Національна академія наук України

ІНСТИТУТ ФІЗИКИ КОНДЕНСОВАНИХ СИСТЕМ

ICMP–07–06E

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RADIATION REACTION
IN 2+1 ELECTRODYNAMICS

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Реакція випромінювання у 2 + 1-електродинаміці

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Реакція випромінювання у $2+1$-електродинаміці

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Анотація. Шукаємо реакцію випромінювання прискореного заряду у тривимірному просторі Мінковського. Обчислюємо енергію, імпульс та момент кількості руху електромагнетного поля точкового заряду, які розбиваємо на причастинкову та радіаційну частини. В рамках процедури перенормування причастинкові доданки абсорбуються імпульсом та моментом імпульсу "голої" частики. Радіаційні доданки, разом з перенормованими індивідуальними характеристиками частинки, утворюють інтеграли руху системи. Аналізуючи їхні похідні, отримуємо ефективне інтегро-диференційне рівняння руху точкового заряду у зовнішньому електромагнетному полі.

Radiation reaction in 2+1 electrodynamics

Yu.Yaremko

Abstract. We discuss the radiation reaction problem for an electric charge moving in flat space-time of three dimensions. We calculate energy-momentum and angular momentum carried by electromagnetic field of the accelerating point-like charge. We decompose them into bound and radiative components. The bound terms are absorbed by individual particle's characteristics within the renormalization procedure. Radiative terms together with already renormalized three-momentum and angular momentum of point-like charge constitute the total conserved quantities of our particle plus field system. Their differential consequences yield the effective (integro-differential) equation of motion of radiating charge in an external electromagnetic field.

1. Introduction

Recently [1,2], there has been considerable interest in renormalization procedure in classical electrodynamics of a point particle moving in flat space-time of arbitrary dimensions. The main task is to derive the analogue of the well-known Lorentz-Dirac equation [3]. The Lorentz-Dirac equation is an equation of motion for a charged particle under the influence of an external force as well as its own electromagnetic field. (For a modern review see [4–6].)

A special attention in [1,2] is devoted to the mass renormalization in $2 + 1$ theory. (Note that electrodynamics in Minkowski space $\mathbb{M}_3$ is quite different from the conventional $3 + 1$ electrodynamics where one space dimension is reduced because of symmetry of specific problem. For example, small charged balls on a plane are interacted inversely with the square of the distance between them, while in $\mathbb{M}_3$ the Coulomb field of a small static charged disk scales as $|\mathbf{r}|^{-1}$.) An essential feature of $2 + 1$ electrodynamics is that Huygens principle does not hold and radiation develops a tail, as it is in curved space-time of four dimensions [7] where electromagnetic waves propagate not just at the speed of light, but all speeds smaller than or equal to it.

In Refs. [1,2] the self-force on a point-like particle is calculated from the local fields in the immediate vicinity of its trajectory. The schemes involve some prescriptions for subtracting away the infinite contributions to the force due to the singular nature of the field on the particle's world line. In [2] the procedure of regularization is based on the methods of functional analysis which are applied to the Tailor expansion of the retarded Green's function. The authors derive the covariant analogue of the Lorentz-Dirac equation which is something other than that obtained in [1]. Both the divergent self-energy absorbed by "bare" mass of point-like charge, and the radiative term which leads an independent existence, are non-local. (They depend not only on the current state of motion of the particle, but also on its past history.)

In this paper we develop a consistent regularization procedure which exploits the symmetry properties of $2 + 1$ electrodynamics. It can be summarized as a simple rule which obeys the spirit of Dirac scheme of decomposition of vector potential of a point-like charge.

According to the scheme proposed by Dirac in his classical paper [3], one can decompose retarded Green's function associated with four-dimensional Maxwell field equation $G^{ret}(x,z) = G^{sym}(x,z) + G^{rad}(x,z)$. The first term, $G^{sym}(x,z)$, is one-half sum of the retarded and the advanced Green's functions; it is just singular as $G^{ret}(x,z)$. The second

one, $G^{rad}(x,z)$, is one-half of the retarded minus one-half of the advanced Green's functions; it satisfies the homogeneous wave equation. Convolving the source with the Green's functions $G^{sym}(x,z)$ and $G^{rad}(x,z)$ yield the singular and the radiative parts of vector potential of a point-like charge, respectively.

The analogous decomposition of Green's function in curved spacetime is much more delicate because of richer causal structure. Detweiler and Whiting [9] modify the singular Green's function by means of two-point function $v(x,z)$ which is symmetric in its arguments. It is constructed from the solutions of the homogeneous wave equation in such a way that a new symmetric Green's function $G^S(x,z) = G^{sym}(x,z) + 1/(8\pi)v(x,z)$ has no support within the null cone.

It is obvious that the physically relevant solution to the wave equation is the retarded solution. In [10] the Lorentz-Dirac equation is derived within the framework of retarded causality. Teitelboim substitutes the retarded Liénard-Wiechert fields in the electromagnetic field's stress-energy tensor and computes the flow of energy-momentum which flows across a tilted hyperplane which is orthogonal to particle's four-velocity at instant of observation. The effective equation of motion is obtained in [10] via consideration of energy-momentum conservation. Similarly, López and Villarroel [11] find out the total angular momentum carried by electromagnetic field of a point-like charge. Outgoing waves carry energy-momentum and angular momentum; the radiation removes energy and angular momentum from the source which then undergoes a radiation reaction. It is shown [12] that the Lorentz-Dirac equation can be derived from the energy-momentum and angular momentum balance equations. In ref. [13] the analogue of the Lorentz-Dirac equation in six dimensions is obtained via analysis of 21 conserved quantities which correspond to the symmetry of an isolated point particle coupled with electromagnetic field. (Firstly this equation was obtained by Kosyakov in [14] via the consideration of energy-momentum conservation. An alternative derivation was produced by Kazinski, Lyakhovich and Sharapov in [2].)

In non-local theories the computation of Noether quantities is highly nontrivial. Quinn and Wald [15] study the energy-momentum conservation for point charge moving in curved spacetime. The Stokes' theorem is applied to the integral of flux of electromagnetic energy over the compact region $V(t^+, t^-)$. It is expanded to the limits $t^- \to -\infty$ and $t^+ \to +\infty$, so that finally the boundary of the integration domain involves smooth spacelike hypersurfaces at the remote past and in the distant future . The spacetime is asymptotically flat here. The authors prove that the net en-

ergy radiated to infinity is equal to the total work done on the particle by the electromagnetic self-force. (DeWitt-Brehme [7,8] radiation-reaction force is meant.) It is shown also [15] that the total work done by the gravitational self-force is equal to the energy radiated (in gravitational waves) by the particle. (The effective equation of motion of a point mass undergoing radiation reaction is obtained in [16]; see also review [18] where the motion of a point electric charge, a point scalar charge, and a point mass in curved spacetime is considered in details.)

In [17] Quinn derives the effective equation of motion of a point particle coupled with scalar field moving in curved spacetime. The author establishes that the total work done by the scalar self-force matches the amount of energy radiated away by the particle.

In the present paper we calculate the total flows of energy-momentum and angular momentum of the retarded field which flow across a hyperplane $\Sigma_t = \{y \in \mathbb{M}_3 : y^0 = t\}$ associated with unmoving observer. The field is generated by a point-like charge arbitrarily moving in flat Minkowski space $\mathbb{M}_3$ of three dimensions. This paper is organized as follows. In Section 2 we recall the retarded and the advanced Green's functions associated with three-dimensional D'Alembert operator. Convolving them with the point source, we derive the retarded and the advanced vector potential and field strengths. In Section 4 we trace a series of stages in calculation of surface integral which gives the energy-momentum carried by the retarded electromagnetic field. (Details are given in Appendices.) We introduce appropriate coordinate system centered on an accelerated world line and we express the components of the Maxwell energy-momentum tensor density in terms of these curvilinear coordinates. We integrate it over the variables which parametrize the surface of integration Σ_t. Resulting expression becomes a combination of two-point functions depending on the state of particle's motion at instants t_1 and t_2 before the observation instant t. They are integrated over particle's world line twice. We arrange them in Section 3. We split the momentum three-vector carried by electromagnetic field into singular and radiative parts by means of Dirac scheme which deals with fields defined on the world line only. All diverging quantities have disappeared into the procedure of mass renormalization while radiative terms lead independent existence. In analogous way we analyze the angular momentum of electromagnetic field. Total energy-momentum and total angular momentum of our particle plus field system depend on already renormalized particle's individual characteristics and radiative parts of "electromagnetic" Noether quantities. Having differentiated the conserved quantities we derive the effective equation of motion of radi-

ating charge. In Section 5 we discuss the result and its implications.

2. Electromagnetic potential and electromagnetic field in 2+1 theory

We consider an electromagnetic field produced by a particle with δ-shaped distribution of the electric charge e moving on a world line $\zeta \subset \mathbb{M}_3$ described by functions $z^\alpha(\tau)$ of proper time τ. The Maxwell equations

$$F^{\alpha\beta}{}_{,\beta} = 2\pi j^\alpha \qquad (2.1)$$

where current density j^α is given by

$$j^\alpha = e \int_{-\infty}^{+\infty} d\tau u^\alpha(\tau)\delta^{(3)}(y - z(\tau)) \qquad (2.2)$$

governs the propagation of the electromagnetic field. $u^\alpha(\tau)$ denotes the (normalized) three-velocity vector $dz^\alpha(\tau)/d\tau$ and $\delta^{(3)}(y - z) = \delta(y^0 - z^0)\delta(y^1 - z^1)\delta(y^2 - z^2)$ is a three-dimensional Dirac distribution supported on the particle's world line ζ. Both the strength tensor $F^{\alpha\beta}$ and the current density j^α are evaluated at a field point $y \in \mathbb{M}_3$. (We choose Minkowski metric tensor $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1)$ which we shall use to raise and lower indices. Greek indices run from 0 to 3, and Latin indices from 1 to 2; summation over repeated indices understood throughout the paper.)

We express the electromagnetic field in terms of a vector potential, $\hat{F} = d\hat{A}$. We impose the Lorentz gauge $A^\alpha{}_{,\alpha} = 0$; then the Maxwell field equation (2.1) becomes

$$\Box A^\alpha = -2\pi j^\alpha. \qquad (2.3)$$

In *2+1 theory* the retarded Green's function associated with the D'Alembert operator $\Box := \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ has the form [1,2]

$$G_{2+1}^{ret}(y, x) = \frac{\theta(y^0 - x^0 - |\mathbf{y} - \mathbf{x}|)}{\sqrt{-2\sigma(y, x)}}. \qquad (2.4)$$

$\theta(y^0 - x^0 - |\mathbf{y} - \mathbf{x}|)$ is step function defined to be one if $y^0 - x^0 \geq |\mathbf{y} - \mathbf{x}|$, and defined to be zero otherwise. Synge's world function $\sigma(y, x)$ is numerically equal to half the squared distance between y and x:

$$\sigma(y, x) = \frac{1}{2}\eta_{\alpha\beta}(y^\alpha - x^\alpha)(y^\beta - x^\beta). \qquad (2.5)$$

The first is y, to which we refer the "field point", while the second is x, to which we refer the "emission point".

(We choose Minkowski metric tensor $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1)$ which we shall use to raise and lower indices. Greek indices run from 0 to 3, and Latin indices from 1 to 2; summation over repeated indices understood throughout the paper.)

While in four-dimensional Minkowski spacetime the retarded Green's function has support on the future light cone of the emission point x, in $2 + 1$ electrodynamics its support extends inside the light cone as well.

Using the retarded Green function (2.4) and the charge-current density (2.2) we construct the retarded Liénard-Wiechert potential in three dimensions. Denoting $K^\mu = y^\mu - z^\mu(\tau)$ the unique timelike (or null) geodesic connecting a field point y to the emission point $z(\tau) \in \zeta$, we arrive at

$$A_\mu^{ret}(y) = e \int_{-\infty}^{+\infty} d\tau\theta(K^0 - |\mathbf{K}|)\frac{u_\mu(\tau)}{\sqrt{-(K \cdot K)}} \qquad (2.6)$$

where "dot" denotes the scalar product of three-vector K on itself (it is equal to double Synge's function of field point y and emission point $z(\tau)$).

