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NEW PROOF OF HÖLDER CONTINUITY OF SOLUTIONS TO SECOND ORDER DIFFERENTIAL EQUATIONS

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Нове доведення непервності за Гельдером розв'язків диференціальних рівнянь другого порядку

Д.В. Портнягін

Анотація. Ми пропонуємо нове, нашу думку дещо простіше доведення неперервності за Гелдером розв'язків задач Діріхле та Коші-Діріхле для диференціальних рівнянь другого порядку еліптичного та параболічного типу в дивергентній формі. Умови на праві частини рівняннь було дещо послаблено. Ідея полягає в отриманні $L^\infty-$оцінок для певної функції від розв'язку.

New proof of Hölder continuity of solutions to second order differential equations

D.V. Portnyagin

Abstract. We propose new and, to our mind, a somewhat simpler proof of Hölder continuity of solutions to Dirichlet and Cauchy-Dirichlet problems for second order elliptic and parabolic equations in divergent form. A conditions on the right-hand sides of equations were somewhat lessen. The idea is to obtain $L^\infty-$estimates for a certain function constructed from the solution.

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1. Introduction.

Ennio De Giorgi was the first to obtain the results on Hölder continuity of solutions of the equation (2.1) for $p = 2$ in [2]. Nash and Moser gave an alternative proofs [9], [7]. They had also extended the results to parabolic equations [10], [8]. For elliptic equation and $p > 2$ Ladyzhenskaya and Ural'tseva generalized the results of DeGiorgi [6]. The question of regularity remained open for degenerate ($p > 2$) parabolic case (3.1) until DiBenedetto come up with his idea of intrinsic scaling [3].

In the present paper we propose a new, alternative proof of Hölder continuity of solutions of the equations (2.1) and (3.1). We think that our proof is simpler. Our method allows to estimate not only the Hölder norm, but also the value of the Hölder exponent α.

We have reduced the hypotheses on the right-hand sides to $f \in L^{p/(p-1)}$ ($u \in W^{1,p}$). Usually it is imposed on the boundary of the domain the condition (A), or the condition of a positive geometrical density, which reads:

the boundary $\partial\Omega$ of the domain Ω is said to satisfy condition (A) provided that there are two positive numbers a_0 and θ_0 such that for any ball K_ρ with the origin on $\partial\Omega$ and radius $\rho \leq a_0$ the inequality

$$\text{mes} K_\rho \cap \Omega \leq (1-\theta_0)\text{mes} K_\rho. \tag{A}$$

takes place.

In our approach we don't impose any conditions on the boundary of the domain.

There is no crucial difference in our proof between degenerate and non-degenerate cases for parabolic equation.

Although we restrict ourselves to the model equations in divergent form, our results can be easily generalized to the general equations and equations in non-divergent form. We left it to the reader.

2. Elliptic case.

We shall consider the equation of the form:

$$\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x), \qquad x \in \Omega. \tag{2.1}$$

Ω is a bounded domain in $\mathbb{R}^n$ (here n is any natural number) and $\partial\Omega$ its boundary; $n > p \geq 2$;

$$f(x) \in L^\eta(\Omega), \quad \eta = \frac{p}{p-1}. \tag{2.2}$$

The boundary conditions of the Dirichlet type are assigned:

$$(u-g)(x) \in W_0^{1,p}(\Omega). \tag{2.3}$$

Definition 2.1. *A measurable vector function u is called a weak solution of problem* (2.1)-(2.3) *if*

$$u \in W^{1,p}(\Omega)$$

and

$$\int\limits_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \int\limits_\Omega f\varphi dx$$

for all testing functions

$$\varphi \in W_0^{1,p}(\Omega).$$

The boundary condition in (2.3) *is meant in the weak sense.*

On the function $g(x)$in boundary data (2.3) we assume

$$g(x) \in H^{\alpha_g}(\partial\Omega),$$

where H is a Hölder space.

