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SPECTRAL PROPERTIES OF FOUR-TIME FERMIONIC GREEN'S FUNCTIONS

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Спектральні властивості чотиричасових ферміонних функцій Ґріна

А.М. Швайка

Анотація. Отримано спектральні співвідношення для чотиричасових ферміонних функцій Ґріна в найзагальнішому вигляді. Виділено аномальні на нульовій частоті внески, які були відомі раніше тільки для бозонних функцій Ґріна, та показано іхній зв'язок з кумулянтами другого порядку розподілу Больцмана. Представлено високочастотні розклади для чотиричасових ферміонних функцій Ґріна для різних напрямків в частотному просторі.

Spectral properties of four-time fermionic Green's functions

A.M. Shvaika

Abstract. The spectral relations for the four-time fermionic Green's functions are derived in the most general case. The terms which correspond to the zero-frequency anomalies, known before only for the bosonic Green's functions, are separated and their connection with the second cumulants of the Boltzmann distribution function is elucidated. The high-frequency expansions of the four-time fermionic Green's functions are provided for different directions in the frequency space.

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Спектральні властивості чотиричасових ферміонних функцій Ґріна

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1. Introduction

One of the main tasks of the quantum many-body theory is, on the one hand, calculation of the observable quantities, which could be measured directly by experiment, and, on the other hand, providing connections between the measured quantities and microscopic properties of the system. It was first noticed by Kubo [1] that linear transport coefficients are expressed in terms of the Fourier transforms of appropriate correlation functions, which relate by spectral relations to the two-time Green's functions. Since that time the Green's function method has been noticed and extensively developed [2–4].

In his seminal article Kubo [1] had also pointed the difference between the isothermal and adiabatic (isolated [5]) response of the many-body system and its connection with the ergodic properties of the system. On the other hand, later it was noticed by Stevens and Toombs [6] that spectral relations must be completed by the special treatment of the additional contribution at zero frequency connected with presence of the conserved quantities [7, 8].

In the Green's function formalism the issue of ergodicity appears as a difficulty in the determination of the zero-frequency bosonic propagators [6–14]. It states that the Fourier transform of the Green's function contains two terms

$$G_{AB}(z) = \widetilde{G}_{AB}(z) - C_{AB}\delta(z), \tag{1}$$

where the first ergodic (Kubo) contribution $\widetilde{G}_{AB}(z)$ is defined by the one-particle bosonic or fermionic density of states

$$\rho_{AB}(\tilde{\omega}) = \frac{1}{Z}\sum_{jf} A_{jf}B_{fj}\left(e^{-\beta\varepsilon_j} \mp e^{-\beta\varepsilon_f}\right)\delta(\tilde{\omega} - \varepsilon_{fj}) \tag{2}$$

through the spectral relation

$$\widetilde{G}_{AB}(z) = \int\limits_{-\infty}^{+\infty} d\tilde{\omega}\, \frac{\rho_{AB}(\tilde{\omega})}{z - \tilde{\omega}} \tag{3}$$

and the second nonergodic term represents the zero-frequency anomaly with $C_{AB} = 0$ for the fermionic functions and $C_{AB} \neq 0$ for the bosonic one. Here, the upper and lower signs correspond to the bosonic and fermionic functions, respectively,

$$A_{jl} = \langle j|\hat{A}|l\rangle \tag{4}$$

are matrix elements of operator $\hat{A}$ between the many-body states with energy difference

$$\varepsilon_{jl} = \varepsilon_j - \varepsilon_l, \tag{5}$$

and

$$Z = \operatorname{Tr} e^{-\beta H} = \sum_j e^{-\beta\varepsilon_j} \tag{6}$$

is partition function.

For the Matsubara Green's function the complex argument z is equal to the bosonic or fermionic Matsubara frequencies $z = i\omega_n$, $\delta(z) = \beta\Delta(i\omega_n)$ with $\Delta(z = 0) = 1$ and 0 in other cases, and for the bosonic functions we have

$$C_{AB} = \frac{1}{Z} \sum_{\substack{jf \\ \varepsilon_j=\varepsilon_f}} e^{-\beta\varepsilon_j} A_{jf} B_{fj}, \tag{7}$$

where the summation is only over the many-body states with equal energies $\varepsilon_j = \varepsilon_f$ including nonergodic contributions and contributions from the conserved quantities.

For the retarded and advanced Green's function one have to replace z by $\omega \pm i0^+$, respectively, and to put $\delta(z) = \delta(\omega)$. Now, the quantity C_{AB} is not well defined and different tricks are used for its calculation [7–9, 11–17].

Nevertheless, even now many textbooks on the quantum statistics and many-body theory do not provide complete discussion of the spectral relations and special treatment of the zero-frequency anomalies. Moreover, because for the two-time Green's functions such anomalies exist only for the bosonic one, no one have even tried to address the problem of the zero-frequency anomalies for the multi-time fermionic Green's functions.

The Kubo's transport theory [1] is not limited to the linear phenomena and provides results to the arbitrary order of external perturbation. Resulting multi-time correlation and Green's functions represents different nonlinear transport phenomena and resonances [18–20]. Besides, multi-time functions also appear as puzzles in different orders of the perturbation theories for many-body systems, e.g., the four-time two-particle one enters the Schwinger-Dyson equation for the one-particle function [21]. Moreover, cross-sections of the inelastic scattering processes can be expressed in terms of the multi-time correlation functions too [22, 23], e.g., for the electronic inelastic light (Raman) scattering the nonresonant, mixed, and resonant responses [24, 25] are connected

with the two-time, three-time, and four-time Matsubara Green's functions [26–28], respectively, and can be rewritten in terms of the multi-time correlation functions.

The spectral relations for the multi-time, i.e., three-time Green's functions of Kubo type were introduced for the first time by Bonch-Bruevich [4, 29] before the zero-frequency anomaly problem was noticed. The spectral relations for the three-time bosonic Matsubara Green's functions with taking into account zero-frequency anomalies were considered by Shvaika [30] and solutions of the reverse problem, i.e., finding of the spectral densities from the known Green's functions, were obtained.

In this article we consider the spectral relations for the four-time fermionic Matsubara Green's functions with special emphasis on the zero-frequency anomalies. It will be shown that despite the obvious statement that there are no zero-frequency anomalies for the separate fermionic frequencies they could exist for the sums of two. In the next section we introduce the four-time correlation functions and spectral densities and separate the terms with different time and frequency dependences. In section 3 we consider the four-time Matsubara Green's functions and show how the zero-frequency anomalies enter and how do they modify the spectral relations. Connection with the generalized cumulants will be considered too. In section 4 we provide the high-frequency asymptotics and in the last section we conclude.

2. Four-time correlation functions and spectral densities

First of all we must introduce the four-time correlation functions. They can be defined in a usual way as

$$K_{ABCD}(t_1, t_2, t_3, t_4) = \langle \hat{A}(t_1)\hat{B}(t_2)\hat{C}(t_3)\hat{D}(t_4)\rangle, \tag{8}$$

where operators $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are of the fermionic type, e.g., ordinary creation and annihilation operators or operators with a more complex commutation relations like the Hubbard one, and

$$\langle \ldots \rangle = \frac{1}{Z} \operatorname{Tr} e^{-\beta H}(\ldots) \tag{9}$$

is a thermodynamical averaging. Here, we consider only the case of the equilibrium many-body systems for which correlation functions are time-shift invariant

$$K_{ABCD}(t_1, t_2, t_3, t_4) = K_{ABCD}(t_1 - t, t_2 - t, t_3 - t, t_4 - t). \tag{10}$$

Spectral density is defined as its Fourier transform

$$I_{ABCD}(\omega_1,\omega_2,\omega_3,\omega_4) = \frac{1}{(2\pi)^3}\int\limits_{-\infty}^{+\infty} d(t_1-t_4)\int\limits_{-\infty}^{+\infty} d(t_2-t_4)\int\limits_{-\infty}^{+\infty} d(t_3-t_4)$$
$$\times K_{ABCD}(t_1,t_2,t_3,t_4)e^{i(\omega_1t_1+\omega_2t_2+\omega_3t_3+\omega_4t_4)}\Delta(\omega_1+\omega_2+\omega_3+\omega_4). \tag{11}$$

Here symbol $\Delta(\omega_1+\omega_2+\omega_3+\omega_4)$ represents the conservation of the total energy (frequency)

$$\omega_1+\omega_2+\omega_3+\omega_4=0, \tag{12}$$

which follows from the time-shift invariance (10). *Below, in all equations we shall keep all four frequencies in order to obtain simple rules for constructing of different contributions, but one has to keep in mind that according to* (12) *only three of them are independent.*