We now turn to the calculation of electromagnetic field $F_{\mu\nu}^{ret} = \partial_\mu A_\nu^{ret} - \partial_\nu A_\mu^{ret}$ generated by an arbitrarily moving point-like charge. It consists of two quite different terms. The first term is due to differentiation of θ-function involved in the vector potential (2.6):

$$F_{\mu\nu}^{(\delta)} = \lim_{\tau \to \tau^{ret}} \frac{e}{\sqrt{-(K \cdot K)}}\frac{u_\mu K_\nu - u_\nu K_\mu}{-(K \cdot u)}. \qquad (2.7)$$

$\tau^{ret}(y)$ denotes the proper-time parameter at point on the world line which links with y by the unique future-directed null geodesic. Since $\tau^{ret}(y)$ is the root of algebraic equation $K^0 - |\mathbf{K}| = 0$ the δ-term (2.7) diverges.

The second term is

$$F_{\mu\nu}^{(\theta)} = -e \int_{-\infty}^{+\infty} d\tau\theta(K^0 - |\mathbf{K}|)\frac{u_\mu K_\nu - u_\nu K_\mu}{[-(K \cdot K)]^{3/2}}$$
$$= -e \int_{-\infty}^{\tau^{ret}(y)} d\tau\frac{u_\mu K_\nu - u_\nu K_\mu}{[-(K \cdot K)]^{3/2}}. \qquad (2.8)$$

We see that the strength tensor $F_{\mu\nu}^{ret}$ of the adjunct electromagnetic field consists of terms proportional to $\delta-$ and $\theta-$functions: $\hat{F}^{ret} = \hat{F}^{(\delta)} + \hat{F}^{(\theta)}$.
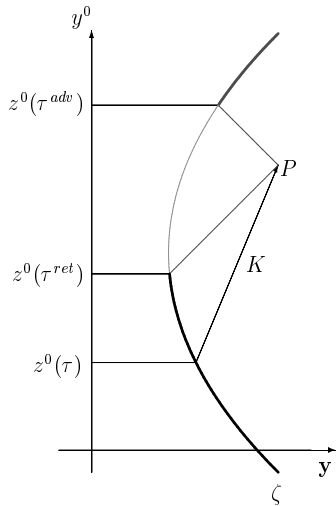
Figure 1. In four dimensions the retarded (advanced) field at observation point $P(y)$ is generated by a single event in spacetime: the intersection of the world line and P's past (future) light cone. In three dimensions the retarded field depends also on the particle's history before $\tau^{ret}(y)$. The advanced field depends on the particle's history after $\tau^{adv}(y)$. The vector K is a vector pointing from the emission point $z(\tau) \in \zeta$ to field point P.

The terms separately are singular. But the singularity, however, can be removed from the sum of $\hat{F}^{(\delta)}$ and $\hat{F}^{(\theta)}$. Using the identity

$$\frac{1}{[-(K \cdot K)]^{3/2}} = \frac{1}{-(K \cdot u)} \frac{d}{d\tau} \frac{1}{\sqrt{-(K \cdot K)}} \tag{2.9}$$

in eq.(2.8) yields

$$F^{(\theta)}_{\mu\nu} = -\frac{e}{\sqrt{-(K \cdot K)}} \frac{u_\mu K_\nu - u_\nu K_\mu}{-(K \cdot u)} \bigg|^{\tau \to \tau^{ret}(y)}_{\tau \to -\infty}$$
$$+ e \int_{-\infty}^{\tau^{ret}(y)} \frac{d\tau}{\sqrt{-(K \cdot K)}} \left\{ \frac{a_\mu K_\nu - a_\nu K_\mu}{-(K \cdot u)} \right. \tag{2.10}$$
$$+ \left. \frac{u_\mu K_\nu - u_\nu K_\mu}{[-(K \cdot u)]^2} [1 + (K \cdot a)] \right\}$$

after integration by parts. Summing up (2.7) and (2.10) and taking into

account that $1/\sqrt{-(K \cdot K)}$ vanishes whenever $\tau \to -\infty$[1], we finally obtain the expression

$$\hat{F}^{ret}(y) = e \int_{-\infty}^{\tau^{ret}(y)} \frac{d\tau}{\sqrt{-(K \cdot K)}} \left\{ \frac{a \wedge K}{r} + \frac{u \wedge K}{r^2} [1 + (K \cdot a)] \right\} \tag{2.11}$$

which is regular on the light cone. (It diverges on the particle's trajectory only.) Symbol $\wedge$ denotes the wedge product. The invariant quantity

$$r = -(K \cdot u) \tag{2.12}$$
$$= -\eta_{\alpha\beta}(y^\alpha - z^\alpha(\tau))u^\beta(\tau)$$

is an affine parameter on the time-like (null) geodesic that links y to $z(\tau)$; it can be loosely interpreted as the time delay between y and $z(\tau)$ as measured by an observer moving with the particle. When $\tau = \tau^{ret}(y)$ parameter r is also the spatial distance between $z(\tau^{ret})$ and y as measured in this momentarily comoving Lorentz frame.

In three dimensions the advanced Green's function

$$G^{adv}_{2+1}(y, x) = \frac{\theta(-y^0 + x^0 - |\mathbf{y} - \mathbf{x}|)}{\sqrt{-2\sigma(y, x)}} \tag{2.13}$$

is nonzero in the past of x. The advanced strength tensor

$$\hat{F}^{adv}(y) = e \int_{\tau^{adv}(y)}^{+\infty} \frac{d\tau}{\sqrt{-(K \cdot K)}} \left\{ \frac{a \wedge K}{r} + \frac{u \wedge K}{r^2} [1 + (K \cdot a)] \right\} \tag{2.14}$$

is generated by the point charge during its entire future history following the advanced time associated with y (see figure 1).

3. Equation of motion of radiating charge

In this section we derive the "three-dimensional" analogue of the Lorentz-Dirac equation via analysis of the energy-momentum and angular momentum balance equations. The momentum three-vector carried by electromagnetic field is calculated in the next Section and in Appendices D, E, and F; the total angular momentum is obtained in Appendix G. We split the Noether quantities into bound (singular) and radiative (regular) parts. Energy-momentum and angular momentum of

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[1] We assume that average velocities are not large enough to initiate particle creation and annihilation, so that "space contribution" $|\mathbf{K}|$ can not match with an extremely large zeroth component K^0.

"bare" sources absorb the bound terms within regularization procedure. Already renormalized characteristics of charged particles are proclaimed to be finite. Together with radiative terms, they constitute the total energy-momentum and angular momentum of our particle plus field system which are properly conserved.

To find out electromagnetic field's energy-momentum we integrate the Maxwell energy-momentum tensor density over the plane $\Sigma_t = \{y \in \mathbb{M}_3 : y^0 = t\}$. The resulting expressions (4.3.6) and (4.3.7), presented in the next Section, can be rewritten in manifestly covariant fashion:

$$p_{\text{em}}^\mu(\tau) = \frac{e^2}{2} \int_{-\infty}^\tau d\tau_2 \frac{u^\mu(\tau_2)}{\sqrt{2\sigma(\tau, \tau_2)}} \tag{3.1}$$

$$+ \left.\begin{array}{c} e^2 \int_{-\infty}^\tau d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \\ e^2 \int_{-\infty}^\tau d\tau_2 \int_{\tau_2}^\tau d\tau_1 \end{array}\right\} \left[-\frac{\partial^2 \sigma}{\partial\tau_1 \partial\tau_2} \frac{q^\mu}{(2\sigma)^{3/2}} \right.$$

$$+ \left. \frac{1}{2}\frac{\partial}{\partial\tau_1}\left(\frac{u_2^\mu}{\sqrt{2\sigma}}\right) - \frac{1}{2}\frac{\partial}{\partial\tau_2}\left(\frac{u_1^\mu}{\sqrt{2\sigma}}\right) \right].$$

(We omit structureless terms which arise due to choice of non-covariant surface of integration.) Index 1 indicates that particle's velocity or position is referred to the instant $\tau_1 \in]-\infty, \tau]$ while index 2 says that the particle's characteristics are evaluated at instant $\tau_2 \leq \tau_1$. Here $q^\mu = z_1^\mu - z_2^\mu$ defines the unique timelike geodesic connecting a field point $z(\tau_1) \in \zeta$ to an emission point $z(\tau_2) \in \zeta$; by σ we mean the Synge's world function (2.5) of z_1 and z_2, taken with opposite sign:

$$\sigma(\tau_1, \tau_2) = -\frac{1}{2}(q \cdot q). \tag{3.2}$$

Two double integrals over (proper) time variables (one about the other) describe integration over the domain $D_\tau = \{(\tau_1, \tau_2) \in \mathbb{R}^2 : \tau_1 \in]-\infty, \tau], \tau_2 \leq \tau_1\}$.

We have to decompose the expression eq.(3.1) into singular and regular parts. Following ref. [2], we postulate that splitting should satisfy the conditions:

- proper non-accelerating limit of singular and regular parts;

- proper short-distance behavior of regular part;

- Poincaré invariance and reparametrization invariance.

By "proper short-distance behavior" we mean the finiteness of integrand at the edge $\tau_2 = \tau_1$ of the domain D_τ.

So, we take one-half of the first term in between the square brackets under the double integral signs in eq.(3.1):

$$-\frac{1}{2}\frac{\partial^2\sigma}{\partial\tau_1\partial\tau_2}\frac{q^\mu}{(2\sigma)^{3/2}} = -\frac{1}{2}\frac{(u_1 \cdot u_2)q^\mu}{[-(q \cdot q)]^{3/2}} \tag{3.3}$$

and add the second term

$$\frac{1}{2}\frac{\partial}{\partial\tau_1}\left(\frac{u_2^\mu}{\sqrt{2\sigma}}\right) = \frac{1}{2}\frac{(u_1 \cdot q)u_2^\mu}{[-(q \cdot q)]^{3/2}}. \tag{3.4}$$

It is convenient to rewrite the resulting expression as follows:

$$p_{12}^\mu = \frac{1}{2}u_{1,\alpha}\frac{-u_2^\alpha q^\mu + u_2^\mu q^\alpha}{[-(q \cdot q)]^{3/2}}. \tag{3.5}$$

We introduce the function

$$G_{\text{ret}}^\mu(\tau_1) = e^2 u_{1,\alpha} \int_{-\infty}^{\tau_1} d\tau_2 \frac{-u_2^\alpha q^\mu + u_2^\mu q^\alpha}{[-(q \cdot q)]^{3/2}} \tag{3.6}$$

which is intimately connected with the retarded field (2.11) generated at point $z(\tau_1) \in \zeta$ by the portion of the world line that corresponds to the interval $-\infty < \tau_2 \leq \tau_1$:

$$G_{\text{ret}}^\mu(\tau_1) = \frac{e^2}{2}a_1^\mu - e u_{1,\alpha}F_{\text{ret}}^{\mu\alpha}(\tau_1). \tag{3.7}$$

(It may be checked via integration by parts.)

Next we take the remaining one-half of the first term and add the third term:

$$-\frac{1}{2}\frac{(u_1 \cdot u_2)q^\mu}{[-(q \cdot q)]^{3/2}} + \frac{1}{2}\frac{(u_2 \cdot q)u_1^\mu}{[-(q \cdot q)]^{3/2}} = \frac{1}{2}u_{2,\alpha}\frac{-u_1^\alpha q^\mu + u_1^\mu q^\alpha}{[-(q \cdot q)]^{3/2}}. \tag{3.8}$$

We change the order of integration of this term over D_τ:

$$\frac{e^2}{2}\int_{-\infty}^\tau d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 u_{2,\alpha}\frac{-u_1^\alpha q^\mu + u_1^\mu q^\alpha}{[-(q \cdot q)]^{3/2}} \tag{3.9}$$

$$= \frac{e^2}{2}\int_{-\infty}^\tau d\tau_2 u_{2,\alpha} \int_{\tau_2}^\tau d\tau_1 \frac{-u_1^\alpha q^\mu + u_1^\mu q^\alpha}{[-(q \cdot q)]^{3/2}}$$

and interchange indices "first" and "second". Taking into account that $q(\tau_2, \tau_1) = -q(\tau_1, \tau_2)$, we finally obtain:

$$-\frac{e^2}{2} \int_{-\infty}^{\tau} d\tau_1 u_{1,\alpha} \int_{\tau_1}^{\tau} d\tau_2 \frac{-u_2^\alpha q^\mu + u_2^\mu q^\alpha}{[-(q \cdot q)]^{3/2}}. \qquad (3.10)$$

The integrand coincides with that under integral sign in the right-hand side of eq.(3.6) while the domain of inner integration corresponds to the interval $\tau_1 \leq \tau_2 \leq \tau$. We introduce the function

$$G_{\mathrm{adv}}^\mu(\tau_1, \tau) = e^2 u_{1,\alpha} \int_{\tau_1}^{\tau} d\tau_2 \frac{-u_2^\alpha q^\mu + u_2^\mu q^\alpha}{[-(q \cdot q)]^{3/2}}. \qquad (3.11)$$

If the instant of observation τ tends to $+\infty$, this function can be rewritten as

$$G_{\mathrm{adv}}^\mu(\tau_1) = \frac{e^2}{2} a_1^\mu - e u_{1,\alpha} F_{\mathrm{adv}}^{\mu\alpha}(\tau_1). \qquad (3.12)$$

The second term is convolution of velocity with the advanced field (2.14) generated at point $z(\tau_1) \in \zeta$ by the portion of the world line that corresponds to the interval $[\tau_1, +\infty[$. The relations (3.7) and (3.12) are symmetric upon future and past.