We also assume that the modulus of the solution $|u|$ is bounded by constant M:

$$|u| \le M \text{ a. e. in } \Omega.$$

We construct the function w:

$$w(x,x') = (u(x)-u(x'))/|x-x'|^\alpha, \quad (x,x') \in \Omega. \tag{2.4}$$

Our proof hinges upon the following self-evident theorem:

Theorem 2.2. *If $w(x) = (u(x)-u(x'))/|x-x'|^\alpha \in L^\infty(\Omega)$ for a. e. $x' \in \Omega$ then $u(x) \in H^\alpha(\Omega)$.*

Instead of working with nested balls we establish the boundedness of function w:

Theorem 2.3. *For the function w as a function of x, defined by* (2.4), *the following estimate is valid*

$$\|w\|_{\infty,\Omega} \le C \qquad \text{for almost all } x' \in \Omega,$$

where were C depends only on the data of the problem, and not on the w and x'.

In order to prove Theorem 2.3 we need the following auxiliary Lemma:

Lemma 2.4. *Let the origin $O \in \partial\Omega$ and w be any function such that $w = 0$ on $\partial\Omega$, α be positive real, then*

$$\int\limits_\Omega |x|^{\alpha p-p}|w|^p \le C(\Omega,p,n)\int\limits_\Omega |x|^{\alpha p}|\nabla w|^p, \tag{2.5}$$

provided the integral on the right exists.

Proof. Let's change to the spherical coordinates $(r,\omega_1,\dots,\omega_{n-1})$, fix r, and use the representation

$$\widetilde{w}(r,\omega_1) = \int\limits_0^\omega \frac{\partial}{\partial\tau}\widetilde{w}(r,\tau)$$

for smooth functions $\widetilde{w}$ approximating w. After raising this equality to the p-th power, applying Hölder inequality, multiplying both sides of it by $r^{\alpha p-p}$, integrating over the domain Ω, and passing to the limit by all such smooth functions, we shall get (2.5). □

Proof of Theorem 2.3. Without loss of generality we may assume that $x' = O$ and $u(x') = 0$.

Step 1. On the first step we shall assume that $O \in \partial\Omega$. Substituting (2.4) into equation (2.1) and using $\operatorname{sgn} w|x|^\alpha(|w|-k_0)_+$, where $k_0 = \sup\limits_{(x,x')\in\partial\Omega} |g(x)-g(x')|/|x-x'|^{\alpha_g} \sup\limits_{(x,x')\in\partial\Omega} |x-x'|^{\alpha_g-\alpha} = (\text{dist } \Omega)^{\alpha_g-\alpha} \sup\limits_{(x,x')\in\partial\Omega} |g(x)-g(x')|/|x-x'|^{\alpha_g}$, $\alpha \le \alpha_g$, as a testing function, after integrating over the domain Ω, and applying Young's inequality, we get:

$$\int\limits_\Omega |x|^{\alpha p}|\nabla w^{k_0}|^p \le \alpha^p C_1(p)\int\limits_\Omega |x|^{\alpha p-p}(w^{k_0})^p +$$
$$+ C_2(p,\alpha)k_0^p\int\limits_\Omega |x|^{(\alpha-1)(p-1)}\chi\{|w|\ge k_0\} + C_3(p,\alpha)\int\limits_\Omega (|f||x|^{1-\alpha})^{\frac{p}{p-1}},$$

where $w^{k_0} = (|w|-k_0)_+$ and $\chi\{|w|\ge k_0\}-$ characteristic function of the set $\{x\in\Omega||w|\ge k_0\}$. After applying Lemma 2.4, choosing $\alpha = \alpha_1$

such that $\alpha_1^p C_1 \le 1/2$, this yields:

$$(k-k_0)^p \text{mes}\ \{w \ge k\}(\text{dist}\ \Omega)^{(\alpha-1)p} \le$$
$$\le \int_\Omega |x|^{\alpha p - p}(w^{k_0})^p \le$$
$$\le C_4(p,\Omega)\int_\Omega |x|^{\alpha p}|\nabla w^{k_0}|^p \le C_5(p,\alpha,\Omega,\|f\|_{\frac{p}{p-1}},k_0), \quad (2.6)$$