In the case of fermionic operators the spectral density (13) includes four different contributions

$$\begin{aligned}
I_{ABCD}(\omega_1,\omega_2,\omega_3,\omega_4) &= \big[\tilde{I}_{ABCD}(\omega_1,\omega_2,\omega_3,\omega_4)\\
&+\delta(\omega_1+\omega_2)\bar{I}_{\overline{AB}\,\overline{CD}}(\omega_1,-\omega_1,\omega_3,-\omega_3)\\
&+\delta(\omega_2+\omega_3)\bar{I}_{\overline{A\,\overline{BC}\,D}}(\omega_1,-\omega_3,\omega_3,-\omega_1)\\
&+\delta(\omega_1+\omega_2)\delta(\omega_2+\omega_3)\bar{\bar{I}}_{\overline{ABCD}}(\omega_1,-\omega_1,\omega_1,-\omega_1)\big]\\
&\times\Delta(\omega_1+\omega_2+\omega_3+\omega_4).
\end{aligned} \tag{13}$$

with different frequency dependences

$$\tilde{I}_{ABCD}(\omega_1,\omega_2,\omega_3,\omega_4) = \frac{1}{\mathcal{Z}}\sum_{\substack{jlfp\\ \varepsilon_j\neq\varepsilon_f\\ \varepsilon_p\neq\varepsilon_l}} e^{-\beta\varepsilon_j}A_{jl}B_{lf}C_{fp}D_{pj}\delta(\varepsilon_{jl}+\omega_1)$$
$$\times\delta(\varepsilon_{lf}+\omega_2)\delta(\varepsilon_{fp}+\omega_3), \tag{14}$$

$$\bar{I}_{\overline{AB}\,\overline{CD}}(\omega_1,-\omega_1,\omega_3,-\omega_3) = \frac{1}{\mathcal{Z}}\sum_{\substack{jlfp\\ \varepsilon_j=\varepsilon_f\\ \varepsilon_p\neq\varepsilon_l}} e^{-\beta\varepsilon_j}A_{jl}B_{lf}C_{fp}D_{pj}\delta(\varepsilon_{jl}+\omega_1)$$
$$\times\delta(\varepsilon_{fp}+\omega_3), \tag{15}$$

$$\bar{I}_{\overline{A\,\overline{BC}\,D}}(\omega_1,-\omega_3,\omega_3,-\omega_1) = \frac{1}{\mathcal{Z}}\sum_{\substack{jlfp\\ \varepsilon_l=\varepsilon_p\\ \varepsilon_j\neq\varepsilon_f}} e^{-\beta\varepsilon_j}A_{jl}B_{lf}C_{fp}D_{pj}\delta(\varepsilon_{jl}+\omega_1)$$

$$\times\delta(\varepsilon_{fp}+\omega_3), \tag{16}$$

$$\bar{\bar{I}}_{\overline{ABCD}}(\omega_1,-\omega_1,\omega_1,-\omega_1) = \frac{1}{\mathcal{Z}}\sum_{\substack{jlfp\\ \varepsilon_f=\varepsilon_j\\ \varepsilon_l=\varepsilon_p}} e^{-\beta\varepsilon_j}A_{jl}B_{lf}C_{fp}D_{pj}\delta(\varepsilon_{jl}+\omega_1). \tag{17}$$

For bosonic operators an additional terms with $\delta(\omega_i)$ could appear [30]. Expression (13) already displays the zero-frequency anomaly — the presence of terms with δ-function factors, which results in different time dependences of the contributions in correlation function

$$\begin{aligned}
K_{ABCD}(t_1,t_2,t_3,t_4) &= \tilde{K}_{ABCD}(t_1,t_2,t_3,t_4)+\bar{K}_{\overline{AB}\,\overline{CD}}(t_1-t_2,t_3-t_4)\\
&+\bar{K}_{\overline{A\,\overline{BC}\,D}}(t_1-t_4,t_3-t_2)+\bar{\bar{K}}_{\overline{ABCD}}(t_1-t_2+t_3-t_4)
\end{aligned} \tag{18}$$

with different asymptotic behavior at large times $t_1\to\infty$, $t_2\to\infty$, $t_3\to\infty$, and $t_4\to\infty$. The first term always goes to zero $\tilde{K}_{ABCD}(t_1,t_2,t_3,t_4)\to 0$, the next two terms $\bar{K}_{\overline{AB}\,\overline{CD}}(t_1-t_2,t_3-t_4)$ and $\bar{K}_{\overline{A\,\overline{BC}\,D}}(t_1-t_4,t_3-t_2)$ are finite for finite differences $|t_1-t_2|\ll\infty$, $|t_3-t_4|\ll\infty$ and $|t_1-t_4|\ll\infty$, $|t_3-t_2|\ll\infty$, respectively, and the last term $\bar{\bar{K}}_{\overline{ABCD}}(t_1-t_2+t_3-t_4)$ is finite for finite values of $|t_1-t_2+t_3-t_4|\ll\infty$.

Besides, the total spectral density (13) as well as each contribution satisfy the following cyclic permutation identities ($\omega_1+\omega_2+\omega_3+\omega_4=0$)

$$\begin{aligned}
&I_{ABCD}(\omega_1,\omega_2,\omega_3,\omega_4) = I_{BCDA}(\omega_2,\omega_3,\omega_4,\omega_1)e^{\beta\omega_1}\\
&= I_{CDAB}(\omega_3,\omega_4,\omega_1,\omega_2)e^{\beta(\omega_1+\omega_2)} = I_{DABC}(\omega_4,\omega_1,\omega_2,\omega_3)e^{-\beta\omega_4}
\end{aligned} \tag{19}$$

and for the given set of operators $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ there are $4!=24$ different correlation functions (18) but only $3!=6$ nonidentical spectral densities (13).

3. Four-time Matsubara Green's function

Now we introduce four-time Matsubara Green's function

$$\begin{aligned}
K_c^{(4)}(\tau_1,\tau_2,\tau_3,\tau_4) &= \langle\mathcal{T}\hat{A}(\tau_1)\hat{B}(\tau_2)\hat{C}(\tau_3)\hat{D}(\tau_4)\rangle,\\
K_c^{(4)}(\tau_1,\tau_2,\tau_3,\tau_4) &= K_c^{(4)}(\tau_1-\tau,\tau_2-\tau,\tau_3-\tau,\tau_4-\tau).
\end{aligned} \tag{20}$$

Due to the imaginary time ordering $\mathcal{T}$ its Fourier transform contains $4!=24$ terms which can be collected into $3!=6$ contributions

$$K_c^{(4)}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4}) = \frac{1}{\beta}\int\limits_0^\beta d\tau_1\int\limits_0^\beta d\tau_2\int\limits_0^\beta d\tau_3\int\limits_0^\beta d\tau_4$$

$$\times\, e^{(i\omega_{n_1}\tau_1+i\omega_{n_2}\tau_2+i\omega_{n_3}\tau_3+i\omega_{n_4}\tau_4)}K_c^{(4)}(\tau_1,\tau_2,\tau_3,\tau_4)$$
$$= \mathfrak{K}_{ABCD}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4}) + \mathfrak{K}_{DCBA}(i\omega_{n_4},i\omega_{n_3},i\omega_{n_2},i\omega_{n_1})$$
$$+ \mathfrak{K}_{ACDB}(i\omega_{n_1},i\omega_{n_3},i\omega_{n_4},i\omega_{n_2}) + \mathfrak{K}_{BDCA}(i\omega_{n_2},i\omega_{n_4},i\omega_{n_3},i\omega_{n_1})$$
$$+ \mathfrak{K}_{ADBC}(i\omega_{n_1},i\omega_{n_4},i\omega_{n_2},i\omega_{n_3}) + \mathfrak{K}_{CBDA}(i\omega_{n_3},i\omega_{n_2},i\omega_{n_4},i\omega_{n_1}), \tag{21}$$

where

$$\mathfrak{K}_{ABCD}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4}) = \frac{1}{\mathcal{Z}}\sum_{jlfp} A_{jl}B_{lf}C_{fp}D_{pj}\mathfrak{P}(j,i\omega_{n_1},l,i\omega_{n_2},f,i\omega_{n_3},p,i\omega_{n_4}) \tag{22}$$

collects terms connected by the cyclic permutations and $i\omega_n = i(2n+1)\pi T$ are fermionic Matsubara frequencies which satisfies constrain

