure 2: Range of parameters for that necessary conditions (2.26) and (2.27) of stability are satisfied is marked in gray. The straight line is defined by the equality $Q = 0$ while the curve by $D = 0$. Curves $D = 0$ and $Q = 0$ intersect in the point $(\kappa, \sigma) = (0, 2/3)$. Thus, for $\vartheta_0 \geq \arcsin(\sqrt{2/3}) \approx 0.95532$ rad $= 54.735^\circ$ the motion is unstable for arbitrary κ.

3. Normalization of non-relativistic Hamiltonian

In this Section we perform the normalization of Hamiltonian (2.11). By this we mean the transformation of canonical variables $\mathbf{x}^T =$

$(q_1, q_2, q_3, p_1, p_2, p_3) \to \mathbf{y}^T = (Q_1, Q_2, Q_3, P_1, P_2, P_3)$ such that transformed Hamiltonian is the sum of three uncoupled harmonic oscillators. According to [12], the canonical transformation $\mathbf{x} = \mathbb{B}\mathbf{y}$ diagonalizing matrix $\mathbb{H}$ is given by $\mathbb{B} = \mathbb{C}\mathbb{D}$ where

$$\mathbb{D} = \begin{bmatrix} \mathrm{i}\mathbb{I}_3 & \mathbb{I}_3 \\ -\mathrm{i}\mathbb{I}_3 & \mathbb{I}_3 \end{bmatrix}, \tag{3.1}$$

and columns of matrix $\mathbb{C}$ are eigenvectors of the initial matrix (2.15) corresponding to eigenvalues $\pm\mathrm{i}\Omega_i$, $i = 1, 2, 3$. We look for a real matrix $\mathbb{B}$ satisfying $\mathbb{B}^T\mathbb{J}\mathbb{B} = \mathbb{J}$. To fulfil this condition we fix the following order of eigenvalues $(\mathrm{i}\Omega_1, -\mathrm{i}\Omega_2, \mathrm{i}\Omega_3, -\mathrm{i}\Omega_1, \mathrm{i}\Omega_2, -\mathrm{i}\Omega_3)$. It is convenient to present the transformation $x = \mathbb{B}y$ as the combination of canonical transformation

$$\begin{bmatrix} q_1 \\ p_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \frac{\kappa/4}{b_1} - \frac{1}{2} & \frac{\kappa/4}{b_2} - \frac{1}{2} & \frac{\kappa/4}{b_3} - \frac{1}{2} \\ \frac{\gamma\kappa/4}{c_1} & \frac{\gamma\kappa/4}{c_2} & \frac{\gamma\kappa/4}{c_3} \end{bmatrix} \begin{bmatrix} A_1\, x_1 \\ A_2\, \pi_2 \\ A_3\, x_3 \end{bmatrix}, \tag{3.2}$$

$$\begin{bmatrix} p_1 \\ q_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2b_1} & 1 - \frac{1}{2b_2} & 1 - \frac{1}{2b_3} \\ \frac{1}{b_1} & \frac{1}{b_2} & \frac{1}{b_3} \\ \frac{\gamma\kappa/4}{c_1} & \frac{\gamma\kappa/4}{c_2} & \frac{\gamma\kappa/4}{c_3} \end{bmatrix} \begin{bmatrix} \Omega_1 A_1\, \pi_1 \\ \Omega_2 A_2\, x_2 \\ \Omega_3 A_3\, \pi_3 \end{bmatrix}, \tag{3.3}$$

and rotations

$$\begin{aligned} \begin{bmatrix} x_1 \\ \pi_1 \end{bmatrix} &= \begin{bmatrix} \sin\phi_1 & \cos\phi_1 \\ -\cos\phi_1 & \sin\phi_1 \end{bmatrix} \begin{bmatrix} Q_1 \\ P_1 \end{bmatrix}, \\ \begin{bmatrix} x_2 \\ \pi_2 \end{bmatrix} &= \begin{bmatrix} \cos\phi_2 & \sin\phi_2 \\ -\sin\phi_2 & \cos\phi_2 \end{bmatrix} \begin{bmatrix} Q_2 \\ P_2 \end{bmatrix}, \\ \begin{bmatrix} x_3 \\ \pi_3 \end{bmatrix} &= \begin{bmatrix} \sin\phi_3 & \cos\phi_3 \\ -\cos\phi_3 & \sin\phi_3 \end{bmatrix} \begin{bmatrix} Q_3 \\ P_3 \end{bmatrix}. \end{aligned} \tag{3.4}$$

We used the notations

$$b_i = \kappa/4 + w_i, \quad c_i = \beta\kappa/2 - w_i, \tag{3.5}$$

where $\beta = 1 - 3\sigma/2$. Phases ϕ_1, ϕ_2, and ϕ_3 are completely arbitrary, while the squared amplitudes are as follows:

$$A_1^2 = \frac{(w_1 - \beta\kappa/2)\,(w_1 + \kappa/4)}{\Omega_1(w_1 - w_2)(w_1 - w_3)},$$

$$A_2^2 = \frac{(\beta\kappa/2 - w_2)(w_2 + \kappa/4)}{\Omega_2(w_1 - w_2)(w_3 - w_2)},$$
$$A_3^2 = \frac{(\beta\kappa/2 - w_3)(w_3 + \kappa/4)}{\Omega_3(w_1 - w_3)(w_3 - w_2)}. \tag{3.6}$$

Normalized Hamiltonian takes the form:

$$H = \frac{1}{2}\Omega_1\left(P_1^2 + Q_1^2\right) - \frac{1}{2}\Omega_2\left(P_2^2 + Q_2^2\right) + \frac{1}{2}\Omega_3\left(P_3^2 + Q_3^2\right). \tag{3.7}$$

The determinant of transformation to normal coordinates is the product of determinants Δ_1 and Δ_2 of the respective 3×3 matrices (3.2) and (3.3). Straightforward calculations yield

$$\Delta_1 = A_1 A_2 A_3 \gamma \frac{\kappa^2}{16}\Upsilon, \quad \Delta_2 = \Omega_1\Omega_2\Omega_3 A_1 A_2 A_3 \gamma \frac{\kappa}{4}\Upsilon, \tag{3.8}$$

where

$$\Upsilon = \frac{\kappa}{4}(2\beta + 1)\frac{(w_2 - w_1)(w_1 - w_3)(w_3 - w_2)}{b_1 b_2 b_3 c_1 c_2 c_3}. \tag{3.9}$$

We recall that functions β and γ are given by Eqs. (2.9). One can directly check that $\Delta = \Delta_1\Delta_2 = 1$.