where mes $\{w \ge k\}$ is a Lebesgues measure of the set $\{x \in \Omega ||w| \ge k\}$, $k > k_0$. The existence of all integrals can be proven in a conventional way approximating by smooth functions and then passing to the limit. Set

$$k_m = k + d - \frac{d}{2^m}, \qquad m = 1,2,3,\ldots, \quad (2.7)$$

where d is any real > 0, $k > k_0$ is to be determined later. We shall show that

$$\text{mes}\ \{w \ge k_m\} \to 0, \quad \text{as } m \to \infty.$$

We have

$$\text{mes}\ \{w \ge k_{m+1}\} = \frac{\text{mes}\ \{w \ge k_{m+1}\}}{(\text{mes}\ \{w \ge k_m\})^2}(\text{mes}\ \{w \ge k_m\})^2 \le$$
$$\le \frac{\text{mes}\ \Omega}{(\text{mes}\ \{w \ge k_m\})^2}(\text{mes}\ \{w \ge k_m\})^2.$$

There are two alternatives. Either (the first alternative)

$$\frac{\text{mes}\ \Omega}{(\text{mes}\ \{w \ge k_m\})^2} \le m$$

for all $m > m_1$, for some $m_1 < \infty$, i. e. there exists such number $m_1 < \infty$; or (the second alternative) for any m, whatever large it be, there exist $\tilde m > m$, such that

$$\frac{\text{mes}\ \Omega}{\tilde m} > (\text{mes}\ \{w \ge k_{\tilde m}\})^2.$$

In the latter case the subsequence mes $\{w \ge k_{\tilde m}\} \to 0$ as $\tilde m \to \infty$, and it is easy to see that by the well-known theorem form calculus, the sequence being nonincreasing, that the whole sequence mes $\{w \ge k_m\} \to 0$ as $m \to \infty$. In the former case we have that there exists m_1 such that

$$\text{mes}\ \{w \ge k_{m+1}\} \le m(\text{mes}\ \{w \ge k_m\})^2 \quad (2.8)$$

for all $m > m_1$. We shall prove by induction that

$$\text{mes}\ \{w \ge k_m\} \le \frac{\text{mes}\ \Omega}{2^m} \quad (2.9)$$

for $m > \tilde{\tilde m}_1 \ge m_1$ in this case. Really, assume for m. From (2.8) we have

$$\text{mes}\ \{w \ge k_{m+1}\} \le m\left(\frac{\text{mes}\ \Omega}{2^m}\right)^2 = \frac{2m\text{mes}\ \Omega}{2^m}\frac{\text{mes}\ \Omega}{2^{m+1}} \le \frac{\text{mes}\ \Omega}{2^{m+1}},$$

provided that $m > \tilde m_1$, where $\tilde m_1$ is such that $\frac{2\tilde m_1 \text{mes}\ \Omega}{2^{\tilde m_1}} = 1$. It remains to check for unit. From (2.6) we have

$$\text{mes}\ \{w \ge k_{\tilde{\tilde m}_1}\} \le \text{mes}\ \{w \ge k\} \le \frac{C_6(p,\alpha,\Omega,\|f\|_{\frac{p}{p-1}},k_0)}{(k-k_0)^p} \le \frac{\text{mes}\ \Omega}{2^{\tilde{\tilde m}_1}},$$

provided that

$$k \ge k_0 + \left(\frac{2^{\tilde{\tilde m}_1}C_6(p,\alpha,\Omega,\|f\|_{\frac{p}{p-1}},k_0)}{\text{mes}\ \Omega}\right)^{1/p}, \quad (2.10)$$

where $\tilde{\tilde m}_1 = \max[m_1, \tilde m_1]$. So, for $m > \tilde{\tilde m}_1$ we have (2.9). Now we are able to get the following estimate

$$\frac{(\text{mes}\ \Omega)^{1/2}}{\sqrt{m}} \ge \frac{\text{mes}\ \Omega}{2^m} > \text{mes}\ \{w \ge k_{\tilde m}\}$$