$$i\omega_{n_1}+i\omega_{n_2}+i\omega_{n_3}+i\omega_{n_4} = 0. \tag{23}$$

In Eq. (22) the cyclic permutations are included through quantity

$$\mathfrak{P}(j,i\omega_{n_1},l,i\omega_{n_2},f,i\omega_{n_3},p,i\omega_{n_4})$$
$$= \frac{1}{\beta}\Bigg[e^{-\beta\varepsilon_j}\int_0^\beta d\tau_1\int_0^{\tau_1} d\tau_2\int_0^{\tau_2} d\tau_3\int_0^{\tau_3} d\tau_4 - e^{-\beta\varepsilon_l}\int_0^\beta d\tau_2\int_0^{\tau_2} d\tau_3\int_0^{\tau_3} d\tau_4\int_0^{\tau_4} d\tau_1$$
$$+ e^{-\beta\varepsilon_f}\int_0^\beta d\tau_3\int_0^{\tau_3} d\tau_4\int_0^{\tau_4} d\tau_1\int_0^{\tau_1} d\tau_2 - e^{-\beta\varepsilon_p}\int_0^\beta d\tau_4\int_0^{\tau_4} d\tau_1\int_0^{\tau_1} d\tau_2\int_0^{\tau_2} d\tau_3\Bigg]$$
$$\times \exp[(\varepsilon_{jl}+i\omega_{n_1})\tau_1+(\varepsilon_{lf}+i\omega_{n_2})\tau_2+(\varepsilon_{fp}+i\omega_{n_3})\tau_3+(\varepsilon_{pj}+i\omega_{n_4})\tau_4], \tag{24}$$

which satisfies the obvious relation

$$\mathfrak{P}(j,i\omega_{n_1},l,i\omega_{n_2},f,i\omega_{n_3},p,i\omega_{n_4}) = -\mathfrak{P}(l,i\omega_{n_2},f,i\omega_{n_3},p,i\omega_{n_4},j,i\omega_{n_1}). \tag{25}$$

In the general case, when all possible nontrivial sums of Matsubara frequencies are nonzero or when there are no eigenstates with the same energy values, function (24) is equal

$$\widetilde{\mathfrak{P}}(j,i\omega_{n_1},l,i\omega_{n_2},f,i\omega_{n_3},p,i\omega_{n_4}) = \Delta(i\omega_{n_1}+i\omega_{n_2}+i\omega_{n_3}+i\omega_{n_4})$$
$$\times\Bigg[\frac{e^{-\beta\varepsilon_j}}{(\varepsilon_{lj}-i\omega_{n_1})(\varepsilon_{fj}-i\omega_{n_1}-i\omega_{n_2})(\varepsilon_{pj}+i\omega_{n_4})}$$
$$-\frac{e^{-\beta\varepsilon_l}}{(\varepsilon_{fl}-i\omega_{n_2})(\varepsilon_{pl}-i\omega_{n_2}-i\omega_{n_3})(\varepsilon_{jl}+i\omega_{n_1})}$$
$$+\frac{e^{-\beta\varepsilon_f}}{(\varepsilon_{if}-i\omega_{n_3})(\varepsilon_{jf}-i\omega_{n_3}-i\omega_{n_4})(\varepsilon_{lf}+i\omega_{n_2})}$$
$$-\frac{e^{-\beta\varepsilon_p}}{(\varepsilon_{jp}-i\omega_{n_4})(\varepsilon_{lp}-i\omega_{n_4}-i\omega_{n_1})(\varepsilon_{fp}+i\omega_{n_3})}\Bigg]. \tag{26}$$

Besides, we must consider several special cases, when we have levels with the same energy value: case of $\varepsilon_j = \varepsilon_f \neq \varepsilon_p, \varepsilon_l$ and $i\omega_{n_1}+i\omega_{n_2} = -i\omega_{n_3}-i\omega_{n_4} = 0$. Now we have additional contribution

$$\left[\lim_{\varepsilon_f\to\varepsilon_j}\lim_{i\omega_{n_2}\to-i\omega_{n_1}} - \lim_{i\omega_{n_2}\to-i\omega_{n_1}}\lim_{\varepsilon_f\to\varepsilon_j}\right]\widetilde{\mathfrak{P}}(j,i\omega_{n_1},l,i\omega_{n_2},f,i\omega_{n_3},p,i\omega_{n_4})$$
$$= \Delta(i\omega_{n_1}+i\omega_{n_2})\Delta(i\omega_{n_3}+i\omega_{n_4})\Delta_{\varepsilon_j,\varepsilon_f}\frac{\beta e^{-\beta\varepsilon_j}}{(\varepsilon_{lj}-i\omega_{n_1})(\varepsilon_{pj}-i\omega_{n_3})}. \tag{27}$$

Another case of $\varepsilon_p = \varepsilon_l \neq \varepsilon_j, \varepsilon_f$ and $i\omega_{n_1}+i\omega_{n_4} = -i\omega_{n_3}-i\omega_{n_2} = 0$ produces different additional contribution

$$\left[\lim_{\varepsilon_p\to\varepsilon_l}\lim_{i\omega_{n_4}\to-i\omega_{n_1}} - \lim_{i\omega_{n_4}\to-i\omega_{n_1}}\lim_{\varepsilon_p\to\varepsilon_l}\right]\widetilde{\mathfrak{P}}(j,i\omega_{n_1},l,i\omega_{n_2},f,i\omega_{n_3},p,i\omega_{n_4})$$
$$= -\Delta(i\omega_{n_1}+i\omega_{n_4})\Delta(i\omega_{n_3}+i\omega_{n_2})\Delta_{\varepsilon_p,\varepsilon_l}\frac{\beta e^{-\beta\varepsilon_l}}{(\varepsilon_{fl}-i\omega_{n_2})(\varepsilon_{jl}-i\omega_{n_4})}. \tag{28}$$

The case of $\varepsilon_j = \varepsilon_f \neq \varepsilon_p = \varepsilon_l$ and $i\omega_{n_1} = -i\omega_{n_2} = i\omega_{n_3} = -i\omega_{n_4}$ does not introduce any additional contributions but it should be considered separately to avoid double counting. Here

$$\Delta_{\varepsilon_j,\varepsilon_f} = \begin{cases} 1, & \varepsilon_j = \varepsilon_f \\ 0, & \varepsilon_j \neq \varepsilon_f \end{cases}; \qquad \bar{\Delta}_{\varepsilon_j,\varepsilon_f} = 1 - \Delta_{\varepsilon_j,\varepsilon_f}. \tag{29}$$

Special consideration of such terms is required because in many cases, e.g., in the numerical calculations, it is very difficult to tune up independently the energies of each many body state ε_j in order to apply tricks like (27) and (28) and they should be incorporated in the theory explicitly. On the other hand, they correspond to the cases when consequent action of two fermionic operators returns many-body system back to the initial state or state with the same energy (true or accidental degeneracy) and represent the elastic scattering collisions and such processes determine the difference between the isothermal (e.g., static) and isolated (Kubo) susceptibilities [5].

Finally, we get

$$\mathfrak{P}(j,i\omega_{n_1},l,i\omega_{n_2},f,i\omega_{n_3},p,i\omega_{n_4}) = \widetilde{\mathfrak{P}}(j,i\omega_{n_1},l,i\omega_{n_2},f,i\omega_{n_3},p,i\omega_{n_4})$$
$$+\,\Delta(i\omega_{n_1}+i\omega_{n_2})\Delta(i\omega_{n_3}+i\omega_{n_4})\Delta_{\varepsilon_j,\varepsilon_f}\frac{\beta e^{-\beta\varepsilon_j}}{(\varepsilon_{lj}-i\omega_{n_1})(\varepsilon_{pj}-i\omega_{n_3})}$$
$$-\,\Delta(i\omega_{n_1}+i\omega_{n_4})\Delta(i\omega_{n_3}+i\omega_{n_2})\Delta_{\varepsilon_p,\varepsilon_l}\frac{\beta e^{-\beta\varepsilon_l}}{(\varepsilon_{fl}-i\omega_{n_2})(\varepsilon_{jl}-i\omega_{n_4})} \qquad (30)$$