3.1. Limit $\vartheta_0 \to 0$

Now we study the limit $\sigma \to 0$ of transformation to normal coordinates (3.2)–(3.6). If the angle of inclination of $\mathbf{B}$ is very small the phase ψ slightly differs from that of ideal Penning trap: $\psi = \psi_0 - \delta$. We expand the argument of arccosine function (2.22) in power series of small parameter σ and equate it to the expression

$$\cos\left[3\left(\psi_0 - \delta\right)\right] = \cos(3\psi_0)\left(1 - 9\delta^2/2\right) + \sin(3\psi_0)3\delta,$$
$$\cos(3\psi_0) = 1 - \frac{\kappa^2(1-\kappa)}{4A_0^3},$$
$$\sin(3\psi_0) = \frac{\kappa}{4A_0^3}\left(\frac{8}{9} - \kappa\right)\sqrt{3(1-\kappa)},$$

where $A_0 = A(\kappa, 0) := 2/3 - \kappa/2$. We truncate the series at δ^2 term because $\sin(3\psi_0)$ vanishes at point $\kappa = 8/9$ where resonance $(2, 1, 2)$ is observed. Solving the quadratic polynomial we obtain

$$\begin{aligned}\delta &= \frac{1}{3}\left\{\tan(3\psi_0) - \left[\tan^2(3\psi_0) - \frac{2\kappa\sigma}{A_0^3}\left(\frac{1}{\cos(3\psi_0)} - \frac{3}{2}A_0\right)\right.\right.\\ &+ \left.\left.\frac{3\kappa^2\sigma^2}{A_0^5}\left(\frac{1}{\cos(3\psi_0)} - \frac{5}{4}A_0\right)\right]^{1/2}\right\}.\end{aligned} \tag{3.10}$$

The expression is valid for any fixed $0 < \kappa < 1$.

If κ differs from 8/9 substantially we restrict ourselves to linear approximations in small parameter σ:

$$\delta = \frac{\kappa\sigma}{(8/9 - \kappa)\sqrt{3(1-\kappa)}}\left(1 + \frac{\kappa(1-\kappa)}{2A_0^2}\right). \tag{3.11}$$

Similarly we develop the matrices (3.2) and (3.3). Calculations are cumbersome but trivial and we do not bother with details. Passing to the limit $\sigma \to 0$ we obtain the following canonical transformation

$$\begin{bmatrix} q_1 \\ p_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2\Omega}} & \frac{1}{\sqrt{2\Omega}} & 0 \\ -\sqrt{\frac{\Omega}{2}} & \sqrt{\frac{\Omega}{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\omega_z}} \end{bmatrix} \begin{bmatrix} x_1 \\ \pi_2 \\ x_3 \end{bmatrix}, \tag{3.12}$$

$$\begin{bmatrix} p_1 \\ q_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\Omega}{2}} & -\sqrt{\frac{\Omega}{2}} & 0 \\ \frac{1}{\sqrt{2\Omega}} & \frac{1}{\sqrt{2\Omega}} & 0 \\ 0 & 0 & \sqrt{\omega_z} \end{bmatrix} \begin{bmatrix} \pi_1 \\ x_2 \\ \pi_3 \end{bmatrix}, \tag{3.13}$$

which normalizes the Hamiltonian governing dynamics in an ideal Penning trap. We denote $\Omega = \sqrt{1-\kappa}/2$.

If κ slightly differs from 8/9 we expand the function (2.24) in power series of small parameter σ and truncate it at the second order term σ^2. We insert it into Eq. (3.10). Applying the resulting expression in matrices (3.2) and (3.3) and passing to the limit $\sigma \to 0$ we obtain the canonical transformation

$$\begin{bmatrix} q_1 \\ p_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{2} & \sqrt{3} & \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{6}}{12} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{12} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ \pi_2 \\ x_3 \end{bmatrix}, \tag{3.14}$$

$$\begin{bmatrix} p_1 \\ q_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{12} & -\frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{12} \\ \frac{\sqrt{6}}{2} & \sqrt{3} & \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} \pi_1 \\ x_2 \\ \pi_3 \end{bmatrix}, \tag{3.15}$$

which normalizes the Hamiltonian governing dynamics in an ideal Penning trap at the resonance $(2, 1, 2)$.

3.2. Resonance $w_1 = w_2$

In this paragraph we look at the critical curve $\sigma_c(\kappa)$ limiting the stability region pictured in Fig. 2. The boundary of this region consists of points in the plane (κ, σ) at which the argument of arcosine function (2.22) is equal to 1. It means that two roots of the spectral equation (2.17) are equal to each other: $w_1 = w_2 := w_0$. The situation is illustrated in Fig. 1 where the resonance is indicated by the capital letter A'.

Vieta's theorem yields the following polynomials:

$$w_0^2 - \frac{2}{3}w_0 + \frac{\kappa}{4}\left(\frac{2}{3}\beta - \frac{\kappa}{4}\right) = 0, \tag{3.16}$$

$$w_0^3 - \frac{1}{2}w_0^2 + \frac{\kappa^3}{64} = 0, \tag{3.17}$$

which define the critical curve (2.29). Appropriate root of the quadratic polynomial (3.16) is

$$w_0(\kappa) = \frac{1}{3} - \frac{1}{2}A_c(\kappa), \tag{3.18}$$

where function $A_c(\kappa)$ is the critical amplitude (2.30) presented previously. Indeed, substituting the right hand side expression for w_0 in the cubic polynomial (3.17) we obtain the cubic polynomial contained in the first square brackets of the Cardano's discriminant (2.26).