for $m > m^*$, where m^* is such that $\frac{(\text{mes}\ \Omega)^{1/2}}{\sqrt{m^*}} = \frac{\text{mes}\ \Omega}{2^{m^*}}$. And it is seen that the first alternative can take place only for $m \le m^*$. Hence we get that m_1 in (2.10) is estimated from above by m^*. Since d is any positive, we can put $d = 0$ in (2.7) to obtain

$$\|w\|_{\infty,\Omega} \le k_0 + \left(\frac{2^{\max[\tilde m_1, m^*]}C_6(p,\alpha,\Omega,\|f\|_{\frac{p}{p-1}},k_0)}{\text{mes}\ \Omega}\right)^{1/p} = C_7, \quad (2.11)$$

where $\tilde m_1$ is such that $\frac{2\tilde m_1 \text{mes}\ \Omega}{2^{\tilde m_1}} = 1$, and m^* is such that $\frac{1}{\sqrt{m^*}} = \frac{(\text{mes}\ \Omega)^{1/2}}{2^{m^*}}$.

Step 2. On this step we assume that $x' \notin \partial\Omega$. We have

$$
\begin{aligned}
&|u(x') - u(x)| = |u(x') - u(x_0) + u(x_0) - u(x)| \le \\
&\le |u(x') - u(x_0)| + |u(x) - u(x_0)| \le C_7|x' - x_0|^\alpha + C_7|x - x_0|^\alpha \le \\
&\le 2C_7 \left(1 + \frac{\text{dist } \Omega}{|x - x'|}\right)^\alpha |x - x'|^\alpha, \quad (2.12)
\end{aligned}
$$

where x_0 is a point on the boundary $\partial\Omega$. Now we act as on the previous step and consider the increasing set of levels

$$k_m = k + d - \frac{d}{2^m}, \qquad m = 1, 2, 3, \ldots, \quad (2.13)$$

where d is any real > 0, k is to be determined later. As on the previous step we consider two alternatives and argue by induction and prove that

$$\text{mes } \{w \ge k_m\} \le \frac{\text{mes } \Omega}{2^m}$$

for $m > \tilde{\tilde{m}}_1 \ge m_1$, and estimate the value of m_1. In order for the statement in the induction argument to obtain on the outset we must impose the condition

$$\text{mes } \{w \ge k_{\max[m^*, \tilde{m}_1]}\} \le \text{mes } \{w \ge k\} \le \text{mes } B(x, x') = \omega(n)|x - x'|^n, \quad (2.14)$$

and

$$\text{mes } B(x, x') = \omega(n)|x - x'|^n \le \frac{\text{mes } \Omega}{2^{\max[m^*, \tilde{m}_1]}}, \quad (2.15)$$

where $B(x, x')$ is a ball centered at x' with radius $|x - x'|$, $\omega(n)$ is a volume of a unit ball in $n-$dimensional space. In order for (2.15) to happen $|x - x'|$ must be

$$|x - x'| \le \left(\frac{\text{mes } \Omega}{2^{\max[m^*, \tilde{m}_1]}\omega(n)}\right)^{1/n}.$$

In order for (2.14) to happen there must be

$$|w|\big|_{\partial B(x, x')} \le 2C_7 \left(1 + \frac{\text{dist } \Omega}{|x - x'|}\right)^\alpha = k.$$

Substituting $|x - x'|$ we get

$$|w(x \in B(x, x'))| \le k = 2C_7 \left(1 + \left(\frac{(\text{dist } \Omega)^n 2^{\max[m^*, \tilde{m}_1]}\omega(n)}{\text{mes } \Omega}\right)^{1/n}\right)^\alpha,$$

where m^* and $\tilde{m}_1$ are as in (2.11). It is seen from (2.12) that for $\tilde{x}$ such that $|\tilde{x} - x'| > |x - x'|$ this estimate is also valid. So we have that

$$\|w\|_{\infty,\Omega} \le k = 2C_7 \left(1 + \left(\frac{(\text{dist } \Omega)^n 2^{\max[m^*, \tilde{m}_1]}\omega(n)}{\text{mes } \Omega}\right)^{1/n}\right)^\alpha.$$

□

3. Parabolic case.

In this section we shall consider the equation of the form:

$$u_t - \text{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x, t), \qquad x \in Q. \quad (3.1)$$