or

$$\mathfrak{P}(j,i\omega_{n_1},l,i\omega_{n_2},f,i\omega_{n_3},p,i\omega_{n_4}) = \Delta(i\omega_{n_1}+i\omega_{n_2}+i\omega_{n_3}+i\omega_{n_4})$$
$$\times\,\bar\Delta_{\varepsilon_j,\varepsilon_f}\bar\Delta_{\varepsilon_l,\varepsilon_p}\Bigg[\frac{e^{-\beta\varepsilon_j}}{(\varepsilon_{lj}-i\omega_{n_1})(\varepsilon_{fj}-i\omega_{n_1}-i\omega_{n_2})(\varepsilon_{pj}+i\omega_{n_4})}$$
$$-\,\frac{e^{-\beta\varepsilon_l}}{(\varepsilon_{fl}-i\omega_{n_2})(\varepsilon_{pl}-i\omega_{n_2}-i\omega_{n_3})(\varepsilon_{jl}+i\omega_{n_1})}$$
$$+\,\frac{e^{-\beta\varepsilon_f}}{(\varepsilon_{pf}-i\omega_{n_3})(\varepsilon_{jf}-i\omega_{n_3}-i\omega_{n_4})(\varepsilon_{lf}+i\omega_{n_2})}$$
$$-\,\frac{e^{-\beta\varepsilon_p}}{(\varepsilon_{jp}-i\omega_{n_4})(\varepsilon_{lp}-i\omega_{n_4}-i\omega_{n_1})(\varepsilon_{fp}+i\omega_{n_3})}\Bigg]$$
$$-\,\Delta(i\omega_{n_1}+i\omega_{n_2}+i\omega_{n_3}+i\omega_{n_4})\Delta_{\varepsilon_j,\varepsilon_f}\bar\Delta_{\varepsilon_l,\varepsilon_p}$$
$$\times\left\{\frac{e^{-\beta\varepsilon_j}}{(\varepsilon_{lj}+i\omega_{n_2})(\varepsilon_{pj}+i\omega_{n_4})}\left[\frac{1}{\varepsilon_{lj}-i\omega_{n_1}}+\frac{1}{\varepsilon_{pj}-i\omega_{n_3}}\right]\right.$$
$$\left.+\,\frac{1}{\varepsilon_{pl}-i\omega_{n_2}-i\omega_{n_3}}\left[\frac{e^{-\beta\varepsilon_l}}{(\varepsilon_{jl}-i\omega_{n_2})(\varepsilon_{jl}+i\omega_{n_1})}-\frac{e^{-\beta\varepsilon_p}}{(\varepsilon_{jp}-i\omega_{n_4})(\varepsilon_{jp}+i\omega_{n_3})}\right]\right\}$$
$$+\,\Delta(i\omega_{n_1}+i\omega_{n_2}+i\omega_{n_3}+i\omega_{n_4})\bar\Delta_{\varepsilon_j,\varepsilon_f}\Delta_{\varepsilon_l,\varepsilon_p}$$
$$\times\left\{\frac{1}{\varepsilon_{fj}-i\omega_{n_1}-i\omega_{n_2}}\left[\frac{e^{-\beta\varepsilon_j}}{(\varepsilon_{lj}-i\omega_{n_1})(\varepsilon_{lj}+i\omega_{n_4})}-\frac{e^{-\beta\varepsilon_f}}{(\varepsilon_{lf}-i\omega_{n_3})(\varepsilon_{lf}+i\omega_{n_2})}\right]\right.$$
$$\left.+\,\frac{e^{-\beta\varepsilon_l}}{(\varepsilon_{jl}+i\omega_{n_1})(\varepsilon_{fl}+i\omega_{n_3})}\left[\frac{1}{\varepsilon_{fl}-i\omega_{n_2}}+\frac{1}{\varepsilon_{jl}-i\omega_{n_4}}\right]\right\}$$
$$+\,\Delta(i\omega_{n_1}+i\omega_{n_2}+i\omega_{n_3}+i\omega_{n_4})\Delta_{\varepsilon_j,\varepsilon_f}\Delta_{\varepsilon_l,\varepsilon_p}$$
$$\times\,\frac{e^{-\beta\varepsilon_j}+e^{-\beta\varepsilon_l}}{(\varepsilon_{lj}-i\omega_{n_1})(\varepsilon_{lj}-i\omega_{n_3})}\left[\frac{1}{\varepsilon_{jl}-i\omega_{n_2}}+\frac{1}{\varepsilon_{jl}-i\omega_{n_4}}\right]$$
$$+\,\Delta(i\omega_{n_1}+i\omega_{n_2})\Delta(i\omega_{n_3}+i\omega_{n_4})\Delta_{\varepsilon_j,\varepsilon_f}\bar\Delta_{\varepsilon_p,\varepsilon_l}\frac{\beta e^{-\beta\varepsilon_j}}{(\varepsilon_{lj}-i\omega_{n_1})(\varepsilon_{pj}-i\omega_{n_3})}$$
$$-\,\Delta(i\omega_{n_1}+i\omega_{n_4})\Delta(i\omega_{n_3}+i\omega_{n_2})\bar\Delta_{\varepsilon_j,\varepsilon_f}\Delta_{\varepsilon_p,\varepsilon_l}\frac{\beta e^{-\beta\varepsilon_l}}{(\varepsilon_{fl}-i\omega_{n_2})(\varepsilon_{jl}-i\omega_{n_4})}.$$

$$+\,\frac{\beta\Delta_{\varepsilon_j,\varepsilon_f}\Delta_{\varepsilon_p,\varepsilon_l}}{(\varepsilon_{lj}-i\omega_{n_1})(\varepsilon_{lj}-i\omega_{n_3})}\big[\Delta(i\omega_{n_1}+i\omega_{n_2})\Delta(i\omega_{n_3}+i\omega_{n_4})e^{-\beta\varepsilon_j}$$
$$-\,\Delta(i\omega_{n_1}+i\omega_{n_4})\Delta(i\omega_{n_3}+i\omega_{n_2})e^{-\beta\varepsilon_l}\big]. \qquad (31)$$

Now we can introduce spectral representations for the four time fermionic Matsubara functions. For the first term in (21) we get ($\omega_4 = -\omega_1-\omega_2-\omega_3$)

$$\mathfrak{K}_{ABCD}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4}) = \widetilde{\mathfrak{K}}_{ABCD}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4})$$
$$+\beta\Delta(i\omega_{n_1}+i\omega_{n_2})\Delta(i\omega_{n_3}+i\omega_{n_4})\int_{-\infty}^{+\infty}d\omega_1\int_{-\infty}^{+\infty}d\omega_3\frac{\bar I_{\overline{AB}\,\overline{CD}}(\omega_1,-\omega_1,\omega_3,-\omega_3)}{(\omega_1-i\omega_{n_1})(\omega_3-i\omega_{n_3})}$$
$$-\beta\Delta(i\omega_{n_1}+i\omega_{n_4})\Delta(i\omega_{n_3}+i\omega_{n_2})\int_{-\infty}^{+\infty}d\omega_1\int_{-\infty}^{+\infty}d\omega_3\frac{\bar I_{\overline{A}\,\overline{BC}\,\overline{D}}(\omega_1,-\omega_3,\omega_3,-\omega_1)}{(\omega_1-i\omega_{n_1})(\omega_3-i\omega_{n_3})}e^{-\beta\omega_1}$$
$$+\beta\Delta(i\omega_{n_1}+i\omega_{n_2})\Delta(i\omega_{n_3}+i\omega_{n_4})\int_{-\infty}^{+\infty}d\omega_1\frac{\bar{\bar I}_{\overline{ABCD}}(\omega_1,-\omega_1,\omega_1,-\omega_1)}{(\omega_1-i\omega_{n_1})(\omega_1-i\omega_{n_3})}$$
$$-\beta\Delta(i\omega_{n_1}+i\omega_{n_4})\Delta(i\omega_{n_3}+i\omega_{n_2})\int_{-\infty}^{+\infty}d\omega_1\frac{\bar{\bar I}_{\overline{ABCD}}(\omega_1,-\omega_1,\omega_1,-\omega_1)}{(\omega_1-i\omega_{n_1})(\omega_1-i\omega_{n_3})}e^{-\beta\omega_1}, \qquad (32)$$

where

$$\widetilde{\mathfrak{K}}_{ABCD}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4}) = \Delta(i\omega_{n_1}+i\omega_{n_2}+i\omega_{n_3}+i\omega_{n_4})$$
$$\times\left(\int_{-\infty}^{+\infty}d\omega_1\int_{-\infty}^{+\infty}d\omega_2\int_{-\infty}^{+\infty}d\omega_3\tilde I_{ABCD}(\omega_1,\omega_2,\omega_3,\omega_4)\right.$$
$$\times\left[\frac{1}{(\omega_1-i\omega_{n_1})(\omega_1+\omega_2-i\omega_{n_1}-i\omega_{n_2})(-\omega_4+i\omega_{n_4})}\right.$$
$$-\,\frac{e^{-\beta\omega_1}}{(\omega_2-i\omega_{n_2})(\omega_2+\omega_3-i\omega_{n_2}-i\omega_{n_3})(-\omega_1+i\omega_{n_1})}$$
$$+\,\frac{e^{-\beta(\omega_1+\omega_2)}}{(\omega_3-i\omega_{n_3})(\omega_3+\omega_4-i\omega_{n_3}-i\omega_{n_4})(-\omega_2+i\omega_{n_2})}$$
$$\left.-\,\frac{e^{\beta\omega_4}}{(\omega_4-i\omega_{n_4})(\omega_4+\omega_1-i\omega_{n_4}-i\omega_{n_1})(-\omega_3+i\omega_{n_3})}\right]$$