The squared axial frequency can be written either $w_3 = 1 - 2w_0$ or $w_3 = \kappa^3/(32w_0^2)$.

The canonical transformation to coordinates which normalize the Hamiltonian (2.11) at this resonance is very cumbersome and we do not present it in the present paper. In normal coordinates this Hamiltonian takes the form

$$\begin{aligned} H_c = &\frac{1}{2}\left(P_1^2 + P_2^2\right) + \sqrt{w_0}\left(P_1Q_2 - P_2Q_1\right) \\ &+ \frac{1}{2}\sqrt{1-2w_0}\left(P_3^2 + Q_3^2\right) . \end{aligned} \tag{3.19}$$

Particle's orbit is the combination of axial harmonic oscillation and parabola in perpendicular plane.

Cylindrical coordinates $Q_1 = r\cos\varphi$ and $Q_2 = r\sin\varphi$ are convenient to present the parabolic motion. In terms of this coordinates the "in-plane" part of Hamiltonian written in the first line of Eq. (3.19) takes the simplified form:

$$H_c' = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - \sqrt{w_0}p_\varphi\,. \tag{3.20}$$

The angular momentum p_φ is conserved as well as the energy E. The parabola which satisfies equations of motion has the form:

$$r^2 = \frac{p_\varphi^2 + T^2}{2\left(E + \sqrt{w_0}p_\varphi\right)},\ \varphi = \arctan\frac{T}{p_\varphi} + \sqrt{w_0}\,(t - t_0) + \varphi_0\,,$$

where $T = 2\left(E + \sqrt{w_0}p_\varphi\right)(t - t_0)$. The orbits become unbounded and a Penning trap loses a charged particle if the angle of inclination of the magnetic field 3-vector reaches the critical value.

4. Relativistic dynamics

Variation of the action $S = \int \mathrm{d}\lambda L$ based on the Lagrangian

$$L = -m\gamma^{-1}(\dot{\mathbf{r}}') - e\dot{x}^0\Phi(\mathbf{r}') + \frac{m}{2}\omega_c\left(x'\dot{y}' - y'\dot{x}'\right), \tag{4.1}$$

produces the Lorentz force equation $ma^\alpha = eF^\alpha{}_\beta u^\beta$ which governs the particle's dynamics in relativistic domain. Electromagnetic field tensor $\hat{F}$ is the combination of constant magnetic field and electric field derived from the "inclined" potential (2.8). As coordinate x^0 is cyclic one, the zeroth momentum is the first integral. We choose the proper time parametrization $\mathrm{d}\tau = \sqrt{(\dot{x}^0)^2 - \left(\dot{x}'\right)^2 - \left(\dot{y}'\right)^2 - \left(\dot{z}'\right)^2}\mathrm{d}\lambda$ such that

$$p_0 = -mu^0 - e\Phi(\mathbf{r}'). \tag{4.2}$$

Obviously, p_0 is the sum of kinetic energy and potential energy taken with opposite sign, i.e., $p_0 = -E$. We express the zeroth component u^0 of the particle four-velocity in the form

$$u^0 = \mathcal{E} - \Phi(\mathbf{r}'), \quad \mathcal{E} = \frac{E}{m}. \tag{4.3}$$

By $\Phi(\mathbf{r}')$ we denote the potential (2.8) rescaled by the factor e/m.

Inserting this into the Lorentz force equation we obtain the system of differential equations

$$\begin{aligned} \dot{u}^1 &= -\left(\mathcal{E} - \Phi\right)\frac{\partial\Phi}{\partial x'} + \omega_c u^2, \\ \dot{u}^2 &= -\left(\mathcal{E} - \Phi\right)\frac{\partial\Phi}{\partial y'} - \omega_c u^1, \\ \dot{u}^3 &= -\left(\mathcal{E} - \Phi\right)\frac{\partial\Phi}{\partial z'}\,. \end{aligned}$$

These equations can be put into Hamiltonian framework. Moreover, it is convenient to introduce the dimensionless variables

$$t=\omega_c\tau,\quad (x',y',z')=\frac{2\omega_c}{\omega_z^2}\,(q_1,q_2,q_3)\,. \tag{4.4}$$

The Hamiltonian is quadratic in momenta

$$H(\mathbf{q},\mathbf{p})=\frac{1}{2}\left(p_1^2+p_2^2+p_3^2\right)+\frac{1}{2}\left(p_1q_2-p_2q_1\right)+V(\mathbf{q}), \tag{4.5}$$

with potential

$$\begin{aligned}V(\mathbf{q})={}&\frac{1-\alpha\eta}{8}q_1^2+\frac{1-\eta}{8}q_2^2+\frac{\beta\eta}{4}q_3^2-\frac{\gamma\eta}{4}q_1q_3\\&-\frac{1}{2}\left(-\frac{1}{2}\alpha q_1^2-\frac{1}{2}q_2^2+\beta q_3^2-\gamma q_1q_3\right)^2,\end{aligned} \tag{4.6}$$

where $\eta=\mathcal{E}\kappa$, $\mathbf{q}=(q_1,q_2,q_3)$ and $\mathbf{p}=(p_1,p_2,p_3)$.

The unit norm 4-velocity condition $(u^0)^2-(\dot{x}')^2-(\dot{y}')^2-(\dot{z}')^2=1$ after this rescaling takes the form $H(\mathbf{q},\mathbf{p})=\varepsilon'$ where

$$\varepsilon'=\frac{1}{24}\,(1-\eta)^2\left(\varepsilon-2\sqrt{\frac{2\lambda}{3(1-\eta)}}\right), \tag{4.7}$$

and ε and λ are parameters defining the trap introduced in [6].