We see that the double integral in eq.(3.1) can be expressed as one-half of G_{ret} minus one-half of G_{adv} integrated over the world line ζ. For this reason we proclaim the expression

$$\begin{aligned} p_{\mathrm{R}}^\mu(\tau) &= \frac{1}{2} \int_{-\infty}^{\tau} d\tau_1 \left[G_{\mathrm{ret}}^\mu(\tau_1) - G_{\mathrm{adv}}^\mu(\tau_1, \tau) \right] \qquad (3.13) \\ &= e^2 \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \left[-\frac{(u_1 \cdot u_2) q^\mu}{[-(q \cdot q)]^{3/2}} + \frac{1}{2} \frac{(u_1 \cdot q) u_2^\mu}{[-(q \cdot q)]^{3/2}} \right. \\ &\quad \left. + \frac{1}{2} \frac{(u_2 \cdot q) u_1^\mu}{[-(q \cdot q)]^{3/2}} \right] \end{aligned}$$

the radiative part of energy-momentum carried by electromagnetic field. The situation is pictured in figure 2.

We evaluate the short-distance behavior of the expression under the double integral in eq.(3.13). Let τ_1 be fixed and $\tau_1 - \tau_2 := \Delta$ be a small parameter. With a degree of accuracy sufficient for our purposes

$$\begin{aligned} \sqrt{-(q \cdot q)} &= \Delta \qquad (3.14) \\ q^\mu &= \Delta \left[u_1^\mu - a_1^\mu \frac{\Delta}{2} + \dot{a}_1^\mu \frac{\Delta^2}{6} \right] \\ u_2^\mu &= u_1^\mu - a_1^\mu \Delta + \dot{a}_1^\mu \frac{\Delta^2}{2}. \end{aligned}$$

Substituting these into integrand of the double integral of eq.(3.13) and passing to the limit $\Delta \to 0$ yields regular expression

$$\lim_{\tau_2 \to \tau_1} \left[\frac{1}{2} u_{1,\alpha} \frac{-u_2^\alpha q^\mu + u_2^\mu q^\alpha}{[-(q \cdot q)]^{3/2}} + \frac{1}{2} u_{2,\alpha} \frac{-u_1^\alpha q^\mu + u_1^\mu q^\alpha}{[-(q \cdot q)]^{3/2}} \right] = \frac{1}{3} (a_1)^2 u_1^\mu - \frac{1}{12} \dot{a}_1^\mu. \qquad (3.15)$$

Therefore the subscript "R" stands for "regular" as well as for "radiative".

Alternatively, choosing the linear superposition

$$p_{\mathrm{S}}^\mu(\tau) = \frac{1}{2} \int_{-\infty}^{\tau} d\tau_1 \left[G_{\mathrm{ret}}^\mu(\tau_1) + G_{\mathrm{adv}}^\mu(\tau_1, \tau) \right] \qquad (3.16)$$

we restore the first term in the right-hand side of eq.(3.1). Indeed, having integrated the half-sum of functions (3.6) and (3.11) over ζ we obtain

$$\begin{aligned} p_{\mathrm{S}}^\mu &= \frac{e^2}{2} \int_{-\infty}^{\tau} d\tau_2 \left. \frac{u_2^\mu}{\sqrt{2\sigma(\tau_1, \tau_2)}} \right|_{\tau_1 = \tau_2}^{\tau_1 = \tau} + \frac{e^2}{2} \int_{-\infty}^{\tau} d\tau_1 \left. \frac{u_1^\mu}{\sqrt{2\sigma(\tau_1, \tau_2)}} \right|_{\tau_2 \to -\infty}^{\tau_2 = \tau_1} \\ &= \frac{e^2}{2} \int_{-\infty}^{\tau} d\tau_2 \frac{u^\mu(\tau_2)}{\sqrt{2\sigma(\tau, \tau_2)}}. \qquad (3.17) \end{aligned}$$

Since this non-local term diverges, in eq.(3.16) the subscript "S" stands for "singular" as well as "symmetric".

In specific case of uniformly moving charge $\sqrt{2\sigma(\tau, \tau_2)} = \tau - \tau_2$. Hence $p_{\mathrm{S}}^\mu(\tau)$ coincides with that obtained in Appendix A where recti-linear uniform motion is considered (see eq.(A.14)). Since the bracketed integrand in (3.13) vanishes if $u^\mu = const$, non-accelerating charge does not radiate.

We therefore introduce the radiative part p_{R} of energy-momentum and postulate that it, and it alone exerts a force on the particle. Singular part, p_{S}, should be coupled with particle's three-momentum, so that "dressed" charged particle would not undergo any additional radiation reaction. Already renormalized particle's individual three-momentum, say p_{part}, together with p_{R} constitute the total energy-momentum of our particle plus field system: $P = p_{\mathrm{part}} + p_{\mathrm{R}}$. Since P does not change with time, its time derivative yields

$$\begin{aligned} \dot{p}_{\mathrm{part}}^\mu(\tau) &= -\dot{p}_{\mathrm{R}}^\mu \qquad (3.18) \\ &= -\frac{e^2}{2} \int_{-\infty}^{\tau} ds \left[u_{\tau,\alpha} \frac{-u_s^\alpha q^\mu + u_s^\mu q^\alpha}{[2\sigma(\tau, s)]^{3/2}} + u_{s,\alpha} \frac{-u_\tau^\alpha q^\mu + u_\tau^\mu q^\alpha}{[2\sigma(\tau, s)]^{3/2}} \right]. \end{aligned}$$

(The overdot means the derivation with respect to proper time τ.) Here index τ indicates that particle's velocity or position is referred to the
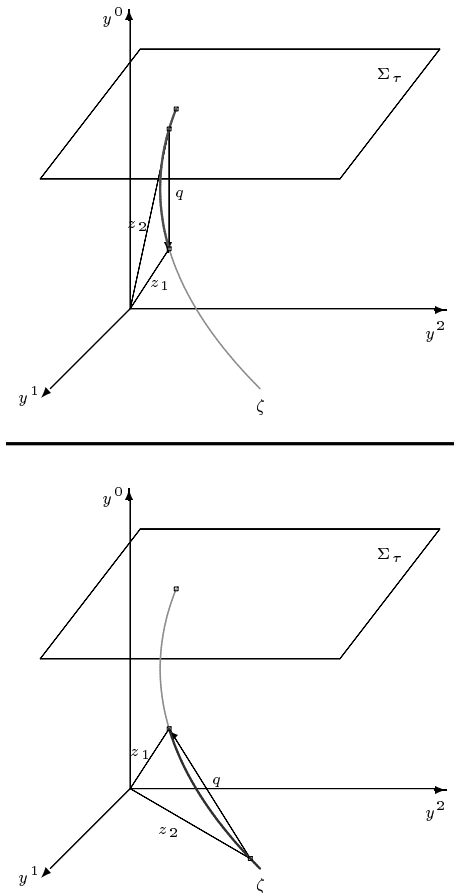
Figure 2. The term (3.6) with integration over the portion of the world line *before* τ_1 we call "retarded". The term (3.11) with integration over the portion of the world line *after* τ_1 we call "advanced". For an observer placed at point $z(\tau_1) \in \zeta$ the regular part (3.13) of electromagnetic field momentum looks as the combination of incoming and outgoing radiation. And yet the retarded causality is not violated. We still consider the interference of outgoing waves presented at the observation instant τ. The electromagnetic field carries information about the charge's past.

observation instant τ while index s says that the particle's characteristics are evaluated at instant $s \leq \tau$.

Our next task is to derive expression which explain how three-momentum of "dressed" charged particle depends on its individual characteristics (velocity, position, mass etc.). We do not make any assumptions about the particle structure, its charge distribution and its size. We only assume that the particle 3-momentum p_{part} is finite. To find out the desired expression we analyze conserved quantities corresponding to the invariance of the theory under proper homogeneous Lorentz transformations. The total angular momentum, say M, consists of particle's angular momentum $z \wedge p_{\text{part}}$ and radiative part of angular momentum carried by electromagnetic field:

$$M^{\mu\nu} = z_\tau^\mu p_{\text{part}}^\nu(\tau) - z_\tau^\nu p_{\text{part}}^\mu(\tau) + M_{\text{R}}^{\mu\nu}(\tau). \qquad (3.19)$$

(Singular part is absorbed by p_{part}.) The last term is calculated in Appendix G:

$$M_R^{\mu\nu} = \frac{e^2}{2} \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \left(z_1^\mu p_{12}^\nu - z_1^\nu p_{12}^\mu + z_2^\mu p_{21}^\nu - z_2^\nu p_{21}^\mu \right) \quad (3.20)$$

where two-point function p_{12}^α is given by eq.(3.5).

Having differentiated (3.19) and inserting eq.(3.18) we arrive at the equation

$$u_\tau \wedge p_{\text{part}} = \frac{e^2}{2} \int_{-\infty}^{\tau} ds \frac{u_\tau \wedge u_s}{\sqrt{2\sigma(\tau, s)}} \qquad (3.21)$$

where symbol $\wedge$ denotes the wedge product. We obtain the system of tree linear equations in three components of particle's momentum. Its rank is equal to 2. Therefore, an arbitrary scalar function $m(\tau)$ arises:

$$p_{\text{part}}^\mu(\tau) = m u^\mu(\tau) + \frac{e^2}{2} \int_{-\infty}^{\tau} ds \frac{u^\mu(s) - u^\mu(\tau)}{\sqrt{2\sigma(\tau, s)}}. \qquad (3.22)$$

(We choose the simplest expression being finite near the point of observation.) We see that, apart from usual velocity term, particle's 3-momentum contains also non-local contribution from particle's electromagnetic field.

The scalar product of particle three-velocity on the first-order time-derivative of particle three-momentum (3.18) is as follows:

$$(\dot{p}_{\text{part}} \cdot u_\tau) = \frac{e^2}{2} \int_{-\infty}^{\tau} ds \left[(u_\tau \cdot u_s) \frac{(u_\tau \cdot q)}{[2\sigma]^{3/2}} + \frac{(u_s \cdot q)}{[2\sigma]^{3/2}} \right]. \qquad (3.23)$$

Since $(u \cdot a) = 0$, the scalar product of particle acceleration on the particle three-momentum (3.22) is given by

$$(p_{\text{part}} \cdot a_\tau) = \frac{e^2}{2} \int_{-\infty}^{\tau} ds \frac{(a_\tau \cdot u_s)}{\sqrt{2\sigma}}. \tag{3.24}$$

Summing up (3.23) and (3.24) we obtain

$$\frac{d}{d\tau}(p_{\text{part}} \cdot u_\tau) = \frac{e^2}{2} \int_{-\infty}^{\tau} ds \left\{ \frac{\partial}{\partial \tau} \left[\frac{(u_\tau \cdot u_s)}{\sqrt{2\sigma}} \right] + \frac{(u_s \cdot q)}{[2\sigma]^{3/2}} \right\}. \tag{3.25}$$

Alternatively, the scalar product of 3-momentum (3.22) and 3-velocity is as follows:

$$(p_{\text{part}} \cdot u_\tau) = -m + \frac{e^2}{2} \int_{-\infty}^{\tau} ds \frac{(u_\tau \cdot u_s) + 1}{\sqrt{2\sigma}}. \tag{3.26}$$

Further we compare its differential consequence with eq.(3.25). A surprising feature of the already renormalized *dynamical* mass m is that it depends on τ:

$$\dot{m} = \frac{e^2}{2} \int_{-\infty}^{\tau} ds \frac{(q \cdot u_\tau) - (q \cdot u_s)}{[2\sigma]^{3/2}}. \tag{3.27}$$

It is interesting, that similar phenomenon occurs in the theory which describes a point-like charge coupled with massless scalar field in flat spacetime of three dimensions [19]. The charge loses its mass through the emission of monopole radiation.