$$
-\int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_3 \bar{I}_{\overline{AB}\,\overline{CD}}(\omega_1,-\omega_1,\omega_3,-\omega_3)
$$
$$
\times\left\{\frac{1}{(\omega_1+i\omega_{n_2})(\omega_3+i\omega_{n_4})}\left[\frac{1}{\omega_1-i\omega_{n_1}}+\frac{1}{\omega_3-i\omega_{n_3}}\right]\right.
$$
$$
\left.+\frac{1}{\omega_3-\omega_1-i\omega_{n_2}-i\omega_{n_3}}\left[\frac{e^{-\beta\omega_1}}{(\omega_1+i\omega_{n_2})(\omega_1-i\omega_{n_1})}-\frac{e^{-\beta\omega_3}}{(\omega_3+i\omega_{n_4})(\omega_3-i\omega_{n_3})}\right]\right\}
$$
$$
+\int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_3 \bar{I}_{\overline{A}\,\overline{BC}\,\overline{D}}(\omega_1,-\omega_3,\omega_3,-\omega_1)
$$
$$
\times\left\{\frac{e^{-\beta\omega_1}}{(\omega_1-i\omega_{n_1})(\omega_3-i\omega_{n_3})}\left[\frac{1}{-\omega_3-i\omega_{n_2}}+\frac{1}{-\omega_1-i\omega_{n_4}}\right]\right.
$$
$$
\left.+\frac{1}{\omega_1-\omega_3-i\omega_{n_1}-i\omega_{n_2}}\left[\frac{1}{(\omega_1+i\omega_{n_4})(\omega_1-i\omega_{n_1})}-\frac{e^{-\beta(\omega_1-\omega_3)}}{(\omega_3+i\omega_{n_2})(\omega_3-i\omega_{n_3})}\right]\right\}
$$
$$
-\int_{-\infty}^{+\infty} d\omega_1 \bar{\bar{I}}_{\overline{ABCD}}(\omega_1,-\omega_1,\omega_1,-\omega_1)\frac{1+e^{-\beta\omega_1}}{(\omega_1-i\omega_{n_1})(\omega_1-i\omega_{n_3})}
$$
$$
\left.\times\left[\frac{1}{\omega_1+i\omega_{n_2}}+\frac{1}{\omega_1+i\omega_{n_4}}\right]\right). \qquad (33)
$$

Similar expressions can be written for other five contributions in (21). One can see from (32) and (33) that spectral densities (14) contribute only in the normal components (33), whereas the spectral densities (15)–(17) contribute in the both normal and anomalous one.

3.1. Zero-frequency anomaly and cumulants

It follows from Eq. (32) that there are two types of contributions: the normal one $\widetilde{\mathfrak{K}}_{ABCD}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4})$ with all frequencies being different and anomalous one with additional constrains on the frequencies. Based on this one can rewrite the four-time Green's function (21) in the form

$$
\begin{aligned}
K_c^{(4)}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4}) &= \widetilde{K}_c^{(4)}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4})\\
&+\beta\Delta(i\omega_{n_1}+i\omega_{n_2})\Delta(i\omega_{n_3}+i\omega_{n_4})\bar{K}_c^{AB,CD}(i\omega_{n_1},i\omega_{n_3})\\
&+\beta\Delta(i\omega_{n_1}+i\omega_{n_3})\Delta(i\omega_{n_4}+i\omega_{n_2})\bar{K}_c^{AC,DB}(i\omega_{n_1},i\omega_{n_4})\\
&+\beta\Delta(i\omega_{n_1}+i\omega_{n_4})\Delta(i\omega_{n_2}+i\omega_{n_3})\bar{K}_c^{AD,BC}(i\omega_{n_1},i\omega_{n_2}),
\end{aligned} \qquad (34)
$$

where

$$
\begin{aligned}
&\widetilde{K}_c^{(4)}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4}) =\\
&= \widetilde{\mathfrak{K}}_{ABCD}(i\omega_{n_1},i\omega_{n_2},i\omega_{n_3},i\omega_{n_4}) + \widetilde{\mathfrak{K}}_{DCBA}(i\omega_{n_4},i\omega_{n_3},i\omega_{n_2},i\omega_{n_1})\\
&+ \widetilde{\mathfrak{K}}_{ACDB}(i\omega_{n_1},i\omega_{n_3},i\omega_{n_4},i\omega_{n_2}) + \widetilde{\mathfrak{K}}_{BDCA}(i\omega_{n_2},i\omega_{n_4},i\omega_{n_3},i\omega_{n_1})\\
&+ \widetilde{\mathfrak{K}}_{ADBC}(i\omega_{n_1},i\omega_{n_4},i\omega_{n_2},i\omega_{n_3}) + \widetilde{\mathfrak{K}}_{CBDA}(i\omega_{n_3},i\omega_{n_2},i\omega_{n_4},i\omega_{n_1})
\end{aligned} \qquad (35)
$$

collects all normal contributions. The anomalous contribution has the form

$$
\bar{K}_c^{AB,CD}(i\omega_{n_1},i\omega_{n_3}) = \frac{1}{Z}\sum_{\substack{jf\\ \varepsilon_j=\varepsilon_f}} e^{-\beta\varepsilon_j} g_{jf}^{AB}(i\omega_{n_1}) g_{fj}^{CD}(i\omega_{n_3}), \qquad (36)
$$

where the quantities

$$
g_{jf}^{AB}(i\omega_{n_1}) = \sum_l \left[\frac{A_{jl}B_{lf}}{i\omega_1-\varepsilon_{lj}}+\frac{B_{jl}A_{lf}}{i\omega_1+\varepsilon_{lj}}\right] \qquad (37)
$$

could be considered as an unaveraged matrix elements of the two-time Green's function (3)

$$
\begin{aligned}
\widetilde{G}_{AB}(i\omega_n) &= \frac{1}{Z}\sum_{jf} e^{-\beta\varepsilon_j}\left[\frac{A_{jf}B_{fj}}{i\omega_n+\varepsilon_{jf}} \mp \frac{B_{jf}A_{fj}}{i\omega_n-\varepsilon_{jf}}\right]\\
&= \frac{1}{Z}\sum_j e^{-\beta\varepsilon_j} g_{jj}^{AB}(i\omega_n).
\end{aligned} \qquad (38)
$$

Anomalous term can be represented as a sum of two contributions

$$
\bar{K}_c^{AB,CD}(i\omega_{n_1},i\omega_{n_3}) = \widetilde{G}_{AB}(i\omega_{n_1})\widetilde{G}_{CD}(i\omega_{n_3}) + \bar{K}_{c,\mathbf{irr}}^{AB,CD}(i\omega_{n_1},i\omega_{n_3}), \qquad (39)
$$

where the reducible part [product of two two-time Green's functions (3)] is separated and the irreducible one is equal

$$
\begin{aligned}
\bar{K}_{c,\mathbf{irr}}^{AB,CD}(i\omega_{n_1},i\omega_{n_3}) = &\sum_{\substack{jfj'f'\\ \varepsilon_j=\varepsilon_f}} \left[b_j^{(1)}\delta_{jj'}\delta_{ff'} - b_j^{(1)} b_{j'}^{(1)}\delta_{jf}\delta_{j'f'}\right]\\
&\times g_{jf}^{AB}(i\omega_{n_1}) g_{f'j'}^{CD}(i\omega_{n_3}),
\end{aligned} \qquad (40)
$$

where

$$b_j^{(1)} = \frac{\partial}{\partial(-\beta\varepsilon_j)} \ln Z = \frac{1}{Z} e^{-\beta\varepsilon_j} \tag{41}$$

could be considered as a first cumulant (Ursell function) [31–33] of the Boltzmann distribution.

Let us consider the case of the non-degenerate states, when there are no different states with the same energy. In this case, expression (40) takes more straight form

$$\bar{K}_{c,\mathbf{irr}}^{AB,CD}(i\omega_{n_1}, i\omega_{n_3}) = \sum_{jf} b_{jf}^{(2)} g_{jj}^{AB}(i\omega_{n_1}) g_{ff}^{CD}(i\omega_{n_3}), \tag{42}$$

where

$$\begin{aligned} b_{jf}^{(2)} &= \frac{\partial}{\partial(-\beta\varepsilon_j)} \frac{\partial}{\partial(-\beta\varepsilon_f)} Z = \frac{\partial b_j^{(1)}}{\partial(-\beta\varepsilon_f)} = \frac{\partial b_f^{(1)}}{\partial(-\beta\varepsilon_j)} \\ &= b_j^{(1)}\delta_{jf} - b_j^{(1)} b_f^{(1)} \end{aligned} \tag{43}$$

is the second cumulant of the Boltzmann distribution. Based on this, one can consider an expression in brackets in (40) as a generalization of the cumulant expansions for the case when a degenerate states are present in the many-body system.