5. Integrability analysis

The main result of this section is following

Theorem 1 *The Hamilton equations of relativistic Penning trap with an inclined magnetic field are non-integrable in the Liouville sense in the class of polynomial functions of coordinates and momenta.*

In order to prove it at first we use the relation between integrability of full Hamiltonian and the integrability of its quasi-homogeneous parts. Let us fix certain weights for coordinates and momenta. Then a polynomial Hamiltonian can be written as a sum of weight homogeneous terms $H=H_{\min}+\cdots+H_{\max}$. If we look for a first integral F which have the decomposition $F=F_{\min}+\cdots+F_{\max}$, then $F_{\min}$ is a first integral of $H_{\min}$, as well as $F_{\max}$ is a first integral of $H_{\max}$. Moreover, thanks to the Ziglin Lemma [13], we can assume that $F_{\min}$ and $H_{\min}$ (or $F_{\max}$ and $H_{\max}$) are functionally independent. This reasoning can be extend to the integrability in the Liouville sense. Namely we have the following implication.

Lemma 1 *If the weight homogeneous system given by Hamiltonian $H=H_{\min}+\cdots+H_{\max}$ is integrable in Liouville sense with first integrals admitting decomposition $F=F_{\min}+\cdots+F_{\max}$, then Hamiltonian systems defined by $H_{\min}$ and $H_{\max}$ are integrable in the Liouville sense.*

The above implies the following.

Lemma 2 *If relativistic Hamiltonian H given in (4.5), admits an additional polynomial first integral, then system given by Hamilton*

$$\mathcal{H}_4=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+\frac{1}{2}\left(\frac{1}{2}\alpha q_1^2+\frac{1}{2}q_2^2-\beta q_3^2+\gamma q_1q_3\right)^2, \tag{5.1}$$

admits an additional first integral polynomial in variables.

Proof. We put weights: 1 for coordinates and 2 for the momenta. Then Hamiltonian (4.5) has decomposition $H=\mathcal{H}_2+\mathcal{H}_3+\mathcal{H}_4$, where $\mathcal{H}_i$ are its quasi-homogeneous parts of degrees 2,3 and 4 and explicit form of $\mathcal{H}_4$ is given by (5.1). □

One can prove non-integrability (4.5) using the implication concerning Hamiltonian $\mathcal{H}_4$ contained in Lemma 2. Now we will show non-integrability of $\mathcal{H}_4$. It is a natural Hamiltonian with standard kinetic energy and with homogeneous potential V_4 of degree 4 in the usual sense i.e. when standard weight 1 for coordinates are used. For natural Hamiltonians with homogeneous potentials V of integer degree k very strong necessary conditions of integrability are known. They were obtained from analysis of properties of differential Galois group of variational equations obtained by a linearisation of Hamilton equations along straight line particular solutions constructed by means of non-zero solutions of the system of algebraic equations

$$V'(d)=d, \tag{5.2}$$

for details see [14]. For a homogeneous potential V of degree 4 these conditions state that all eigenvalues of Hessian $V''(d)$ must belong to the following set

$$\begin{aligned}\mathcal{M}_4={}&\{p(2p-1)\mid p\in\mathbb{Z}\}\cup\left\{\frac{3}{8}+2p(p+1)\mid p\in\mathbb{Z}\right\}\\&\cup\left\{-\frac{1}{8}+\frac{2}{9}(1+3p)^2\mid p\in\mathbb{Z}\right\}.\end{aligned}$$

For $V=V_4$ equation (5.2) has the following solutions

$$d_{1,2}=\left(q_1,\mathrm{i}\,\frac{\sqrt{2+3q_1^2+4\beta}}{\sqrt{1+2\beta}},\pm\frac{q_1\sqrt{2-2\beta}}{\sqrt{1+2\beta}}\right),$$

$$d_3 = \left(\frac{1}{3}\sqrt{3\beta - 3}, 0, \frac{1}{6}\sqrt{-12\beta - 6}\right),$$

where in $d_{1,2}$ variable q_1 is arbitrary.

Spectra of Hessian V_4 at these points are the following: $\operatorname{spect}(V_4''(d_1)) = \operatorname{spect}(V_4''(d_2)) = \{-2, 1, 3\}$, and $\operatorname{spect}(V_4''(d_3)) = \{-\frac{1}{2}, -\frac{1}{2}, 3\}$. In spectra of Hessians $V''(d_i)$, for $i = 1, 2$ eigenvalues $1, 3 \in \mathcal{M}_4$ but $-2 \notin \mathcal{M}_4$. Hessian $V''(d_3)$ has double eigenvalue $-1/2 \notin \mathcal{M}_4$ and $3 \in \mathcal{M}_4$ and is diagonalisable. Since not all eigenvalues belong to $\mathcal{M}_4$ thus Hamiltonian $\mathcal{H}_4$ is nonintegrable in the Liouville sense. Non-integrability of Hamiltonian $\mathcal{H}_4$ immediately implies non-integrability of H for all values of ϑ_0. This is in accordance with non-integrability result for classical Penning trap given in [6].

We can also ask about existence of just one additional first integral of $\mathcal{H}_4$. Necessary conditions for the existence of additional first integrals for Hamiltonian systems with homogeneous potentials are formulated in [15]. These conditions for $\mathcal{H}_4$ say that at each point d_i apart from $\lambda_3 = 3 \in \mathcal{M}_4$ one more eigenvalue of $V''(d_i)$ must belong to $\mathcal{M}_4$ or difference $\frac{1}{4}\left(\sqrt{1 + 8\lambda_1} - \sqrt{1 + 8\lambda_2}\right)$ must be integer. We note that these conditions are satisfied. It also means that necessary conditions for additional first integral of whole relativistic system are satisfied. We know that such additional first integral exists for $\vartheta_0 = 0$, namely z-th component of angular momentum.