Having integrated derivative (3.27) over the world line ζ we obtain

$$m = m_0 + \frac{e^2}{2} \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \left[\frac{\partial}{\partial \tau_1} \left(\frac{1}{\sqrt{2\sigma}} \right) + \frac{\partial}{\partial \tau_2} \left(\frac{1}{\sqrt{2\sigma}} \right) \right]$$
$$= m_0 + \frac{e^2}{2} \int_{-\infty}^{\tau} \frac{ds}{\sqrt{2\sigma(\tau, s)}} \tag{3.28}$$

where m_0 is an infinite "bare" mass of the particle. Inserting this into (3.22) we arrive at the equality $p_{\text{part}}^\mu(\tau) = m_0 u_\tau^\mu + p_S^\mu$ which shows that particle's momentum renormalization agrees with the renormalization of mass.

The main goal of the present paper is to compute the effective equation of motion of radiating charge in $2+1$ dimensions. To do it we replace $\dot{p}_{\text{part}}^\mu$ in left-hand side of eq.(3.18) by differential consequence of eq.(3.22) where the right-hand side of eq.(3.27) substitutes for $\dot{m}$. At the end of a

straightforward calculations, we obtain

$$ma_\tau^\mu = \frac{e^2}{2} a_\tau^\mu - e^2 \int_{-\infty}^{\tau} ds \left[u_{\tau,\alpha} \frac{-u_s^\alpha q^\mu + u_s^\mu q^\alpha}{[2\sigma]^{3/2}} - \frac{1}{2} \frac{a_\tau^\mu}{\sqrt{2\sigma}} \right] \tag{3.29}$$
$$= \frac{e^2}{2} a_\tau^\mu - G_{\text{ret}}^\mu(\tau) + \frac{e^2}{2} a_\tau^\mu \int_{-\infty}^{\tau} \frac{ds}{\sqrt{2\sigma(\tau, s)}}.$$

The first term in the right-hand side of this equation looks horribly irrelevant. The relation (3.7) prompts that the retarded Lorentz self-force should be substituted for the combination of the first (local) and of the second (non-local) terms in the right-hand side of eq.(3.29). If an external electromagnetic field $\hat{F}_{\text{ext}}$ is applied, the equation of motion of radiating charge in $2 + 1$ theory becomes

$$ma_\tau^\mu = eu_{\tau,\alpha} F_{\text{ret}}^{\mu\alpha}(\tau) + \frac{e^2}{2} a_\tau^\mu \int_{-\infty}^{\tau} \frac{ds}{\sqrt{2\sigma(\tau, s)}} + eu_{\tau,\alpha} F_{\text{ext}}^{\mu\alpha} \tag{3.30}$$

where

$$F_{\text{ret}}^{\mu\alpha}(\tau) = e \int_{-\infty}^{\tau} \frac{ds}{\sqrt{2\sigma}} \left\{ \frac{u_s^\mu q^\alpha - u_s^\alpha q^\mu}{r^2} [1 + (a_s \cdot q)] + \frac{a_s^\mu q^\alpha - a_s^\alpha q^\mu}{r} \right\}$$
$$= \int_{-\infty}^{\tau} ds f^{\mu\alpha}(\tau, s) \tag{3.31}$$

is the field strengths at point $z(\tau) \in \zeta$ generated by portion of the world line *before* the observation instant τ. The non-local term in eq.(3.30) which is proportional to particle's acceleration $a(\tau)$ arises also in [2]. It provides proper short-distance behavior of the radiation back-reaction. If $s \to \tau$ the integrand tends to three-dimensional analog of the Abraham radiation reaction vector:

$$\lim_{s \to \tau} \left[eu_{\tau,\alpha} f^{\mu\alpha}(\tau, s) + \frac{e^2}{2} \frac{a_\tau^\mu}{\sqrt{2\sigma(\tau, s)}} \right] = \frac{2}{3} e^2 \left(\dot{a}^\mu - a^2 u^\mu \right). \tag{3.32}$$

(All quantities on the right-hand side refer to the instant of observation τ.)

If one moves the second term to the left-hand side of eq.(3.30), they restore unphysical motion equation which follows from variational principle: it involves an infinite "bare" mass and divergent Lorentz self-force.

4. Energy-momentum of electromagnetic field in 2+1 dimensions

In this section we trace a series of stages in calculation of the surface integral

$$p_{\text{em}}^\nu(\tau) = \int_\Sigma d\sigma_\mu T^{\mu\nu} \tag{4.1}$$

which gives the energy-momentum carried by electromagnetic field of a point-like source arbitrarily moving in $\mathbb{M}_3$. In Appendix D, Appendix E and Appendix F we perform the computation in detail.

In eq.(4.1) $d\sigma_\mu$ is the vectorial surface element on an arbitrary space-like surface Σ. The electromagnetic field's stress-energy tensor $\hat{T}$ has the components

$$2\pi T^{\mu\nu} = F^{\mu\lambda}F^\nu{}_\lambda - 1/4\eta^{\mu\nu}F^{\kappa\lambda}F_{\kappa\lambda} \tag{4.2}$$

where $\hat{F}$ is the non-local strength tensor (2.11).

4.1. Coordinate system

In general, the rate of radiation does not depend on the shape of Σ. We choose the simplest plane $\Sigma_t = \{y \in \mathbb{M}_3 : y^0 = t\}$ associated with unmoving inertial observer. If parametrization of the world line is provided by a disjoint union of planes Σ_t, particle's velocity takes the form $u^\mu = \gamma v^\mu, v^\mu = (1, \dot{z}^i)$, and acceleration $a^\mu = \gamma^4(\mathbf{v}\dot{\mathbf{v}})v^\mu + \gamma^2\dot{v}^\mu$; factor $\gamma = 1/\sqrt{1 - \mathbf{v}^2}$. (The "overdot" indicates differentiation with respect to t.) Electromagnetic field (2.11) takes the form

$$\hat{F}^{ret}(y) = e \int\limits_{-\infty}^{t^{ret}(y)} \frac{dt}{\sqrt{-(K \cdot K)}} \left\{ \frac{\dot{v} \wedge K}{r} + \frac{v \wedge K}{r^2} \left[\gamma^{-2} + (K \cdot \dot{v}) \right] \right\} \tag{4.1.1}$$

where $\dot{v}^\mu = (0, \dot{v}^i)$ and $r = -(K \cdot v)$. (Although we use the same notation, r should not be confused with manifestly covariant parameter (2.12).)

Huygens principle does not hold in three dimensions and radiation develops a tail (see figure 3). In $3D$ the circle $C(z(0), t) = \{y \in \mathbb{M}_3 : (y^0)^2 = \sum_{i=1}^2 (y^i - z^i(0))^2, y^0 = t\}$ is filled up by electromagnetic radiation even if interval $\triangle t \to 0$. (The period of time during which the point source emanates is meant.) So, a point $z(t_1) \in \zeta$ produces the disk of radius $t - t_1$ in the observation plane $\Sigma_t = \{y \in \mathbb{M}_3 : y^0 = t\}$. This property reflects the fact that in $\mathbb{M}_3$ the electromagnetic field at y is generated by the portion of the world line that corresponds to the interval $-\infty < \tau < \tau^{ret}(y)$; this represents the past history of the particle.
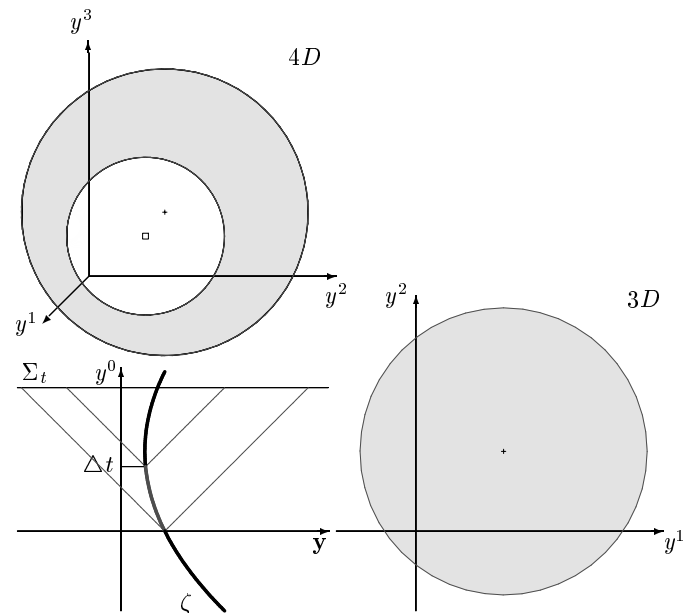
Figure 3. Let the point source radiates within the interval $[0, \triangle t]$. In four dimensions the support of the Maxwell energy-momentum tensor density in hyperplane $y^0 = t$ is in between two spheres centered at points $z^i(0)$ (cross symbol) and $z^i(\triangle t)$ (box symbol) with radii t and $t - \triangle t$, respectively. In three dimensions the radiation fills the disk with radius t centered at point $z^i(0)$ (cross symbol) even if the interval shrinks to zero.

We introduce coordinate system associated with two points on a particle's world line labelled by instants t_1 and t_2 (see figure 4). Flat space-time $\mathbb{M}_3$ becomes a disjoint union of planes $\Sigma_t = \{y \in \mathbb{M}_3 : y^0 = t\}$. A plane Σ_t is a union of (retarded) disks centered on a world line of the particle. The disk

$$C(z(t_a), t - t_a) = \{y \in \mathbb{M}_3 : y^0 - t_a \geq \sqrt{\sum_i (y^i - z^i(t_a))^2}, y^0 = t\} \tag{4.1.2}$$

is bounded by the intersection of the future light cone generated by null rays emanating from $z(t_a) \in \zeta$ in all possible directions, and plane Σ_t. The circular spot (4.1.2) is filled up by circles of radii $R \in [0, t - t_a]$ centered on points on a line connecting points $z^i(t_1)$ and $z^i(t_2)$. Points in
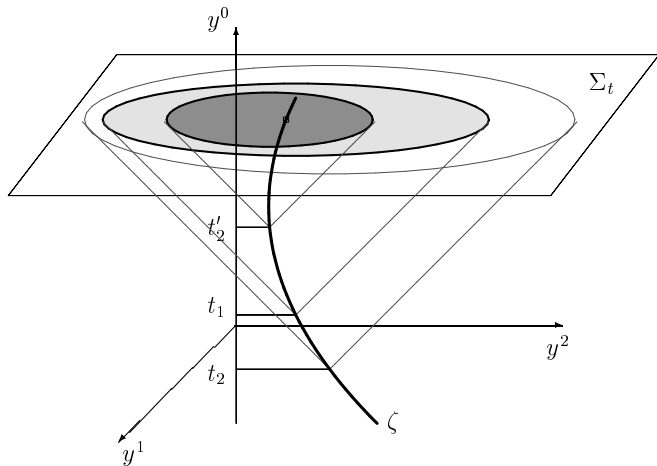
Figure 4. Outgoing electromagnetic waves generated by the portion of the world line that corresponds to the interval $-\infty < t_2 < t_1$ combine within the gray disk with radius $k_1^0 = t - t_1$. Their contribution is given by the first fourfold integral in eq.(4.1.4). If the domain of integration $t_1 < t_2 \leq t$ the waves joint together inside the dark disk with radius $k_2^0 = t - t_2'$. The second fourfold integral in eq.(4.1.4) describes them.

an $R-$circle are distinguished by polar angle φ. We define the coordinate transformation locally written as

$$y^0 = t \tag{4.1.3}$$
$$y^i = \alpha z^i(t_1) + \beta z^i(t_2) + R\omega^i{}_j n^j$$

where $\alpha + \beta = 1$ and $n^j = (\cos\varphi, \sin\varphi)$. Orthogonal matrix ω is given by eq.(B.2) (see Appendix B). It rotates space axes till new y^1-axis be directed along two-vector $\mathbf{q} := \mathbf{z}(t_1) - \mathbf{z}(t_2)$.