$Q = (0, T] \times \Omega$; $S = \partial\Omega \times (0, T]$; $\partial Q \equiv \{\Omega \times \{0\}\} \bigcup \{\partial\Omega \times (0, T]\}$; Ω is a bounded domain in $\mathbb{R}^n$ (here n is any natural number); $x \in \Omega$; $t \in (0, T]$; $T > 0$; $n > p \ge 2$;

$$f(x, t) \in L^\eta(Q), \quad \eta = \frac{p}{p - 1}. \quad (3.2)$$

The boundary conditions of the Dirichlet type are assigned:

$$\begin{cases} (u - g)(x, t) \in W_0^{1,p}(\Omega) & \text{a.e.} \quad t \in (0, T), \\ (u)(x, 0) = (u_0)(x). \end{cases} \quad (3.3)$$

A solution to equation (3.1) with Dirichlet data (3.3) is understood in the weak sense, as in [3].

Definition 3.1. *A measurable vector function u is called a weak solution of problem* (3.1)-(3.3) *if*

$$u \in C(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$$

and for all $t \in (0, T]$

$$\int_\Omega u\varphi(x, t)dx + \iint_{\Omega\times(0,t]} \{-u\varphi_\tau + |\nabla u|^{p-2}\nabla u \nabla\varphi\}dxd\tau =$$

$$= \int_\Omega u_0\varphi(x, 0)dx + \iint_{\Omega\times(0,t]} f\varphi dxd\tau$$

for all testing functions

$$\varphi \in W^{1,2}(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega)).$$

The boundary condition in (3.3) *is meant in the weak sense.*

On the function $g(x,t)$, $u_0(x)$ in boundary data (3.3) we assume

$$g(x,t) \in H^{\alpha_g,\beta_g}(S), \quad u_0^{\mu}(x) \in H^{\alpha_0}(\overline{\Omega} \times \{0\}),$$

where H is a Hölder space.

We also assume that the modulus of the solution $|u|$ is bounded by constant M:

$$|u| \le M \text{ a. e. in } Q.$$

We construct the function w:

$$w(x,x',t,t') = (u(x,t) - u(x',t'))/(|x-x'|^{\alpha} + |t-t'|^{\beta}), \qquad (x,t),\ (x',t') \in Q. \quad (3.4)$$

Our proof hinges upon the following self-evident theorem:

Theorem 3.2. *If $w(x,t) = (u(x,t) - u(x',t'))/(|x-x'|^{\alpha} + |t-t'|^{\beta}) \in L^{\infty}(Q)$ for a. e. $(x',t') \in Q$ then $u(x,t) \in H^{\alpha,\beta}(Q)$.*

Instead of working with nested cylinders we establish the boundedness of function w:

Theorem 3.3. *For the function w as a function of (x,t), defined by (3.4), the following estimate is valid*

$$\|w\|_{\infty,Q} \le C \qquad \text{for almost all } (x',t') \in Q,$$

where were C depends only on the data of the problem, and not on the w, x' and t'.

In order to prove Theorem 3.3 we need the following auxiliary Lemma:

Lemma 3.4. *Let the origin $O \in \partial\Omega$ and w be any function such that $w = 0$ on $\partial\Omega$, α be positive real, then*

$$\int_0^T \int_{\Omega} |x|^{\alpha-p}|w|^p \le C(\Omega,p,n) \int_0^T \int_{\Omega} |x|^{\alpha}|\nabla w|^p, \qquad (3.5)$$

provided the integral on the right exists.

The proof is analogous to that of Lemma 2.4.

Proof of Theorem 3.3. Without loss of generality we may assume that $x' = O$, $t' = 0$, and $u(x',t') = 0$.