6. Averaging of the relativistic Hamiltonian

The relativistic Hamiltonian system given by (4.5) is non-integrable. In order to investigate general features of its dynamics we transform it to a simpler form called a normal form. Non-dimensional Hamiltonian (4.5) can be decomposed into homogeneous parts with respect of variables $\mathbf{x} = (q_1, q_2, q_3, p_1, p_2, p_3)$

$$H = H_2 + H_4, \tag{6.1}$$

where

$$\begin{aligned} H_2 = {} & \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(p_1 q_2 - p_2 q_1) \\ & + \frac{1 - \alpha\eta}{8} q_1^2 + \frac{1 - \eta}{8} q_2^2 + \frac{\beta\eta}{4} q_3^2 - \frac{\gamma\eta}{4} q_1 q_3, \quad \text{and} \end{aligned} \tag{6.2}$$

$$H_4 = -\frac{1}{2}\left(-\frac{1}{2}\alpha q_1^2 - \frac{1}{2} q_2^2 + \beta q_3^2 - \gamma q_1 q_3\right)^2. \tag{6.3}$$

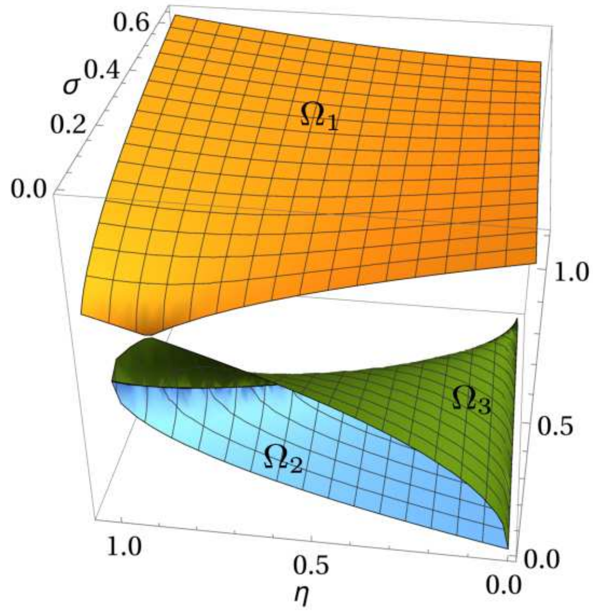


Figure 3: Dependence of Ω_1, Ω_2 and Ω_3 on η and σ. For almost all values od parameters Ω_3 is much greater than Ω_1 and Ω_2. When we cut this plot with plane $\sigma = \text{const}$, then we obtain primed curves of Fig. 1.

Normalization starts from lowest order parts of Hamiltonian, i.e. from quadratic part H_2. But we note that H_2 coincides with the non-relativistic Hamiltonian given in (2.11) provided we substitute $\kappa \to \eta = \mathcal{E}\kappa$. Normalization of the non-relativistic Hamiltonian was made in Sec. 3 and we only recall final result

$$\mathcal{H}_2 = \frac{\Omega_1}{2}(P_1^2 + Q_1^2) - \frac{\Omega_2}{2}(P_2^2 + Q_2^2) + \frac{\Omega_3}{2}(P_3^2 + Q_3^2) = \sum_{i=1}^{3} \omega_i I_i, \tag{6.4}$$

where $I_i = \frac{1}{2}(P_i^2 + Q_i^2)$ and $\omega_1 = \Omega_1$, $\omega_2 = -\Omega_2$, $\omega_3 = \Omega_3$. Plots of $\Omega_i = \sqrt{w_i}$ as functions of parameters σ and η are given in Fig. 3. For generic values of parameters these three frequencies are different, see Fig. 3 but for some specific values of parameters (σ, η) they become linearly dependent on $\mathbb{Z}$. The eigenfrequencies $\Omega_1, \Omega_2, \Omega_3$ satisfy a resonance

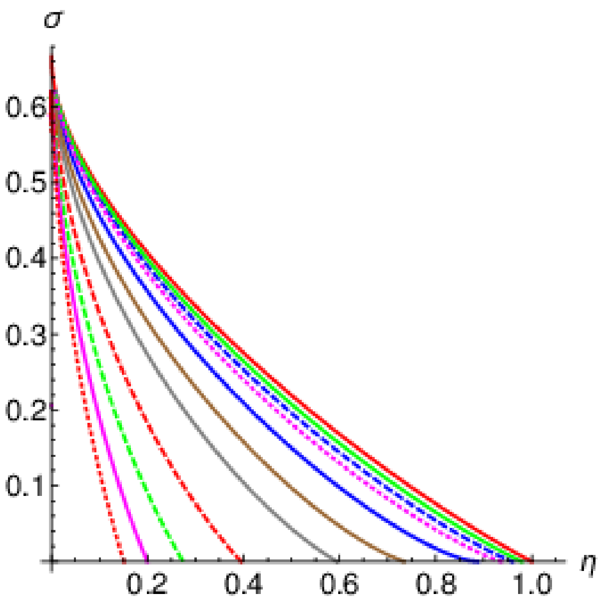


Figure 4: Resonance curves of various orders plotted in parameter's plane

relation of order $l > 0$ if there exist integers k_i such that

$$k_1\Omega_1 + k_2\Omega_2 + k_3\Omega_3 = 0 \quad \text{and} \quad |k_1| + |k_2| + |k_3| = l. \tag{6.5}$$

Since for most of values of parameters $\Omega_1 \gg \Omega_2$ and $\Omega_1 \gg \Omega_3$, in Fig. 4 we only present resonances of orders 2–8 with $k_1 = 0$ involving only frequencies Ω_2 and Ω_3.

Considerations of Sec. 3 enable to plot immediately region of linear stability of equilibrium as in Fig. 2 with only changed label $\kappa \to \eta = \kappa\mathcal{E}$. After normalization of quadratic part of Hamiltonian the higher order parts of H can be normalized (simplified) by means of sequence of nonlinear canonical transformations. However for the considered system another analysis is more useful. Since one of frequencies, namely Ω_1 is significantly greater than others averaging over corresponding fast variable φ_1 is very natural. To this aim we make a linear transformation of variables $(q_1, q_2, q_3, p_1, p_2, p_3) \to (Q_1, Q_2, Q_3, P_1, P_2, P_3)$ in Hamiltonian H (6.1) given by (3.2), (3.3) and (3.4) that gives new Hamiltonian $\widetilde{H}(Q_1, Q_2, Q_3, P_1, P_2, P_3)$. Its quadratic part is normalized as in (6.4) but

this transformation enables to identify in whole Hamiltonian $\widetilde{H}$ terms depending on fast oscillations defined by angle variable φ_1 as

$$Q_1 = \sqrt{2I_1}\sin\varphi_1, \quad P_1 = \sqrt{2I_1}\cos\varphi_1.$$

The explicit form of the average Hamiltonian $H_{\rm aver}(Q_2, Q_3, P_2, P_3, I_1)$ is given in Eq. (6.6). Quantities b_i, c_i and A_i are given in (3.5) and (3.6) and in these formulae substitution $\kappa \to \eta = \kappa\mathcal{E}$ is made. For simplicity we chose identity rotations transformations in (3.4) i.e. $\phi_1 = \phi_3 = \pi/2$ and $\phi_2 = 0$.