The integration of energy and momentum densities over two-dimensional plane $y^0 = const$ means the study of interference of outgoing electromagnetic waves emitted by different points on particle's world line (see figure 4). Note that the retarded field is generated by portion of the world line ζ that corresponds to the particle's history *before* $t^{ret}(y)$. Since the stress-energy tensor is quadratic in field strengths, we should *twice* integrate it over ζ. There are also two variables which parametrize Σ_t. In

curvilinear coordinates $(t, t_1, t_2, s, \varphi)$ the surface integral (4.1) becomes

$$p_{\text{em}}^\alpha = \int_{\Sigma_t} d\sigma_0 T^{0\alpha} \tag{4.1.4}$$

$$= \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{0}^{k_1^0} dR \int_{0}^{2\pi} d\varphi J t_{12}^{0\alpha} + \int_{-\infty}^{t} dt_1 \int_{t_1}^{t} dt_2 \int_{0}^{k_2^0} dR \int_{0}^{2\pi} d\varphi J t_{12}^{0\alpha}$$

with Jacobian

$$J = \left(1 - q\frac{\partial\beta}{\partial R}\cos\varphi\right) R \tag{4.1.5}$$
$$= \left(1 + q\frac{\partial\alpha}{\partial R}\cos\varphi\right) R.$$

The integrand

$$2\pi t_{12}^{\alpha\beta} = f_{(1)}^{\alpha\lambda} f_{(2)\lambda}^{\beta} - \frac{1}{4}\eta^{\alpha\beta} f_{(1)}^{\mu\nu} f_{\mu\nu}^{(2)} \tag{4.1.6}$$

describes the combination of field strength densities at $y \in \Sigma_t$

$$\hat{f}_{(a)}(y) = \frac{e}{\sqrt{-(K_a \cdot K_a)}} \left(\frac{\dot{v}_a \wedge K_a}{r_a} + \frac{v_a \wedge K_a}{(r_a)^2}c_a\right) \tag{4.1.7}$$

generated by emission points $z(t_1) \in \zeta$ and $z(t_2) \in \zeta$. Symbol c_a denotes the factor $\gamma_a^{-2} + (K_a \cdot \dot{v}_a)$ involved in eq.(4.1.1).

The first multiple integral calculates the interference of the disk emanated by fixed point $z(t_1) \in \zeta$ with radiation generated by all the points *before* the instant t_1. The second fourfold integral gives the contribution of points *after* t_1 (see figure 4).

It is worth noting that time variables t_1 and t_2 parametrize the same world line ζ. Coordinate transformation (4.1.3) is invariant with respect to the following reciprocity:

$$\Upsilon : t_1 \leftrightarrow t_2, \alpha \leftrightarrow \beta, \varphi \mapsto \varphi + \pi. \tag{4.1.8}$$

This symmetry provides identity of domains of fourfold integrals in energy-momentum (4.1.4).

It is obvious that the support of double integral $\int_{-\infty}^{t} dt_1 \int_{t_1}^{t} dt_2$ coincides with the support of the integral $\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1$. Since instants t_1 and t_2 label different points at the same world line ζ, one can interchanges the indices "first" and "second" in the second fourfold integral of eq.(4.1.4). Via interchanging of these indices we finally obtain

$\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2$ instead of initial $\int_{-\infty}^{t} dt_1 \int_{t_1}^{t} dt_2$. Taking into account these circumstances in the expression (4.1.4) for energy-momentum carried by electromagnetic field we finally obtain

$$p_{\text{em}}^{\alpha} = \int\limits_{-\infty}^{t} dt_1 \int\limits_{-\infty}^{t_1} dt_2 \int\limits_{0}^{k_1^0} dR \int\limits_{0}^{2\pi} d\varphi \, J t^{0\alpha} \qquad (4.1.9)$$

where new stress-energy tensor is symmetric in the pair of indices 1 and 2:

$$2\pi t^{0\alpha} = 2\pi \left(t_{12}^{0\alpha} + t_{21}^{0\alpha} \right) \qquad (4.1.10)$$
$$= f_{(1)}^{0\lambda} f_{(2)\lambda}^{\alpha} + f_{(2)}^{0\lambda} f_{(1)\lambda}^{\alpha} - \frac{1}{4}\eta^{0\alpha} \left[f_{(1)}^{\mu\nu} f_{\mu\nu}^{(2)} + f_{(2)}^{\mu\nu} f_{\mu\nu}^{(1)} \right].$$

4.2. Angular integration of energy-momentum tensor density

We see that it is sufficient to consider the situation when $t_1 \geq t_2$. The smaller disk $C_1(z_1, t - t_1) \subset C_2(z_2, t - t_2)$ is filled up by non-concentric circles with radii $R \in [0, k_1^0]$ (see figures 4,5). To calculate the total flows (4.1.9) of electromagnetic field energy and momentum which flow across the plane Σ_t we should integrate the Maxwell energy-momentum tensor density (4.1.10) over angular variable φ, over radius R and, finally, over time variables t_1 and t_2. Integration over φ is not a trivial matter. The difficulty resides mostly with norms $\|K_a\|^2 = -\eta_{\alpha\beta} K_a^{\alpha} K_a^{\beta}$ of separation vectors $K_a = y - z_a$ which result elliptic integrals. To avoid dealing with them we modify the coordinate transformation (4.1.3). We fix the parameter β in such a way that the norm $\|K_1\|^2$ becomes proportional to the norm $\|K_2\|^2$:

$$\|K_1\|^2 = -\frac{\beta}{\alpha}\|K_2\|^2. \qquad (4.2.1)$$

Keeping in mind identity $\alpha + \beta = 1$, we arrive at the quadratic algebraic equation on β which does not contain the angle variable:

$$R^2 = \alpha(k_1^0)^2 + \beta(k_2^0)^2 - \alpha\beta\mathbf{q}^2. \qquad (4.2.2)$$

We choose the root which vanishes when $R = k_1^0$:

$$\beta = \frac{1}{2\mathbf{q}^2} \left(-(k_2^0)^2 + (k_1^0)^2 + \mathbf{q}^2 + \sqrt{D} \right) \qquad (4.2.3)$$
$$D = \left[(k_2^0)^2 - (k_1^0)^2 - \mathbf{q}^2 \right]^2 - 4\mathbf{q}^2 \left[(k_1^0)^2 - R^2 \right].$$

If $\mathbf{q}^2$ tends to zero while $t_1 \neq t_2$, it becomes the unique root of the linear equation on β originated from eq.(4.2.2) with $\mathbf{q}^2 = 0$.
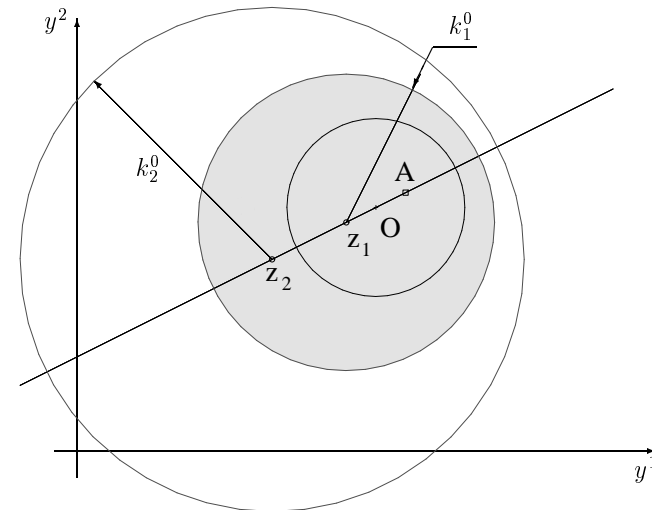
Figure 5. The interference picture in a plane Σ_t. The points $z(t_1) \in \zeta$ and $z(t_2) \in \zeta, t_2 < t_1$, emanate the radiation which filled up the disks centered at z_1 and z_2, respectively. The gray disk with radius $k_1^0 = t - t_1$ is filled up by non-concentric circles centered at the line crossing both the points z_1 and z_2. If parameter β vanishes the circle is centered at z_1; its radius is equal to k_1^0. If $\beta = \beta_0 < 0$ the circle reduces to the point A labelled by the box symbol. In case of intermediate value $\beta_0 < \beta < 0$ we have the circle of radius R with center at point O between z_1 and A.

If $R = 0$ the $R-$circle reduces to the point A with coordinates $(z_1^i - \beta_0 q^i)$ where $\beta_0 = \beta|_{R=0}$. If $R = k_1^0$ then $\beta = 0$ and the circle is centered at $z^i(t_1)$ (see figure 5).

Changing the variable of integration from R to β transforms two inner integrals in the fourfold integral (4.1.9) as follows:

$$\int\limits_{0}^{k_1^0} dR \int\limits_{0}^{2\pi} d\varphi \left(1 - q\frac{\partial\beta}{\partial R}\cos\varphi \right) R = \int\limits_{\beta_0}^{0} d\beta \int\limits_{0}^{2\pi} d\varphi \left(\frac{1}{2}\frac{\partial R^2}{\partial\beta} - qR\cos\varphi \right).$$
$$(4.2.4)$$

Having differentiated eq.(4.2.2) with respect to β we obtain new Jacobian

$$J = (1/2)\left[(k_2^0)^2 - (k_1^0)^2 - \mathbf{q}^2 \right] + \beta\mathbf{q}^2 - qR\cos\varphi. \qquad (4.2.5)$$

It is straightforward to substitute the field densities (4.1.7) evaluated

at instants t_1 and t_2 into expression (4.1.10) to calculate the integrand of the multiple integral (4.1.9). It is of great importance that the square of norm $\|K_a\|$ is proportional to J (see eqs.(B.6) derived in Appendix B). For the first term, $Jt_{12}^{\alpha\beta}$, we obtain the following cumbersome expression

$$
\begin{aligned}
Jt_{12}^{\alpha\beta} = \frac{e^2}{2} I & \left\{ \hat{T}_{12}^{\alpha\beta} \left(\frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) + \hat{T}_1^{\alpha} \left(v_2^{\beta} \frac{\partial \sigma}{\partial t_1} \right) + \hat{T}_2^{\beta} \left(v_1^{\alpha} \frac{\partial \sigma}{\partial t_2} \right) \right. \\
& + \hat{T}^0(v_1^{\alpha} v_2^{\beta} \sigma) - C_1^{\alpha} v_2^{\beta} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} - D_1^{\alpha} v_2^{\beta} \frac{\partial^3 \sigma}{\partial t_1^2 \partial t_2} - B_2^{\beta} v_1^{\alpha} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \\
& - D_2^{\beta} v_1^{\alpha} \frac{\partial^3 \sigma}{\partial t_1 \partial t_2^2} - B^0 v_1^{\alpha} v_2^{\beta} \frac{\partial \sigma}{\partial t_1} - C^0 v_1^{\alpha} v_2^{\beta} \frac{\partial \sigma}{\partial t_2} \\
& \left. - D^0 \left(\dot{v}_1^{\alpha} v_2^{\beta} \frac{\partial \sigma}{\partial t_2} + v_1^{\alpha} \dot{v}_2^{\beta} \frac{\partial \sigma}{\partial t_1} + v_1^{\alpha} v_2^{\beta} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) \right\} \\
& - \frac{e^2}{2} I' \hat{T}^J (v_1^{\alpha} v_2^{\beta}) - \frac{e^2}{4} \eta^{\alpha\beta} \left\{ I \hat{T}^0(\lambda) - I' \hat{T}^J(\lambda_0) \right\}.
\end{aligned}
\tag{4.2.6}
$$

Functions

$$
\lambda = \sigma \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} - \frac{\partial \sigma}{\partial t_1} \frac{\partial \sigma}{\partial t_2}, \qquad \lambda_0 = \frac{\partial^2 \sigma}{\partial t_1 \partial t_2}
\tag{4.2.7}
$$

depend on Synge's world function (3.2) of two timelike related points, $z(t_1) \in \zeta$ and $z(t_2) \in \zeta$, taken with opposite sign. Symbols I, I' denote β-dependent factors

$$
I = \frac{1}{\sqrt{-\beta\alpha}}, \qquad I' = \sqrt{\frac{-\beta}{\alpha}} + \sqrt{\frac{\alpha}{-\beta}}.
\tag{4.2.8}
$$

Each second order differential operator

$$
\hat{T}^a = D^a \frac{\partial^2}{\partial t_1 \partial t_2} + B^a \frac{\partial}{\partial t_1} + C^a \frac{\partial}{\partial t_2} + A^a
\tag{4.2.9}
$$

has been labelled according to its dependence on combination of components of the separation vectors K_1 and K_2 pictured in figure 6 (see Appendix B), or on Jacobian (4.2.5). The components of these vectors are involved in φ-dependent coefficients

$$
D^a = \frac{1}{2\pi} \frac{a}{r_1 r_2}, \qquad B^a = \frac{1}{2\pi} \frac{ac_2}{r_1 (r_2)^2}
$$
$$
C^a = \frac{1}{2\pi} \frac{ac_1}{(r_1)^2 r_2}, \qquad A^a = \frac{1}{2\pi} \frac{ac_1 c_2}{(r_1)^2 (r_2)^2}
\tag{4.2.10}
$$

where factor a is replaced by $K_1^{\alpha} K_2^{\beta}, K_1^{\alpha}, K_2^{\beta}, J$ or 1 for $\hat{T}_{12}^{\alpha\beta}, \hat{T}_1^{\alpha}, \hat{T}_2^{\beta}, \hat{T}^J$ or $\hat{T}^0$, respectively.