In the many-body theory cumulant contributions appear in a natural way in the strong coupling approaches [34–40] and in some cases they are the only contributions which enter an expressions for the dynamical response, e.g., dynamical charge susceptibilities [41, 42] and cross-sections of the inelastic light (Raman) and x-ray scattering [26–28] for the Falicov-Kimball model [43].

3.2. Analytic continuation and reverse engineering problem

Next, we can perform an analytic continuation from the Matsubara frequencies to the real one and for each term in (32) we will get different sets of the branch cuts as for single frequencies

$$i\omega_{n_\alpha} \to \omega_\alpha \pm i0^+, \quad \alpha = 1, 2, 3, 4 \tag{44}$$

as for sums of two frequencies

$$i\omega_{n_\alpha} + i\omega_{n_\gamma} \to \omega_\alpha + \omega_\gamma \pm i0^+. \tag{45}$$

Differences in the analytic properties of each term in (32) and, as a result, in (22) allow to solve the reverse engineering problem: extracting

all spectral densities from the single Green's function. To do this, one have to extract consequently nonanalyticities at all brunch cuts of type (44) and (45) but in different order, which will produce a set of equations for the unknown spectral densities. The procedure is very cumbersome and will be not presented here, see for the details Ref. [30].

4. High frequency asymptotics

In many applications, e.g., for the correctness checking of analytic approximations or for the memory consumption limitations for storing of the high frequency tails in numerical calculations, it is useful to have the high frequency asymptotics of the four time fermionic Matsubara Green's functions. It is obvious that for different directions in the three-frequency space defined by constrain (23) one can observe different asymptotic behavior and below we shall present results for some cases.

4.1. $|i\omega_n| \sim \Omega$, $|i\omega_n + i\omega_m| \sim \Omega$, $\Omega \gg E$

First of all we consider the most general case when each Matsubara's frequency $|i\omega_n| \sim \Omega$ as well as each nontrivial sum of Matsubara frequencies $|i\omega_n + i\omega_m| \sim \Omega$ with taking into account constrain (23) are much larger in modulus then possible many-body state energy differences $\Omega \gg E = \max|\varepsilon_j - \varepsilon_l|$. The first $1/\Omega^3$ order terms in the high frequency expansion of (21) using (30) are equal

$$\begin{aligned} & K_c^{(4)}(i\omega_{n_1}, i\omega_{n_2}, i\omega_{n_3}, i\omega_{n_4}) \longrightarrow \\ & \frac{\langle\{[\{A,D\},B],C\}\rangle}{i\omega_{n_3} i\omega_{n_4}(i\omega_{n_2}+i\omega_{n_3})} + \frac{\langle\{[A,\{B,D\}],C\}\rangle}{i\omega_{n_3} i\omega_{n_4}(i\omega_{n_1}+i\omega_{n_3})} \\ & + \frac{\langle\{[C,\{A,D\}],B\}\rangle}{i\omega_{n_2} i\omega_{n_4}(i\omega_{n_2}+i\omega_{n_3})} + \frac{\langle\{A,[\{B,D\},C]\}\rangle}{i\omega_{n_1} i\omega_{n_4}(i\omega_{n_1}+i\omega_{n_3})} \\ & + \frac{\langle\{B,[\{C,D\},A]\}\rangle}{i\omega_{n_2} i\omega_{n_4}(i\omega_{n_1}+i\omega_{n_2})} + \frac{\langle\{A,[B,\{C,D\}]\}\rangle}{i\omega_{n_1} i\omega_{n_4}(i\omega_{n_1}+i\omega_{n_2})}, \end{aligned} \tag{46}$$

where we have introduced anticommutators $\{X_1, X_2\} = X_1X_2 + X_2X_1$ and commutators $[X, Y] = XY - YX$ of operators, and in the case of the ordinary fermionic creation and annihilation operators it is equal to zero (but it is not correct for the Hubbard operators).

The next $1/\Omega^4$ order contributions are equal

$$\mathfrak{K}_{ABCD}(i\omega_{n_1}, i\omega_{n_2}, i\omega_{n_3}, i\omega_{n_4}) \longrightarrow \frac{1}{i\omega_{n_1}(i\omega_{n_1}+i\omega_{n_2})i\omega_{n_4}} \times \left\{ \frac{\langle [A,H]BCD \rangle}{i\omega_{n_1}} + \frac{\langle [AB,H]CD \rangle}{i\omega_{n_1}+i\omega_{n_2}} + \frac{\langle ABC[D,H] \rangle}{i\omega_{n_4}} \right\} \tag{47}$$

where we have used identity

$$\varepsilon_{lj} \langle j|\hat{A}|l\rangle = \langle j|[\hat{A},H]|l\rangle. \tag{48}$$

The presence of different frequency denominators does not allow to collapse the total expression (21) in a compact form like (46).

4.2. $|i\omega_{n_1}| \sim E$, $|i\omega_n| \sim \Omega$, $|i\omega_n + i\omega_m| \sim \Omega$, $\Omega \gg E$

Next we consider the case of the finite frequency value $|i\omega_{n_1}| \sim E$. Other Matsubara frequencies $|i\omega_n| \sim \Omega$ ($n = 2, 3, 4$) as well as each nontrivial sum of Matsubara frequencies $|i\omega_n + i\omega_m| \sim \Omega$ are much larger in modulus then possible energy differences $\Omega \gg E = \max|\varepsilon_j - \varepsilon_l|$. The first terms in the high frequency expansion of (21) using (30) are equal