Step 1. On the first step we shall assume that $O \in \partial\Omega \times \{0\}$. Substituting (3.4) into equation (3.1) and using $\operatorname{sgn} w(|x|^{\alpha} + t^{\beta})(|w| - k_0)_+$, where

$$\sup_{(x,t),(x',t')\in S} \frac{|g(x,t) - g(x',t')|}{(|x-x'|^{\alpha_g} + |t-t'|^{\beta_g})} \times$$
$$\times \sup_{(x,t),(x',t')\in S} \frac{(|x-x'|^{\alpha_g} + |t-t'|^{\beta_g})}{(|x-x'|^{\alpha} + |t-t'|^{\beta})} +$$
$$+ \sup_{(x,x')\in\Omega} \frac{|u_0(x) - u_0(x')|}{|x-x'|^{\alpha_0}} \cdot \sup_{(x,x')\in\Omega} \frac{|x-x'|^{\alpha_0}}{|x-x'|^{\alpha}} \le$$
$$\le \sup_{(x,t),(x',t')\in S} \frac{|g(x,t) - g(x',t')|}{(|x-x'|^{\alpha_g} + |t-t'|^{\beta_g})} \cdot ((\text{dist } \Omega)^{\alpha_g - \alpha} + T^{\beta_g - \beta}) +$$
$$+ \sup_{(x,x')\in\Omega} \frac{|u_0(x) - u_0(x')|}{|x-x'|^{\alpha_0}} \cdot (\text{dist } \Omega)^{\alpha_0 - \alpha} = k_0,$$

$\alpha \le \min[\alpha_g, \alpha_0]$, $\beta \le \beta_g$, as a testing function, after integrating over the domain $\Omega \times (0,T]$, and applying Young's inequality, we get:

$$\int_{\Omega} (|x|^{\alpha} + t^{\beta})^2 (w^{k_0})^2 \Big|_{t=T} + \beta k_0 \int_Q t^{\beta-1}(|x|^{\alpha} + t^{\beta}) w^{k_0} +$$
$$+ \int_Q (|x|^{\alpha} + t^{\beta})^p |\nabla w^{k_0}|^p \le \alpha^p C_1(p) \int_Q |x|^{\alpha p - p}(w^{k_0})^p +$$
$$+ C_2(p,\alpha) k_0^p \int_Q |x|^{(\alpha-1)(p-1)} \chi\{|w| \ge k_0\} + C_3(p,\alpha) \int_Q (|f||x|^{1-\alpha})^{\frac{p}{p-1}},$$

where $w^{k_0} = (|w| - k_0)_+$ and $\chi\{|w| \ge k_0\}-$ characteristic function of the set $\{(x,t) \in Q | |w| \ge k_0\}$. After applying Lemma 3.4, choosing $\alpha = \alpha_1$ such that $\alpha_1^p C_1 \le 1/2$, this yields:

$$(k - k_0)^p \text{mes } \{w \ge k\} (\text{dist } \Omega)^{(\alpha-1)p} \le$$
$$\le \int_Q |x|^{\alpha p - p}(w^{k_0})^p \le$$
$$\le C_4(p,\Omega) \int_Q |x|^{\alpha p} |\nabla w^{k_0}|^p \le C_5(p,\alpha,\Omega,T,\|f\|_{\frac{p}{p-1}},k_0),$$

where mes $\{w \geq k\}$ is a Lebesgues measure of the set $\{(x,t) \in Q||w| \geq k\}$, $k > k_0$. The existence of all integrals can be proven in a conventional way approximating by smooth functions and then passing to the limit. As in the previous section set

$$k_m = k + d - \frac{d}{2^m}, \qquad m = 1, 2, 3, \ldots,$$

where d is any real > 0, $k > k_0$ is to be determined later. As in the previous section considering two alternatives and arguing by induction we can prove that

$$\|w\|_{\infty,Q} \leq k_0 + \left(\frac{2^{\max[\tilde{m}_1, m^*]} C_6(p, \alpha, \Omega, \|f\|_{\frac{p}{p-1}}, k_0)}{\text{mes } \Omega} \right)^{1/p} = C_7, \quad (3.6)$$

where $\tilde{m}_1$ is such that $\frac{2\tilde{m}_1 \text{mes } Q}{2^{\tilde{m}_1}} = 1$, and m^* is such that $\frac{1}{\sqrt{m^*}} = \frac{(\text{mes } Q)^{1/2}}{2^{m^*}}$.