Integration of the electromagnetic field's stress-energy tensor over the angular variable is the key to the problem. All the φ-dependent constructions are concentrated in the coefficients of differential operators of type (4.2.9). We introduce new operator

$$
\hat{\mathcal{T}}^a = \mathcal{D}^a \frac{\partial^2}{\partial t_1 \partial t_2} + \mathcal{B}^a \frac{\partial}{\partial t_1} + \mathcal{C}^a \frac{\partial}{\partial t_2} + \mathcal{A}^a
\tag{4.2.11}
$$

where calligraphic letters denote the coefficients (4.2.10) integrated over φ. (The integration is performed in Appendix C).

To distinguish the partial derivatives in time variables we rewrite the operator (4.2.11) as the sum of the second-order differential operator

$$
\hat{\Pi}^a = \frac{\partial^2}{\partial t_1 \partial t_2} \mathcal{D}^a + \frac{\partial}{\partial t_1} \left(\mathcal{B}^a - \frac{\partial \mathcal{D}^a}{\partial t_2} \right) + \frac{\partial}{\partial t_2} \left(\mathcal{C}^a - \frac{\partial \mathcal{D}^a}{\partial t_1} \right)
\tag{4.2.12}
$$

and the "tail"

$$
\pi^a = \frac{\partial^2 \mathcal{D}^a}{\partial t_1 \partial t_2} - \frac{\partial \mathcal{B}^a}{\partial t_1} - \frac{\partial \mathcal{C}^a}{\partial t_2} + \mathcal{A}^a.
\tag{4.2.13}
$$

For a smooth function $f(t_1, t_2)$ we have

$$
\hat{\mathcal{T}}^a(f) = \hat{\Pi}^a(f) + f\pi^a.
\tag{4.2.14}
$$

In Appendix C we derive the relations

$$
\pi^0 = 0, \qquad \pi^J = 0
\tag{4.2.15}
$$
$$
\pi_1^{\alpha} = v_1^{\alpha} \left(\mathcal{B}^0 - \frac{\partial \mathcal{D}^0}{\partial t_2} \right), \qquad \pi_2^{\beta} = v_2^{\beta} \left(\mathcal{C}^0 - \frac{\partial \mathcal{D}^0}{\partial t_1} \right)
$$
$$
\pi_1^{\alpha J} = v_1^{\alpha} \left(\mathcal{B}^J - \frac{\partial \mathcal{D}^J}{\partial t_2} \right), \qquad \pi_2^{\beta J} = v_2^{\beta} \left(\mathcal{C}^J - \frac{\partial \mathcal{D}^J}{\partial t_1} \right)
$$
$$
\pi_{12}^{\alpha\beta} = v_1^{\alpha} \left(\mathcal{B}_2^{\beta} - \frac{\partial \mathcal{D}_2^{\beta}}{\partial t_2} \right) + v_2^{\beta} \left(\mathcal{C}_1^{\alpha} - \frac{\partial \mathcal{D}_1^{\alpha}}{\partial t_1} \right) - v_1^{\alpha} v_2^{\beta} \mathcal{D}^0
$$

which allow us to rewrite the integral of (4.2.6) over φ in terms of differential operators $\hat{\Pi}^a$:

$$
\begin{aligned}
\int_0^{2\pi} d\varphi J t_{12}^{\alpha\beta} = \frac{e^2}{2} I & \left\{ \hat{\Pi}_{12}^{\alpha\beta} \left(\frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) + \hat{\Pi}_1^{\alpha} \left(v_2^{\beta} \frac{\partial \sigma}{\partial t_1} \right) + \hat{\Pi}_2^{\beta} \left(v_1^{\alpha} \frac{\partial \sigma}{\partial t_2} \right) \right. \\
& \left. + \hat{\Pi}^0 (v_1^{\alpha} v_2^{\beta} \sigma) - \frac{\partial}{\partial t_1} \left(D_1^{\alpha} v_2^{\beta} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) \right. \\
& \left. - \frac{\partial}{\partial t_2} \left(D_2^{\beta} v_1^{\alpha} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) - \frac{\partial}{\partial t_1} \left(D^0 v_1^{\alpha} v_2^{\beta} \frac{\partial \sigma}{\partial t_2} \right) \right.
\end{aligned}
\tag{4.2.16}
$$

$$- \frac{\partial}{\partial t_2} \left(D^0 v_1^\alpha v_2^\beta \frac{\partial \sigma}{\partial t_1} \right) \Bigg\}$$
$$- \frac{e^2}{2} I' \hat{\Pi}^J (v_1^\alpha v_2^\beta) - \frac{e^2}{4} \eta^{\alpha\beta} \left\{ I \hat{\Pi}^0(\lambda) - I' \hat{\Pi}^J(\lambda_0) \right\}.$$

Since operator (4.2.12) is the combination of partial derivatives in time variables, the angular integration gives the key to the problem.

Putting $\alpha = 0$ and $\beta = i$ in (4.2.16) we obtain the first term of the mixed space-time components of the stress-energy tensor (4.1.10). We add the term where indices "first" and "second" are interchanged. Since zeroth components of the separation three-vectors K_1 and K_2 do not depend on φ, the final expression get simplified:

$$\int_0^{2\pi} d\varphi J t^{0i} = \frac{e^2}{2} I \left[\hat{\Pi}_1^i \left(\frac{\partial \lambda_2}{\partial t_1} \right) + \hat{\Pi}_2^i \left(\frac{\partial \lambda_1}{\partial t_2} \right) + \hat{\Pi}^0 \left(v_2^i \lambda_1 + v_1^i \lambda_2 \right) \right.$$
$$\left. - \frac{\partial}{\partial t_1} \left(v_2^i \frac{\partial \lambda_1}{\partial t_2} \mathcal{D}^0 \right) - \frac{\partial}{\partial t_2} \left(v_1^i \frac{\partial \lambda_2}{\partial t_1} \mathcal{D}^0 \right) \right]$$
$$- \frac{e^2}{2} I' \hat{\Pi}^J (v_1^i + v_2^i) \qquad (4.2.17)$$

where

$$\lambda_1 = k_1^0 \frac{\partial \sigma}{\partial t_1} + \sigma, \qquad \lambda_2 = k_2^0 \frac{\partial \sigma}{\partial t_2} + \sigma. \qquad (4.2.18)$$

Similarly we derive zeroth component t^{00}. Putting $\alpha = 0$ and $\beta = 0$ in (4.2.16) we obtain the first term of energy density. Since it is symmetric in the pair of indices 1 and 2, the second term, t_{21}^{00}, doubles it. The integral of energy density t^{00} over the angular variable has the form

$$\int_0^{2\pi} d\varphi J t^{00} = e^2 \left[I \hat{\Pi}^0(\kappa) - I' \hat{\Pi}^J(\mu) \right] \qquad (4.2.19)$$

where

$$\kappa = k_1^0 k_2^0 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} + k_1^0 \frac{\partial \sigma}{\partial t_1} + k_2^0 \frac{\partial \sigma}{\partial t_2} - \frac{1}{2} \frac{\partial \sigma}{\partial t_1} \frac{\partial \sigma}{\partial t_2} + \sigma \mu \quad (4.2.20)$$
$$\mu = \frac{1}{2} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} + 1.$$

We see that the integration of electromagnetic field's stress-energy tensor over φ yields integrals being functions of the end points only. In the next subsection we classify them and consider the problem of integration over the remaining variables.

4.3. Integration over time variables and β

Our purpose in this section is to develop the mathematical tools required in a surface integration of energy-momentum tensor density in $(2+1)-$electrodynamics. Integration over angle variable results the combination of partial derivatives in time variables:

$$p_{\text{em}}^\alpha(t) = \left. \begin{matrix} e^2 \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \\[2mm] e^2 \int_{-\infty}^t dt_2 \int_{t_2}^t dt_1 \end{matrix} \right\} \left(\frac{\partial^2 G_{12}^\alpha}{\partial t_1 \partial t_2} + \frac{\partial G_1^\alpha}{\partial t_1} + \frac{\partial G_2^\alpha}{\partial t_2} \right). \quad (4.3.1)$$

By virtue of the equality

$$\int_{\beta_0}^0 d\beta \frac{\partial G(\beta, t_1, t_2)}{\partial t_a} = \frac{\partial}{\partial t_a} \left[\int_{\beta_0}^0 d\beta G(\beta, t_1, t_2) \right] + G(\beta_0, t_1, t_2) \frac{\partial \beta_0(t_1, t_2)}{\partial t_a} \tag{4.3.2}$$

the triple integral (4.3.1) can be rewritten as follows:

$$p_{\text{em}}^\alpha(t) = e^2 \left[\lim_{k_1^0 \to 0} \int_{\beta_0}^0 d\beta G_{12}^\alpha \right]_{t_2 \to -\infty}^{t_2 = t} + e^2 \int_{-\infty}^t dt_2 \lim_{k_1^0 \to 0} \left[\int_{\beta_0}^0 d\beta G_1^\alpha \right]$$
$$+ e^2 \int_{-\infty}^t dt_2 \lim_{k_1^0 \to 0} \left[G_{12}^\alpha |_{\beta = \beta_0} \frac{\partial \beta_0}{\partial t_2} \right] \tag{4.3.3}$$
$$- e^2 \int_{-\infty}^t dt_2 \lim_{\triangle t \to 0} \int_{\beta_0}^0 d\beta \left[\frac{\partial G_{12}^\alpha}{\partial t_2} + G_1^\alpha \right]_{k_1^0 = k_2^0 - \triangle t}$$
$$+ e^2 \int_{-\infty}^t dt_1 \lim_{\triangle t \to 0} \left[\int_{\beta_0}^0 d\beta G_2^\alpha \right]_{k_2^0 = k_1^0 + \triangle t}$$
$$- e^2 \int_{-\infty}^t dt_1 \lim_{t_2 \to -\infty} \left[\int_{\beta_0}^0 d\beta G_2^\alpha \right]$$
$$+ \left. \begin{matrix} e^2 \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \\[2mm] e^2 \int_{-\infty}^t dt_2 \int_{t_2}^t dt_1 \end{matrix} \right\} \left(\left[\frac{\partial G_{12}^\alpha}{\partial t_2} + G_1^\alpha \right]_{\beta = \beta_0} \frac{\partial \beta_0}{\partial t_1} \right.$$
$$+ \left. G_2^\alpha |_{\beta = \beta_0} \frac{\partial \beta_0}{\partial t_2} \right).$$