$$H_{\rm aver} = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{H}(\sqrt{2I_1}\sin\varphi_1, \sqrt{2I_1}\cos\varphi_1, Q_2, Q_3, P_2, P_3)\mathrm{d}\varphi_1$$

$$= \Omega_1 I_1 - \frac{1}{2}\Omega_2(Q_2^2 + P_2^2) + \frac{1}{2}\Omega_3(Q_3^2 + P_3^2)$$

$$- \frac{1}{256 b_1^4 b_2^4 b_3^4 c_1^4 c_2^4 c_3^4}\Big[2b_1^4\Big(8b_2^2b_3^2c_2^2(A_2P_2 + A_3Q_3)^2\beta c_3^2$$

$$+ 4c_2^2\Big(A_3^2(P_3^2\Omega_3^2 - b_3^2Q_3^2)b_2^2 + 2A_2A_3b_3(P_3Q_2\Omega_2\Omega_3 - b_2b_3P_2Q_3)b_2$$

$$+ A_2^2b_3^2(Q_2^2\Omega_2^2 - b_2^2P_2^2)\Big)c_3^2 + 2b_2^2b_3^2c_2(A_2P_2 + A_3Q_3)(A_2c_3P_2$$

$$+ A_3c_2Q_3)\gamma^2\eta c_3 - \frac{1}{2}b_2^2b_3^2(A_2c_3P_2 + A_3c_2Q_3)^2\beta\gamma^2\eta^2\Big)^2 c_1^4$$

$$+ 4A_1^2b_1^2b_2^2b_3^2c_2^2c_3^2I_1\Big(A_3^2b_2^2\Big(\Big(2c_3^2P_3^2\Big(8(2\beta - 1)c_1^2 + 4\gamma^2\eta c_1 - \beta\gamma^2\eta^2\Big)\Omega_3^2$$

$$- \frac{1}{16}b_3^2Q_3^2\Big(3\eta^4\gamma^6 + 6\beta(8c_1 + 8c_3 - \eta)\eta^3\gamma^4 - 6\eta^2(8c_1^2 + 32c_3c_1 + 8c_3^2$$

$$+ \eta^2)\gamma^4 - 768c_1c_3(c_1 + c_3)\beta\eta\gamma^2 + 32\Big((24c_3^2 + 12\eta c_3 + \eta^2)c_1^2$$

$$+ 4c_3\eta(3c_3 + \eta)c_1 + c_3^2\eta^2\Big)\gamma^2 - 2304c_1^2c_3^2 + 1536c_1^2c_3^2\beta\Big)\Big)b_1^2 + 4c_1^2\Omega_1^2\times$$

$$\times\Big(\frac{1}{2}b_3^2\Big(8(2\beta - 1)c_3^2 + 4\gamma^2\eta c_3 - \beta\gamma^2\eta^2\Big)Q_3^2 + 12c_3^2P_3^2\Omega_3^2\Big)\Big)c_2^2$$

$$+ 2A_2A_3b_2b_3c_3\Big(\Big(2c_2c_3P_3Q_2\Big(8(2\beta - 1)c_1^2 + 4\gamma^2\eta c_1 - \beta\gamma^2\eta^2\Big)\Omega_2\Omega_3$$

$$- \frac{1}{16}b_2b_3P_2Q_3\Big(3\eta^4\gamma^6 + 6\beta(8c_1 + 4c_2 + 4c_3 - \eta)\eta^3\gamma^4 - 6\eta^2\Big(8c_1^2$$

$$+ 16(c_2 + c_3)c_1 + \eta^2 + 8c_2c_3\Big)\gamma^4 - 384c_1(2c_2c_3 + c_1(c_2 + c_3))\beta\eta\gamma^2$$

$$+ 32\Big((6c_2(4c_3 + \eta) + \eta(6c_3 + \eta))c_1^2 + 2\eta(c_3\eta + c_2(6c_3 + \eta))c_1 + c_2c_3\eta^2\Big)\gamma^2$$

$$- 2304c_1^2c_2c_3 + 1536c_1^2c_2c_3\beta\Big)\Big)b_1^2 + 2c_1^2\Omega_1^2\Big(b_2b_3P_2Q_3\Big(- \beta\eta^2\gamma^2$$

$$
\begin{aligned}
&+2(c_2+c_3)\eta\gamma^2-8c_2c_3+16c_2c_3\beta\Big)+24c_2c_3P_3Q_2\Omega_2\Omega_3\Big)\Big)c_2\\
&+A_2^2b_3^2c_3^2\Big(\Big(2c_2^2Q_2^2\Big(8(2\beta-1)c_1^2+4\gamma^2\eta c_1-\beta\gamma^2\eta^2\Big)\Omega_2^2-\frac{1}{16}b_2^2P_2^2\Big(3\eta^4\gamma^6\\
&+6\beta(8c_1+8c_2-\eta)\eta^3\gamma^4-6\eta^2(8c_1^2+32c_2c_1+8c_2^2+\eta^2)\gamma^4-768c_1c_2\times\\
&\times(c_1+c_2)\beta\eta\gamma^2+32\Big((24c_2^2+12\eta c_2+\eta^2)c_1^2+4c_2\eta(3c_2+\eta)c_1+c_2^2\eta^2\Big)\gamma^2\\
&-2304c_1^2c_2^2+1536c_1^2c_2^2\beta\Big)\Big)b_1^2+4c_1^2\Omega_1^2\Big(\frac{1}{2}b_2^2\Big(8(2\beta-1)c_2^2+4\gamma^2\eta c_2\\
&-\beta\gamma^2\eta^2\Big)P_2^2+12c_2^2Q_2^2\Omega_2^2\Big)\Big)\Big)c_1^2+A_1^4b_2^4b_3^4c_2^4c_3^4I_1^2\Big(3\Big((4-8\beta)c_1^2-2\gamma^2\eta c_1\\
&+\frac{1}{2}\beta\gamma^2\eta^2\Big)^2b_1^4+4c_1^2\Big(8(2\beta-1)c_1^2+4\gamma^2\eta c_1-\beta\gamma^2\eta^2\Big)\Omega_1^2b_1^2+48c_1^4\Omega_1^4\Big)\Big].
\end{aligned}
\tag{6.6}
$$