$$\begin{aligned}
&\frac{1}{(i\omega_{n_3}+i\omega_{n_4})i\omega_{n_4}}\left\{\mathfrak{F}_{A,BCD}(i\omega_{n_1})+\frac{\mathfrak{F}_{A,B[CD,H]}(i\omega_{n_1})}{i\omega_{n_3}+i\omega_{n_4}}+\frac{\mathfrak{F}_{A,BC[D,H]}(i\omega_{n_1})}{i\omega_{n_4}}\right\}\\
&-\frac{1}{(i\omega_{n_3}+i\omega_{n_2})i\omega_{n_2}}\left\{\mathfrak{F}_{A,DCB}(i\omega_{n_1})+\frac{\mathfrak{F}_{A,D[CB,H]}(i\omega_{n_1})}{i\omega_{n_3}+i\omega_{n_2}}+\frac{\mathfrak{F}_{A,DC[B,H]}(i\omega_{n_1})}{i\omega_{n_2}}\right\}\\
&+\frac{1}{(i\omega_{n_4}+i\omega_{n_2})i\omega_{n_2}}\left\{\mathfrak{F}_{A,CDB}(i\omega_{n_1})+\frac{\mathfrak{F}_{A,C[DB,H]}(i\omega_{n_1})}{i\omega_{n_4}+i\omega_{n_2}}+\frac{\mathfrak{F}_{A,CD[B,H]}(i\omega_{n_1})}{i\omega_{n_2}}\right\}\\
&-\frac{1}{(i\omega_{n_4}+i\omega_{n_3})i\omega_{n_3}}\left\{\mathfrak{F}_{A,BDC}(i\omega_{n_1})+\frac{\mathfrak{F}_{A,B[DC,H]}(i\omega_{n_1})}{i\omega_{n_4}+i\omega_{n_3}}+\frac{\mathfrak{F}_{A,BD[C,H]}(i\omega_{n_1})}{i\omega_{n_3}}\right\}\\
&+\frac{1}{(i\omega_{n_2}+i\omega_{n_3})i\omega_{n_3}}\left\{\mathfrak{F}_{A,DBC}(i\omega_{n_1})+\frac{\mathfrak{F}_{A,D[BC,H]}(i\omega_{n_1})}{i\omega_{n_2}+i\omega_{n_3}}+\frac{\mathfrak{F}_{A,DB[C,H]}(i\omega_{n_1})}{i\omega_{n_3}}\right\}\\
&-\frac{1}{(i\omega_{n_2}+i\omega_{n_4})i\omega_{n_4}}\left\{\mathfrak{F}_{A,CBD}(i\omega_{n_1})+\frac{\mathfrak{F}_{A,C[BD,H]}(i\omega_{n_1})}{i\omega_{n_2}+i\omega_{n_4}}+\frac{\mathfrak{F}_{A,CB[D,H]}(i\omega_{n_1})}{i\omega_{n_4}}\right\}\\
&-\frac{1}{i\omega_{n_2}(i\omega_{n_2}+i\omega_{n_3})}\left\{\mathfrak{F}_{BCD,A}(-i\omega_{n_1})+\frac{\mathfrak{F}_{[BC,H]D,A}(-i\omega_{n_1})}{i\omega_{n_2}+i\omega_{n_3}}+\frac{\mathfrak{F}_{[B,H]CD,A}(-i\omega_{n_1})}{i\omega_{n_2}}\right\}\\
&+\frac{1}{i\omega_{n_4}(i\omega_{n_4}+i\omega_{n_3})}\left\{\mathfrak{F}_{DCB,A}(-i\omega_{n_1})+\frac{\mathfrak{F}_{[DC,H]B,A}(-i\omega_{n_1})}{i\omega_{n_4}+i\omega_{n_3}}+\frac{\mathfrak{F}_{[D,H]CB,A}(-i\omega_{n_1})}{i\omega_{n_4}}\right\}\\
&-\frac{1}{i\omega_{n_3}(i\omega_{n_3}+i\omega_{n_4})}\left\{\mathfrak{F}_{CDB,A}(-i\omega_{n_1})+\frac{\mathfrak{F}_{[CD,H]B,A}(-i\omega_{n_1})}{i\omega_{n_3}+i\omega_{n_4}}+\frac{\mathfrak{F}_{[C,H]DB,A}(-i\omega_{n_1})}{i\omega_{n_3}}\right\}\\
&+\frac{1}{i\omega_{n_2}(i\omega_{n_2}+i\omega_{n_4})}\left\{\mathfrak{F}_{BDC,A}(-i\omega_{n_1})+\frac{\mathfrak{F}_{[BD,H]C,A}(-i\omega_{n_1})}{i\omega_{n_2}+i\omega_{n_4}}+\frac{\mathfrak{F}_{[B,H]DC,A}(-i\omega_{n_1})}{i\omega_{n_2}}\right\}\\
&-\frac{1}{i\omega_{n_4}(i\omega_{n_4}+i\omega_{n_2})}\left\{\mathfrak{F}_{DBC,A}(-i\omega_{n_1})+\frac{\mathfrak{F}_{[DB,H]C,A}(-i\omega_{n_1})}{i\omega_{n_4}+i\omega_{n_2}}+\frac{\mathfrak{F}_{[D,H]BC,A}(-i\omega_{n_1})}{i\omega_{n_4}}\right\}
\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{i\omega_{n_3}(i\omega_{n_3}+i\omega_{n_2})}\left\{\mathfrak{F}_{CBD,A}(-i\omega_{n_1})+\frac{\mathfrak{F}_{[CB,H]D,A}(-i\omega_{n_1})}{i\omega_{n_3}+i\omega_{n_2}}+\frac{\mathfrak{F}_{[C,H]BD,A}(-i\omega_{n_1})}{i\omega_{n_3}}\right\}\\
&+\frac{\langle CDAB\rangle}{i\omega_{n_3}(i\omega_{n_3}+i\omega_{n_4})i\omega_{n_2}}-\frac{\langle DABC\rangle}{i\omega_{n_4}(i\omega_{n_4}+i\omega_{n_1})i\omega_{n_3}}+\frac{\langle BADC\rangle}{i\omega_{n_2}(i\omega_{n_2}+i\omega_{n_1})i\omega_{n_3}}\\
&-\frac{\langle CBAD\rangle}{i\omega_{n_3}(i\omega_{n_3}+i\omega_{n_2})i\omega_{n_4}}+\frac{\langle DBAC\rangle}{i\omega_{n_4}(i\omega_{n_4}+i\omega_{n_2})i\omega_{n_3}}-\frac{\langle BACD\rangle}{i\omega_{n_2}(i\omega_{n_2}+i\omega_{n_1})i\omega_{n_4}}\\
&+\frac{\langle CABD\rangle}{i\omega_{n_3}(i\omega_{n_3}+i\omega_{n_1})i\omega_{n_4}}-\frac{\langle DCAB\rangle}{i\omega_{n_4}(i\omega_{n_4}+i\omega_{n_3})i\omega_{n_2}}+\frac{\langle BCAD\rangle}{i\omega_{n_2}(i\omega_{n_2}+i\omega_{n_3})i\omega_{n_4}}\\
&-\frac{\langle CADB\rangle}{i\omega_{n_3}(i\omega_{n_3}+i\omega_{n_1})i\omega_{n_2}}+\frac{\langle DACB\rangle}{i\omega_{n_4}(i\omega_{n_4}+i\omega_{n_1})i\omega_{n_2}}-\frac{\langle BDAC\rangle}{i\omega_{n_2}(i\omega_{n_2}+i\omega_{n_4})i\omega_{n_3}}.
\end{aligned} \tag{49}$$

Here we have introduced function

$$\mathfrak{F}_{A,X}(i\omega_{n_1}) = \frac{1}{\mathcal{Z}}\sum_{jl} e^{-\beta\varepsilon_j}\frac{A_{jl}X_{lj}}{\varepsilon_{lj}-i\omega_{n_1}} = -\int_{-\infty}^{+\infty} d\omega n_+(-\omega)\frac{\rho_{AX}(\omega)}{i\omega_{n_1}-\omega}, \tag{50}$$

where $\rho_{AX}(\omega)$ is fermionic density of states (2) which can be obtained from the corresponding Green's functions and

$$n_+(\omega) = \frac{1}{e^{\beta\omega}+1}$$

is the Fermi distribution function. The spectral representation for function (50) differs from the one for the Matsubara and retarded (advanced) Green's functions (3) by the Fermi factor $n_+(\omega)$ and is similar to the one for the so-called "half" Green's functions [44].

One can imagine that first terms in braces in (49) produce contributions of the order $1/\Omega^2$ whereas other one are of the order $1/\Omega^3$, but it could be shown that in the case of ordinary creation and annihilation fermionic operators the total contribution of these terms is of the order $|i\omega_{n_1}|/\Omega^3$, that is of the same order as other terms are.

The cases of $|i\omega_{n_2}| \sim E$, $|i\omega_{n_3}| \sim E$, or $|i\omega_{n_4}| \sim E$ can be obtained from the above expression by the corresponding permutation of operators and frequencies.

Next we consider the cases of the finite values of the sums of two Matsubara frequencies.

4.3. $|i\omega_{n_1} + i\omega_{n_2}| \sim E$, $|i\omega_n| \sim \Omega$, $|i\omega_n + i\omega_m| \sim \Omega$, $\Omega \gg E$

First we put that only one sum of two frequencies is finite $|i\omega_{n_1} + i\omega_{n_2}| = |i\omega_{n_3} + i\omega_{n_4}| \sim E$, including the case of $i\omega_{n_1} + i\omega_{n_2} = -i\omega_{n_3} - i\omega_{n_4} =$

0. Single Matsubara frequencies $|i\omega_n| \sim \Omega$ as well as other sums of Matsubara frequencies $|i\omega_n + i\omega_m| \sim \Omega$ are much larger in modulus then possible energy differences $\Omega \gg E$. The first $1/\Omega^2$ order terms in the high frequency expansion of (21) using (31) are equal

$$
\begin{aligned}
&\frac{\mathfrak{B}_{CD,AB}(i\omega_{n_3}+i\omega_{n_4})}{i\omega_{n_3}i\omega_{n_2}}+\frac{\mathfrak{B}_{AB,CD}(i\omega_{n_1}+i\omega_{n_2})}{i\omega_{n_1}i\omega_{n_4}}+\beta\Delta(i\omega_{n_1}+i\omega_{n_2})\frac{C_{AB,CD}}{i\omega_{n_1}i\omega_{n_3}}\\
&+\frac{\mathfrak{B}_{BA,DC}(i\omega_{n_2}+i\omega_{n_1})}{i\omega_{n_2}i\omega_{n_3}}+\frac{\mathfrak{B}_{DC,BA}(i\omega_{n_4}+i\omega_{n_3})}{i\omega_{n_4}i\omega_{n_1}}+\beta\Delta(i\omega_{n_4}+i\omega_{n_3})\frac{C_{DC,BA}}{i\omega_{n_4}i\omega_{n_2}}\\
&-\left[\frac{\mathfrak{B}_{BA,CD}(i\omega_{n_2}+i\omega_{n_1})}{i\omega_{n_2}i\omega_{n_4}}+\frac{\mathfrak{B}_{CD,BA}(i\omega_{n_3}+i\omega_{n_4})}{i\omega_{n_3}i\omega_{n_1}}+\beta\Delta(i\omega_{n_3}+i\omega_{n_4})\frac{C_{CD,BA}}{i\omega_{n_3}i\omega_{n_2}}\right]\\
&-\left[\frac{\mathfrak{B}_{DC,AB}(i\omega_{n_4}+i\omega_{n_3})}{i\omega_{n_4}i\omega_{n_2}}+\frac{\mathfrak{B}_{AB,DC}(i\omega_{n_1}+i\omega_{n_2})}{i\omega_{n_1}i\omega_{n_3}}+\beta\Delta(i\omega_{n_1}+i\omega_{n_2})\frac{C_{AB,DC}}{i\omega_{n_1}i\omega_{n_4}}\right]
\end{aligned}
\tag{51}
$$