In Appendix C we calculate the functions

$$G_{12}^0 = I \mathcal{D}^0 \kappa - I' \mathcal{D}^J \mu \tag{4.3.4}$$

$$G_1^0 = -\sqrt{\frac{-\beta}{\alpha}}\kappa\frac{\mathbf{v}_1^2}{\|r_1\|^3} - I'\mu\frac{\partial}{\partial\beta}\left(\frac{\beta}{\|r_1\|}\right)$$

$$G_2^0 = \sqrt{\frac{\alpha}{-\beta}}\kappa\frac{\mathbf{v}_2^2}{\|r_2\|^3} - I'\mu\frac{\partial}{\partial\beta}\left(\frac{\alpha}{\|r_2\|}\right)$$

$$G_{12}^i = \frac{I}{2}\left[\frac{\partial\lambda_1}{\partial t_2}\mathcal{D}_2^i + \frac{\partial\lambda_2}{\partial t_1}\mathcal{D}_1^i + \left(v_1^i\lambda_2 + v_2^i\lambda_1\right)\mathcal{D}^0\right] - \frac{I'}{2}\left(v_1^i + v_2^i\right)\mathcal{D}^J$$

$$G_1^i = \frac{I}{2}\frac{\beta}{\|r_1\|^3}\left[\frac{\partial\lambda_1}{\partial t_2}\left(\alpha q^i\mathbf{v}_1^2 + r_1^0 v_1^i\right) + \frac{\partial\lambda_2}{\partial t_1}\left(-\beta q^i\mathbf{v}_1^2 + r_1^0 v_1^i\right)\right.$$
$$\left. + \left(v_1^i\lambda_2 + v_2^i\lambda_1\right)\mathbf{v}_1^2\right] - \frac{I'}{2}\left(v_1^i + v_2^i\right)\frac{\partial}{\partial\beta}\left(\frac{\beta}{\|r_1\|}\right) \qquad (4.3.5)$$

$$G_2^i = \frac{I}{2}\frac{\alpha}{\|r_2\|^3}\left[\frac{\partial\lambda_1}{\partial t_2}\left(\alpha q^i\mathbf{v}_2^2 + r_2^0 v_2^i\right) + \frac{\partial\lambda_2}{\partial t_1}\left(-\beta q^i\mathbf{v}_2^2 + r_2^0 v_2^i\right)\right.$$
$$\left. + \left(v_1^i\lambda_2 + v_2^i\lambda_1\right)\mathbf{v}_2^2\right] - \frac{I'}{2}\left(v_1^i + v_2^i\right)\frac{\partial}{\partial\beta}\left(\frac{\alpha}{\|r_2\|}\right)$$

involved in these integrals.

$1°$. **Integrals where** $\mathbf{t}_1 \to t$. Equality (B.10) implies that the lower limit β_0 tends to 0 if $k_1^0 = t - t_1$ vanishes. The upper limit is equal to zero too. Then the integral over parameter β vanishes whenever an expression under integral sign is smooth. So, we must limit our computations to the singular terms only. They are performed in Appendix D; these integrals do not contribute in the energy-momentum at all.

$2°$. **Integrals where** $\mathbf{t}_1 = \mathbf{t}_2$. According to eq.(B.10), the equality $t_1 = t_2$ yields $\sin\vartheta_0 = 1$ and lower limit $\beta_0 = -\tan\vartheta_0$ tends to $-\infty$. The small parameter is the positively valued difference $\triangle t = t_1 - t_2$. The integration is performed in Appendix E; the resulting terms belong to the bound part of energy-momentum. (To that which is permanently "attached" to the charge and is carried along with it.)

$3°$. **Integrals where** $\mathbf{t}_2 \to -\infty$. Equality (B.10) implies that the lower limit β_0 tends to 0 if $k_2^0 = t - t_2$ increases extremely. Then the integral over parameter β vanishes whenever an expression under integral sign is smooth. So, we must limit our computations to the singular terms only. They are performed in Appendix E; the resulting terms belong to the bound electromagnetic "cloud" which can not be separated from the charged particle.

$4°$. **Integrals at point where** $\beta = \beta_0$. In this case the radius of the smallest circle pictured in figure 5 vanishes and it reduces to the point A. The contribution in p_{em}^α is given by the last line of eq.(4.3.3). In Appendix F we present the integrand as the combination of partial derivatives in time variables and non-derivative "tail". After integration

over t_1 or t_2, the derivatives are coupled with bound terms obtained in Appendix E; the sum is absorbed by three-momentum of "bare" particle within renormalization procedure. The "tail" contains radiative terms which detach themselves from the charge and lead independent existence.

Summing up all the contributions $2° - 4°$ we finally obtain

$$p_{\text{em}}^0(t) = e^2\frac{1 + 1/2\sqrt{1-\mathbf{v}_1^2}}{1 + \sqrt{1-\mathbf{v}_1^2}}\frac{1}{\sqrt{1-\mathbf{v}_1^2}}\Bigg|_{t_1\to-\infty}^{t_1=t} \qquad (4.3.6)$$
$$+ \frac{e^2}{2}\int_{-\infty}^t dt_2\frac{1}{\sqrt{2\sigma(t,t_2)}}$$
$$+ e^2\int_{-\infty}^t dt_1\int_{-\infty}^{t_1} dt_2\left[-\frac{(v_1\cdot v_2)q^0}{(2\sigma)^{3/2}} + \frac{1}{2}\frac{(v_1\cdot q)}{(2\sigma)^{3/2}} + \frac{1}{2}\frac{(v_2\cdot q)}{(2\sigma)^{3/2}}\right]$$

$$p_{\text{em}}^i(t) = e^2\frac{1 + 1/2\sqrt{1-\mathbf{v}_1^2}}{1 + \sqrt{1-\mathbf{v}_1^2}}\frac{v_1^i}{\sqrt{1-\mathbf{v}_1^2}}\Bigg|_{t_1\to-\infty}^{t_1=t} \qquad (4.3.7)$$
$$+ \frac{e^2}{2}\int_{-\infty}^t dt_2\frac{v_2^i}{\sqrt{2\sigma(t,t_2)}}$$
$$+ e^2\int_{-\infty}^t dt_1\int_{-\infty}^{t_1} dt_2\left[-\frac{(v_1\cdot v_2)q^i}{(2\sigma)^{3/2}} + \frac{1}{2}\frac{(v_1\cdot q)v_2^i}{(2\sigma)^{3/2}} + \frac{1}{2}\frac{(v_2\cdot q)v_1^i}{(2\sigma)^{3/2}}\right].$$

The finite terms which depend on the end points only are non-covariant. They express the "deformation" of electromagnetic "cloud" due to the choice of coordinate-dependent hole around the particle in the integration surface Σ_t. We neglect these structureless terms. The single integrals describe covariant singular part of energy-momentum carried by electromagnetic field. The first term in between the square brackets of eqs.(4.3.6) and (4.3.7) can not be rewritten as the partial derivative in t_1 or t_2. It determines the radiation reaction in $2+1$ electrodynamics.

5. Conclusions

In the present paper, we calculate the total flows of (retarded) electromagnetic field energy, momentum and angular momentum which flow across the plane $\Sigma_t = \{y \in \mathbb{M}_3 : y^0 = t\}$. The field is generated by a point-like electric charge arbitrarily moving in flat spacetime of three dimensions. The computation is not a trivial matter, since the Maxwell

energy-momentum tensor density evaluated at field point $y \in \Sigma_t$ is non-local. In odd dimensions the retarded field is generated by portion of the world line ζ that corresponds to the particle's history *before* $t^{ret}(y)$. Since the stress-energy tensor is quadratic in field strengths, we should *twice* integrate it over ζ. We integrate it also over two variables which parametrize Σ_t in order to calculate energy-momentum and angular momentum which flow across this plane. Thanks to integration we reduce the support of the retarded and advanced Green's functions to particle's trajectory.

The Dirac scheme which manipulates fields *on the world line only* is the key point of investigation. By fields we mean the convolution $u_\mu(\tau_1)F^{\mu\nu}_{(\vartheta)}$ of three-velocity and non-local part (2.8) of the retarded strength tensor evaluated at point $z(\tau_1) \in \zeta$; the torque of this "Lorentz θ-force" arises in electromagnetic field's total angular momentum. (Singular δ-term (2.7) is defined on the light cone; it is meaningless since both the field point, $z(t_1)$, and the emission point, $z(t_2)$, lie on the time-like world line). The retarded and the "advanced" quantities arise naturally. The retarded Lorentz self-force as well as its torque contain integration over the portion of the world line which corresponds to the interval $-\infty < t_2 \leq t_1$. Domain of integration of their "advanced" counterparts corresponds to the interval $t_1 \leq t_2 \leq t$.

Noether quantity G^α_{em} carried by electromagnetic field consists of terms of two quite different types: (i) singular, G^α_{S}, which is permanently "attached" to the source and carried along with it; (ii) radiative, G^α_{R}, which detaches itself from the charge and leads independent existence. The former is the half-sum of retarded and "advanced" expressions, integrated over ζ, while the latter is the integral of one-half of the retarded quantity minus one-half of the "advanced" one. Within regularization procedure the bound terms G^α_{S} are coupled with energy-momentum and angular momentum of "bare" source, so that already renormalized characteristics G^α_{part} of charged particle are proclaimed to be finite. Noether quantities which are properly conserved become:

$$G^\alpha = G^\alpha_{\text{part}} + G^\alpha_{\text{R}}.$$

Energy-momentum balance equations define the change of particle's three-momentum under the influence of an external electromagnetic field where loss of energy due to radiation is taken into account. The angular momentum balance equations explain how this already renormalized three-momentum depend on particle's individual characteristics. They constitute the system of three linear equations in three components of particle's momentum. Its rank is equal to 2, so that arbitrary scalar

function arises naturally. It can be interpreted as a *dynamical* mass of "dressed" charge which is proclaimed to be finite. A surprising feature is that this mass depends on the particle's history before the instant of observation when the charge is accelerated. Already renormalized particle's momentum contains, apart from usual velocity term, also non-local contribution from point-like particle's electromagnetic field.

Having substituted this expression in the energy-momentum balance equations we derive three-dimensional analogue of the Lorentz-Dirac equation

$$ma^\mu_\tau = eu_{\tau,\alpha}F^{\mu\alpha}_{\text{ret}}(\tau) + \frac{e^2}{2}a^\mu_\tau \int_{-\infty}^{\tau} \frac{ds}{\sqrt{2\sigma(\tau,s)}} + eu_{\tau,\alpha}F^{\mu\alpha}_{\text{ext}}.$$

The loss of energy due to radiation is determined by work done by Lorentz force of point-like charge acting upon itself. Non-local term which is proportional to particle's acceleration provides finiteness of the self-action. Third term describes influence of an external field.

In this paper we develop convenient technique which allows us to integrate non-local stress-energy tensor over the spacelike plane. The next step will be to implement this technique to a point particle coupled to massive scalar field following an arbitrary trajectory on a flat spacetime. The Klein-Gordon field generated by the scalar charge holds energy near the particle. This circumstance makes unclear the procedure of decomposition of the energy-momentum into bound and radiative parts.

In [20] the remarkable correspondence is established between dynamical equations which govern behavior of superfluid ^{4}He films and Maxwell equations for electrodynamics in 2+1 dimensions (see also refs. [21,22])[2]. Perhaps the effective equation of motion (3.30) will be useful in study of phenomena in superfluid dynamics which correspond to the radiation friction.