We note that one can use averaging procedure for values of parameters for that there is no resonance relations (6.5), otherwise errors are significant. After averaging Hamiltonian system with two degrees of freedom is obtained for which one can easily check qualitatively its dynamics by means of Poincaré sections. In particular we are interested in regions near origin where the trap is centered and checking whether in this region phase curves moves along quasi-periodic orbits.

Below we present plots for a trap characterized by parameters $\lambda = 0.0082$, $\varepsilon' = 0.00475635$, $\eta = \kappa\mathcal{E} = 0.393323$, that corresponds to parameters $\sigma = 0.648324$ and $\varepsilon = 0.5$ analyzed for ideal trap in paper [6]. Figures 5 show time evolution of variables Q_2, Q_3, P_2, P_3 governed by non-averaged Hamiltonian (plotted in gray) and after averaging (plotted in black dotted) for trap inclined under angle $\vartheta_0 = 25^\circ$. As initial conditions were taken $Q_1(0) = -0.09234498$, $Q_2(0) = -0.01282565$, $Q_3(0) = -0.0113905$, $P_1(0) = -0.04068128$, $P_2(0) = 0.005233979$, $P_3(0) = -0.007263822$. For variables Q_2 and P_2 averaging gives quite good approximation of full dynamics, for variables Q_3 and P_3 about $t = 400$ averaged and described by non-averaged Hamiltonian curves start to diverge. If we make similar plots with the same initial conditions for almost ideal trap with $\vartheta_0 = 1^\circ$, then we obtain good accordance for almost twice longer time.

Let us note that average Hamiltonian is not natural one i.e. it is not the sum of kinetic and potential energy but is a polynomial of the fourth order in the momenta. Regions of possible motions are obtained from conditions that $H_{\text{aver}} = \varepsilon'$, where ε' is defined in (4.7), has a real solution for P_3. Polynomial of fourth order can possess four of two real roots. Regions of possible motions for inclination angles $\vartheta_0 = 15^\circ$ and

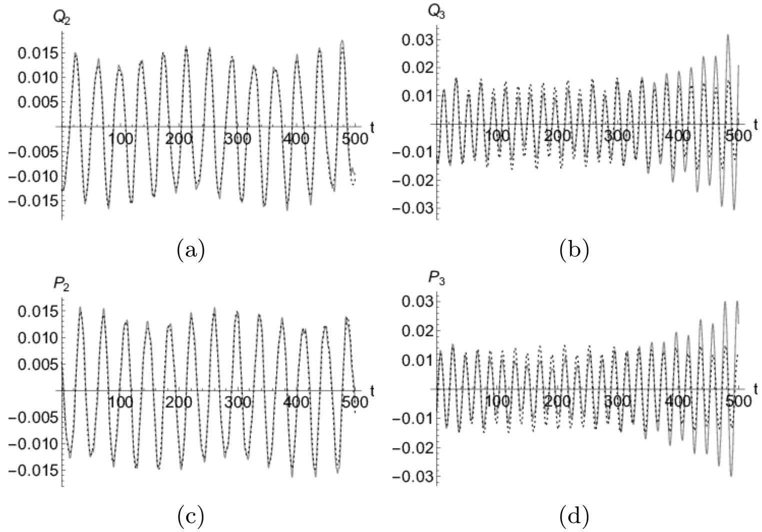


Figure 5: Comparision of time evolution for phase variables before averaging (in grey) with corresponding variables after averaging (in black and dotted). Inclination angle is $\vartheta_0 = 25^\circ$

$I_1 = 0.00551846$, and $\vartheta_0 = 25^\circ$ and $I_1 = 0.0050913$ are given in Figs. 6a and 7a, respectively. Regions of four real roots are denoted by dark gray and of two roots by light gray. Figs. 6b and 7b shows branches of algebraic function for P_3 when we move along axis Q_2 and Figs. 6c and 7c when we move along axis P_2 respectively.

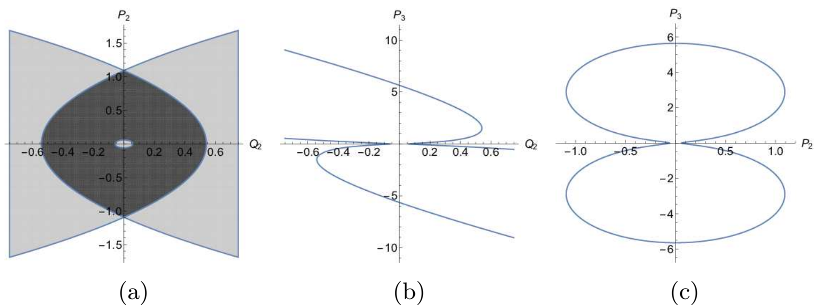
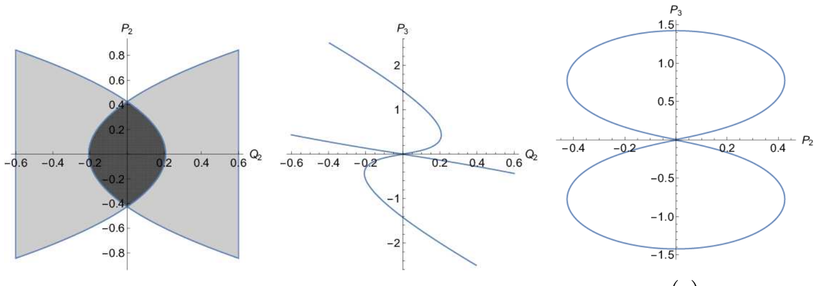