Step 2. On this step we assume that $(x', t') \notin \partial\Omega \times \{0\}$. As in the previous section we have

$$\begin{aligned}
|u(x',t') - u(x,t)| &= |u(x',t') - u(x_0,0) + u(x_0,0) - u(x,t)| \leq \\
&\leq |u(x',t') - u(x_0,0)| + |u(x,t) - u(x_0,0)| \leq \\
&\leq C_7(|x' - x_0|^\alpha + |t'|^\beta) + C_7(|x - x_0|^\alpha + |t|^\beta) \leq \\
&\leq 2C_7 \left(1 + \frac{(\text{dist } \Omega)^\alpha + T^\beta}{(|x - x'|^\alpha + |t - t'|^\beta)}\right) (|x - x'|^\alpha + |t - t'|^\beta) \leq \\
&\leq 2C_7 \left(1 + \frac{n^{1/(n+1)}[(\text{dist } \Omega)^\alpha + T^\beta]}{(n+1)^{1/(n+1)}(|x - x'|^{\alpha n}|t - t'|^\beta)^{1/(n+1)}}\right) \times \\
&\times (|x - x'|^\alpha + |t - t'|^\beta) \leq \\
&\leq C_8 \left(1 + \frac{1}{(|x - x'|^n |t - t'|)^{\max[\alpha,\beta]/(n+1)}}\right) (|x - x'|^\alpha + |t - t'|^\beta),
\end{aligned} \quad (3.7)$$

where x_0 is a point on the boundary $\partial\Omega \times \{0\}$. In much the same way as in the elliptic case we obtain

$$\begin{aligned}
\text{mes } \{w \geq k_{\max[m^*, \tilde{m}_1]}\} &\leq \text{mes } \{w \geq k\} \leq \\
&\leq \text{mes } Cy(x, x', t, t') = 2\omega(n)|x - x'|^n |t - t'|, \quad (3.8)
\end{aligned}$$

and

$$\text{mes } Cy(x, x', t, t') = 2\omega(n)|x - x'|^n |t - t'| \leq \frac{\text{mes } Q}{2^{\max[m^*, \tilde{m}_1]}}, \quad (3.9)$$

where $Cy(x, x', t, t')$ is a cylinder centered at (x', t') with radius $|x - x'|$ and hight $2|t - t'|$, $\omega(n)$ is a volume of a unit ball in $n-$dimensional space. In order for (3.9) to happen $|x - x'|^n |t - t'|$ must be

$$|x - x'|^n |t - t'| \leq \frac{\text{mes } Q}{2^{\max[m^*, \tilde{m}_1]} 2\omega(n)}.$$

In order for (3.8) to happen there must be

$$|w|\big|_{\partial Cy(x,x',t,t')} \leq C_8 \left(1 + \frac{1}{(|x - x'|^n |t - t'|)^{\max[\alpha,\beta]/(n+1)}}\right) = k,$$

where $\partial Cy(x, x', t, t')$ is a surface of the cylinder $Cy(x, x', t, t')$, i. e. lateral area with bases. Substituting here $|x - x'|^n |t - t'|$ we get

$$|w(x \in Cy(x, x', t, t'))| \leq k = C_8 \left(1 + \frac{(2^{\max[m^*, \tilde{m}_1]} 2\omega(n))^{\max[\alpha,\beta]/(n+1)}}{(\text{mes } Q)^{\max[\alpha,\beta]/(n+1)}}\right),$$

where m^* and $\tilde{m}_1$ are as in (3.6). It is seen from (3.7) that for $\tilde{x}$ such that $|\tilde{x} - x'| > |x - x'|$ and $\tilde{t}$ such that $|\tilde{t} - t'| > |t - t'|$ this estimate is also valid. And so we have that

$$\|w\|_{\infty,Q} \leq k = C_8 \left(1 + \frac{(2^{\max[m^*, \tilde{m}_1]} 2\omega(n))^{\max[\alpha,\beta]/(n+1)}}{(\text{mes } Q)^{\max[\alpha,\beta]/(n+1)}}\right).$$

□

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