In this expression we have introduced the bosonic "half" Green's function [44] using

$$
\mathfrak{B}_{Y_1,Y_2}(i\omega_\nu) = \int_{-\infty}^{+\infty} d\omega n_-(-\omega)\frac{\rho_{Y_1,Y_2}(\omega)}{i\omega_\nu - \omega}, \tag{52}
$$

where

$$
n_-(\omega) = \frac{1}{e^{\beta\omega}-1}
$$

is the Bose distribution function, $\rho_{Y_1,Y_2}(\omega)$ is bosonic density of states (2), and C_{AB} is an anomalous contribution defined by (7).

The cases of $|i\omega_{n_1} + i\omega_{n_3}| = |i\omega_{n_2} + i\omega_{n_4}| \sim E$ or $|i\omega_{n_1} + i\omega_{n_4}| = |i\omega_{n_2} + i\omega_{n_3}| \sim E$ can be obtained from the above expressions by the appropriate permutation of operators and frequencies.

4.4. $|i\omega_{n_1} + i\omega_{n_2}| \sim E$, $|i\omega_{n_2} + i\omega_{n_3}| \sim E$, $|i\omega_n| \sim \Omega$, $|i\omega_n + i\omega_m| \sim \Omega$, $\Omega \gg E$

The last case which we consider is the case of large frequencies $|i\omega_n| \sim \Omega$ but finite sums $|i\omega_{n_1} + i\omega_{n_2}| = |i\omega_{n_3} + i\omega_{n_4}| \sim E$ and $|i\omega_{n_2} + i\omega_{n_3}| = |i\omega_{n_4} + i\omega_{n_1}| \sim E$ ($\Omega \gg E$). The first $1/\Omega^2$ order terms in the high frequency expansion of (21) using (31) are equal

$$
\begin{aligned}
&\frac{\mathfrak{B}_{CD,AB}(i\omega_{n_3}+i\omega_{n_4})}{i\omega_{n_3}i\omega_{n_2}}+\frac{\mathfrak{B}_{AB,CD}(i\omega_{n_1}+i\omega_{n_2})}{i\omega_{n_1}i\omega_{n_4}}+\beta\Delta(i\omega_{n_1}+i\omega_{n_2})\frac{C_{AB,CD}}{i\omega_{n_1}i\omega_{n_3}}\\
&-\left[\frac{\mathfrak{B}_{DA,BC}(i\omega_{n_4}+i\omega_{n_1})}{i\omega_{n_4}i\omega_{n_3}}+\frac{\mathfrak{B}_{BC,DA}(i\omega_{n_2}+i\omega_{n_3})}{i\omega_{n_2}i\omega_{n_1}}+\beta\Delta(i\omega_{n_2}+i\omega_{n_3})\frac{C_{BC,DA}}{i\omega_{n_2}i\omega_{n_4}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mathfrak{B}_{BA,DC}(i\omega_{n_2}+i\omega_{n_1})}{i\omega_{n_2}i\omega_{n_3}}+\frac{\mathfrak{B}_{DC,BA}(i\omega_{n_4}+i\omega_{n_3})}{i\omega_{n_4}i\omega_{n_1}}+\beta\Delta(i\omega_{n_4}+i\omega_{n_3})\frac{C_{DC,BA}}{i\omega_{n_4}i\omega_{n_2}}\\
&-\left[\frac{\mathfrak{B}_{AD,CB}(i\omega_{n_1}+i\omega_{n_4})}{i\omega_{n_1}i\omega_{n_2}}+\frac{\mathfrak{B}_{CB,AD}(i\omega_{n_3}+i\omega_{n_2})}{i\omega_{n_3}i\omega_{n_4}}+\beta\Delta(i\omega_{n_3}+i\omega_{n_2})\frac{C_{CB,AD}}{i\omega_{n_3}i\omega_{n_1}}\right]\\
&-\left[\frac{\mathfrak{B}_{BA,CD}(i\omega_{n_2}+i\omega_{n_1})}{i\omega_{n_2}i\omega_{n_4}}+\frac{\mathfrak{B}_{CD,BA}(i\omega_{n_3}+i\omega_{n_4})}{i\omega_{n_3}i\omega_{n_1}}+\beta\Delta(i\omega_{n_3}+i\omega_{n_4})\frac{C_{CD,BA}}{i\omega_{n_3}i\omega_{n_2}}\right]\\
&-\left[\frac{\mathfrak{B}_{DC,AB}(i\omega_{n_4}+i\omega_{n_3})}{i\omega_{n_4}i\omega_{n_2}}+\frac{\mathfrak{B}_{AB,DC}(i\omega_{n_1}+i\omega_{n_2})}{i\omega_{n_1}i\omega_{n_3}}+\beta\Delta(i\omega_{n_1}+i\omega_{n_2})\frac{C_{AB,DC}}{i\omega_{n_1}i\omega_{n_4}}\right]\\
&+\frac{\mathfrak{B}_{BC,AD}(i\omega_{n_2}+i\omega_{n_3})}{i\omega_{n_2}i\omega_{n_4}}+\frac{\mathfrak{B}_{AD,BC}(i\omega_{n_1}+i\omega_{n_4})}{i\omega_{n_1}i\omega_{n_3}}+\beta\Delta(i\omega_{n_1}+i\omega_{n_4})\frac{C_{AD,BC}}{i\omega_{n_1}i\omega_{n_2}}\\
&+\frac{\mathfrak{B}_{DA,CB}(i\omega_{n_4}+i\omega_{n_1})}{i\omega_{n_4}i\omega_{n_2}}+\frac{\mathfrak{B}_{CB,DA}(i\omega_{n_3}+i\omega_{n_2})}{i\omega_{n_3}i\omega_{n_1}}+\beta\Delta(i\omega_{n_3}+i\omega_{n_2})\frac{C_{CB,DA}}{i\omega_{n_3}i\omega_{n_4}}.
\end{aligned}
\tag{53}
$$

The other cases can be obtained by the appropriate permutation of operators and frequencies.

The order of magnitude of the terms in high frequency expansion strongly depends on the way how we increase the frequencies:

1. for the general case of $|i\omega_n| \sim \Omega$ and $|i\omega_n + i\omega_m| \sim \Omega$ ($\Omega \gg E$) we have contributions of the order $1/\Omega^4$;

2. for the case when one Matsubara's frequency, e.g., $|i\omega_{n_1}| \sim E$, is finite and all other are large $|i\omega_n| \sim \Omega$ and $|i\omega_n + i\omega_m| \sim \Omega$ we have contributions of the order $1/\Omega^3$;

3. for the case when one or two sums of Matsubara frequencies are finite we have contributions of the order $1/\Omega^2$ with additional spikes when these sums of frequencies are equal to zero.

5. Summary

In conclusion, we have presented a general approach of derivation of the spectral relations for the four-time fermionic Green's functions completed by the consideration of the zero-frequency anomalies. It is known that for the two-time Green's functions such anomalies contribute only in the bosonic functions and does not exist for the fermionic one. Here we have shown that zero-frequency anomalous terms are present in the spectral representations for the multi-time fermionic Green's functions when sum of any two fermionic Matsubara frequencies is equal to zero.

Equation (21) together with (32) and (33) gives spectral representation of the four-time fermionic Matsubara Green's function in terms of the spectral densities (14)–(17). Special consideration of the processes involving states with the same energy values (the same states or true or accidental degeneracy) is required in order to get the correct spectral representations and correct expressions of the anomalous nonergodic contributions, which appear to be connected by Eq. (40) with the second cumulants of the Boltzmann distribution function.

An algorithm of analytic continuations for the solution of reverse engineering problem: extracting of the spectral densities from the known expressions for four-time Matsubara Green's functions is described.

In addition, it is shown that high-frequency expansions for the four-time fermionic Green's functions demonstrate different asymptotic behavior and have different order of magnitude from Ω^{-4} to Ω^{-2} for different directions in the frequency space.

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