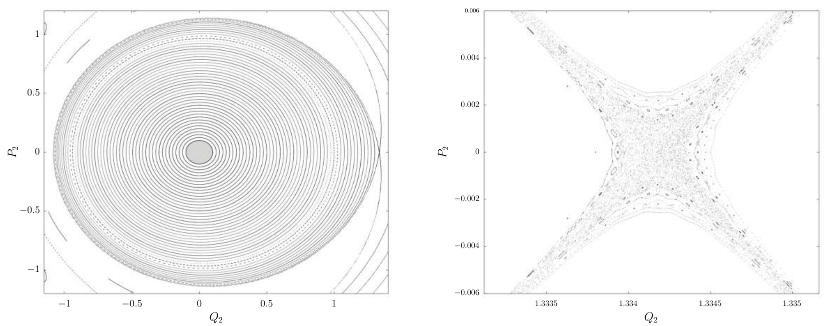
Figure 6: a) Region of possible motions on section $Q_3 = 0$ for inclination angle $\vartheta_0 = 15^\circ$ and $I_1 = 0.00551846$. b) Solutions for P_3 as a function of Q_2 for $Q_3 = P_2 = 0$. c) Solutions for P_3 as a function of P_2 for $Q_3 = Q_2 = 0$.



(a) (b) (c)

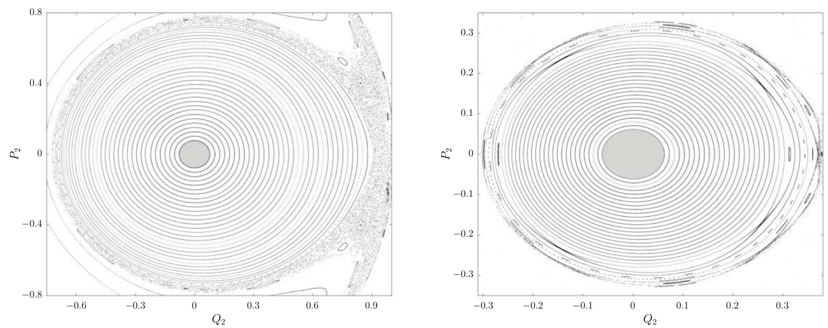
Figure 7: a) Region of possible motions on section $Q_3 = 0$ for inclination angle $\vartheta_0 = 25^\circ$ and $I_1 = 0.0050913$. b) Solutions for P_3 as a function of Q_2 for $Q_3 = P_2 = 0$. c) Solutions for P_3 as a function of P_2 for $Q_3 = Q_2 = 0$.

In Fig. 6 one can note a region near the origin when only two real roots exist (in light gray) that is surrounded by greater region with four real roots (in dark gray) and for sufficiently big absolute values of Q_2 rest only two real roots (in light gray). These small regions with two roods near the origin are radii about 0.075, 0.1 and 0.055 for inclinations angles 1°, 5° and 15° and disappear for 25°. Let us note that solutions with smaller values of magnitudes of P_3 disappear. Fact that in the neighbourhood of origin rest only solutions with higher magnitude of P_3 implies that solutions escapes from this area that is really visible on Poincaré sections in Fig. 8 – 9. In contrary for trap with inclination angle $\vartheta_0 = 25^\circ$ trajectories pass through this neighbourhood, see Fig. 10.



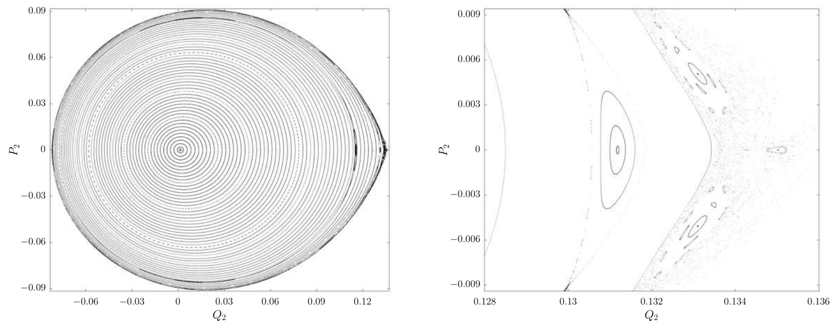
(a) region of regular behaviour near origin (b) magnification of chaotic region

Figure 8: Poincaré sections for trap with inclination angle $\vartheta_0 = 1^\circ$.



(a) inclination angle $\vartheta_0 = 5^\circ$ (b) inclination angle $\vartheta_0 = 15^\circ$

Figure 9: Regions of regular behavior near origin for traps with inclination angles 5 and 15 degrees



(a) region of regular behavior near origin (b) magnification of chaotic region

Figure 10: Poincaré sections for trap with inclination angle $\vartheta_0 = 25^\circ$

7. Conclusions

In this paper we propose the new design of the Penning trap and describe unique possibility of the wide-range tuning and control the relevant modes' characteristics. Indeed, even very small misalignment of the magnetic field 3-vector yields shifts in the trap's eigenfrequencies [2,3,9]. We propose non-relativistic as well as relativistic description of a charged particle in such a trap. For non-relativistic description we we found characteristic frequencies of linear system for an arbitrary angle of inclination and we determine the region of stability in space of controlling parameters. We show that if ϑ_0 exceeds the specific critical value ϑ_c for a fixed κ, the charge's motion becomes unstable. Relativistic description leads to non-linear equations of motion that are non-integrable. However, analysis of its averaged Hamiltonians shows macroscopic regions neighborhoods near origins that behavior of trajectories of charged particles is regular.

Besides, if the misalignment angle reaches a specific value *before* the critical angle, the corrected cyclotron frequency and axial frequency change abruptly. We can imagine the experiment when the trapping parameter κ and inclination angle ϑ_0 change continuously and pass through extreme curve in the region of stability. The experiment should answer the question whether the frequencies' "jumps" result a broken phase trajectory or not.

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