Acknowledgments

Appendix A. Uniformly moving charge: conserved quantities

The simplest field is generated by unmoved charge placed at the coordinate origin. Putting $z = (t, 0, 0)$ and $u = (1, 0, 0)$ in eq.(2.11), one can

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[2]I wish to thank O.Derzhko for drawing these papers to my attention.

derive that the only non-trivial components of static field are:

$$F_{i0} = e \int\limits_{-\infty}^{y^0 - r} \frac{dt}{\sqrt{(y^0 - t)^2 - r^2}} \frac{y^i}{(y^0 - t)^2} \tag{A.1}$$

$$= -e\frac{y^i}{r^2} \frac{\sqrt{(y^0 - t)^2 - r^2}}{y^0 - t} \bigg|_{t \to -\infty}^{t = y^0 - r} = e\frac{y^i}{r^2}.$$

$r := \sqrt{(y^1)^2 + (y^2)^2}$ is the distance to the charge. Having performed Poincaré transformation being the combination of translation and Lorentz transformation we find out the field generated by uniformly moving charge:

$$F_{\alpha\beta} = e\frac{u_\alpha k_\beta - u_\beta k_\alpha}{r}. \tag{A.2}$$

Here $r = -\eta_{\alpha\beta}(y^\alpha - z^\alpha(s))u^\beta$ is the retarded distance where particle's position $z^\alpha(\tau) = z_0^\alpha + u^\alpha\tau$ is referred to the retarded instant of time s. We denote k^α the null-vector $y^\alpha - z^\alpha(s)$ rescaled by the retarded distance, i.e.

$$k^\alpha = \frac{y^\alpha - z^\alpha(s)}{r}. \tag{A.3}$$

It is straightforward to substitute (A.2) in (4.2) to calculate the electromagnetic field's stress-energy tensor:

$$2\pi T^{\alpha\beta} = \frac{e^2}{r^2}\left(u^\alpha k^\beta + u^\beta k^\alpha - k^\alpha k^\beta + \frac{1}{2}\eta^{\alpha\beta}\right). \tag{A.4}$$

Now we calculate the electromagnetic field momentum

$$p_{\text{em}}^\nu(t) = \int_{\Sigma_t} d\sigma_\mu T^{\mu\nu} \tag{A.5}$$

where an integration surface Σ_t is a surface of constant y^0. We start with coordinate transformation $(y^0, y^1, y^2) \mapsto (r, s, \varphi)$ locally given by

$$y^\alpha = z^\alpha(s) + rk^\alpha, \qquad k^\alpha = \Lambda^\alpha{}_{\alpha'}n^{\alpha'} \tag{A.6}$$

where $n^{\alpha'} = (1, \cos\varphi, \sin\varphi)$. The Lorentz matrix Λ determines the transformation to the particle's comoving Lorentz frame where the particle is at rest. To adopt these curvilinear coordinates to the integration surface $\Sigma_t = \{y \in \mathbb{M}_3 : y^0 = t\}$ we replace the retarded distance r by the expression

$$r = \frac{t - s}{k^0} \tag{A.7}$$

where t is the observation time. On rearrangement, the final coordinate transformation $(y^0, y^1, y^2) \mapsto (t, s, \varphi)$ looks as follows:

$$y^0 = t, \qquad y^i = z^i(s) + (t - s)\frac{k^i}{k^0}. \tag{A.8}$$

Differentiation of this coordinate transformation yields differential chart

$$\vec{e}_t = \frac{1}{k^0}\left(k^0 \vec{e}_0 + k^i \vec{e}_i\right) \tag{A.9}$$

$$\vec{e}_s = \left(v^i - \frac{k^i}{k^0}\right)\vec{e}_i$$

$$\vec{e}_\varphi = (t - s)\left(\frac{k_\varphi^i}{k^0} - \frac{k^i}{(k^0)^2}k_\varphi^0\right)\vec{e}_i$$

where $k_\varphi^\alpha = \partial k^\alpha/\partial\varphi$ and $v^i = \gamma^{-1}u^i$. Their scalar products are the components of metric tensor g of Minkowski space $\mathbb{M}_3$ as it is viewed in curvilinear coordinates (A.8). To calculate the determinant of g it is sufficient to know that

$$g_{tt} = 0, \quad g_{t\varphi} = 0, \quad g_{ts} = -\frac{\gamma^{-1}}{k^0}, \quad g_{\varphi\varphi} = \frac{(t - s)^2}{(k^0)^2}. \tag{A.10}$$

The surface element is given by $d\sigma_0 = \sqrt{-g}dsd\varphi$ where

$$\sqrt{-g} = \gamma^{-1}\frac{t - s}{(k^0)^2} \tag{A.11}$$

is the Jacobian of coordinate transformation (A.8). Electromagnetic field momentum (A.5) takes the form:

$$p_{\text{em}}^\beta = \frac{e^2}{2\pi} \int\limits_{-\infty}^{t} ds \int\limits_{0}^{2\pi} d\varphi \frac{\gamma^{-1}}{t - s}\left(u^0 k^\beta + u^\beta k^0 - k^0 k^\beta + (1/2)\eta^{0\beta}\right). \tag{A.12}$$

The angular integration can be handled via the relations

$$\int\limits_{0}^{2\pi} d\varphi k^\alpha = 2\pi u^\alpha, \qquad \int\limits_{0}^{2\pi} d\varphi k^\alpha k^\beta = 3\pi u^\alpha u^\beta + \pi\eta^{\alpha\beta}. \tag{A.13}$$

After trivial calculations we arrive at the logarithmic divergence

$$p_{\text{em}}^\beta = \frac{e^2}{2} \int\limits_{-\infty}^{t} ds \frac{u^\beta}{t - s} \tag{A.14}$$

as could be expected for the two-dimensional Coulomb potential.

Appendix B. Coordinate system

Coordinate transformation (4.1.3) is associated with two points, $z(t_1)$ and $z(t_2)$, on an accelerated world line ζ (see figure 6). Its differentiation yields differential chart

$$\vec{e}_t = \vec{e}_0 - q\frac{\partial \beta}{\partial t}\vec{e}_{1'}$$

$$\vec{e}_R = n^{j'}\vec{e}_{j'} - q\frac{\partial \beta}{\partial R}\vec{e}_{1'}$$

$$\vec{e}_\varphi = s n_\varphi^{j'}\vec{e}_{j'} \tag{B.1}$$

where $\mathbf{n} = (\cos\varphi, \sin\varphi)$, $\mathbf{n}_\varphi = (-\sin\varphi, \cos\varphi)$ and $\vec{e}_{j'} = \vec{e}_i \omega^i{}_{j'}$. Orthogonal matrix

$$\omega = \begin{pmatrix} n_q^1 & -n_q^2 \\ n_q^2 & n_q^1 \end{pmatrix} \tag{B.2}$$

where $n_q^i = q^i/q$ rotates space axes till new $x-$axis be directed along two-vector $\mathbf{q} := \mathbf{z}(t_1) - \mathbf{z}(t_2)$ (we denote $q := \sqrt{\mathbf{q}^2}$). In new coordinates three-vectors $K_a = y - z(t_a), a = 1, 2$, pointing from points of emanation $z(t_a) := z_a$ to an observation point $y \in \Sigma_t$ (see figure 6) have the following form:

$$K_a^0 = t - t_a := k_a^0, \qquad K_a^i = \omega^i{}_j k_a^j \tag{B.3}$$

where

$$k_1^1 = -\beta q + R\cos\varphi, \qquad k_1^2 = R\sin\varphi \tag{B.4}$$
$$k_2^1 = \alpha q + R\cos\varphi, \qquad k_2^2 = R\sin\varphi.$$

The norms $\|K_a\| = \sqrt{-(K_a \cdot K_a)}$ of the separation vectors K_1 and K_2 pictured in figure 6 contain the angular variable:

$$-(K_1 \cdot K_1) = (k_1^0)^2 - \beta^2\mathbf{q}^2 + 2\beta qR\cos\varphi - R^2,$$
$$-(K_2 \cdot K_2) = (k_2^0)^2 - \alpha^2\mathbf{q}^2 - 2\alpha qR\cos\varphi - R^2. \tag{B.5}$$

Substituting of the right-hand side of eq.(4.2.2) for R^2 in these expressions and comparing them with Jacobian (4.2.5) leads to the important relations:

$$\|K_1\|^2 = -2\beta J, \qquad \|K_2\|^2 = 2\alpha J. \tag{B.6}$$

They immediately follow:

$$\frac{J}{\sqrt{-(K_1 \cdot K_1)}\sqrt{-(K_2 \cdot K_2)}} = \frac{1}{2\sqrt{-\beta\alpha}}. \tag{B.7}$$
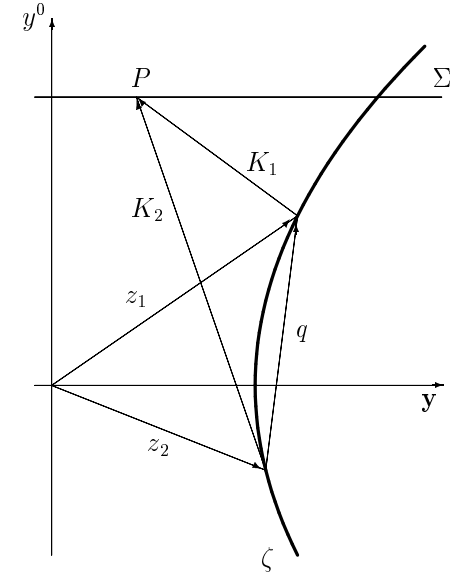
Figure 6. The separation vector K_a is a vector pointing from point of emission $z(t_a) := z_a$ to point of observation $P \in \Sigma_t$ with coordinates (y^0, y^1, y^2). The integrand $t^{0\alpha}$ involves also one-half of square of three-vector $q := z_1 - z_2 = K_2 - K_1$ (double Synge's function), and its partial derivatives with respect to time variables. Space components of q determine the orthogonal matrix (B.2).

To concretely compute integrals over β we clarify mathematical sense of this parameter. Since eq.(B.6), we parametrize the ratio $\|K_1\|/\|K_2\|$ by angle variable ϑ:

$$\sin\vartheta = \sqrt{\frac{-\beta}{\alpha}}. \tag{B.8}$$

Having inserted it in the expression (4.2.2) we obtain algebraic equation on $\sin^2\vartheta$. Its solution is as follows

$$\cos 2\vartheta = \frac{-(k_1^0)^2 + \mathbf{q}^2 + R^2 + \sqrt{D}}{(k_2^0)^2 - R^2} \tag{B.9}$$

where D is defined by eq.(4.2.3). If $R = k_1^0$ then $\vartheta = 0$ and parameter β vanishes. If $R = 0$ we obtain the lower limit of integral over β. We denote it $\beta_0 = -\tan^2\vartheta_0$ where

$$\sin\vartheta_0 = \frac{1}{2k_2^0}\left(\sqrt{2\Sigma} - \sqrt{2\sigma}\right) \tag{B.10}$$

and

$$\Sigma = \frac{1}{2}\left(k_2^0 + k_1^0\right)^2 - \frac{1}{2}\mathbf{q}^2, \qquad \sigma = \frac{1}{2}\left(k_2^0 - k_1^0\right)^2 - \frac{1}{2}\mathbf{q}^2. \qquad (B.11)$$

Appendix C. Integration over angular variable

To calculate the energy-momentum (4.1.9) carried by electromagnetic field we should perform the integration over angle firstly. When facing this problem it is convenient to mark out φ-dependent terms in expressions under the integral sign. In the Maxwell energy-momentum tensor density we distinguish the second-order differential operator $\hat{T}_a$ with φ-dependent coefficients (see eq.(4.2.9)). Having integrated this operator over φ we obtain the operator $\hat{\mathcal{T}}_a$ which can be decomposed into combination of partial derivatives in time variables $\hat{\Pi}_a$ given by eq.(4.2.12) and "tail" π_a of type (4.2.13).

This Appendix concerned with the computation of the "tails". Equipped with them we express the integrand as combination of partial derivatives in t_1 and t_2. It allows us to integrate the electromagnetic field's stress-energy tensor over the time variables as well as over β.

To implement this strategy we must first integrate the coefficients (4.2.10) over the angle variable. We start with the simplest one

$$\mathcal{D}^a = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{a}{r_1 r_2} \qquad (C.1)$$

where φ-dependent numerator a is equal to Jacobian (4.2.5) or to 1. Our task is rewrite the integrand as a sum of term with denominator r_1 and term with denominator r_2. To do it I introduce a new layer of mathematical formalism and develop convenient technique.

We introduce null-vector $n = (1, \cos\varphi, \sin\varphi)$ which belongs to the vector space, say V, such that $\mathbf{i}_0$, $\mathbf{i}_1$, and $\mathbf{i}_2$ is its linear basis. We shall use $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1)$ and its inverse $\eta^{\alpha\beta} = \text{diag}(-1, 1, 1)$ to lower and raise indices, respectively. We introduce the pairing

$$(\cdot) : V \times V \to \mathbb{R} \qquad (C.2)$$
$$(\mathsf{a} \cdot \mathsf{b}) \mapsto \eta_{\alpha\beta}\mathsf{a}^\alpha\mathsf{b}^\beta \qquad (C.3)$$

which will be called the "scalar product".

We express the φ-dependent constructions

$$r_a = -(K_a \cdot v_a), \qquad c_a = \gamma_a^{-2} + (K_a \cdot \dot{v}_a). \qquad (C.4)$$

as the scalar products $-(\mathsf{r}_a \cdot n)$ and $(\mathsf{c}_a \cdot n)$, respectively. We shall use sans-serif symbols for the components of time-like three-vectors $\mathsf{r}_a \in V$ and $\mathsf{c}_a \in V$

$$\mathsf{r}_1^0 = k_1^0 + \beta(\mathbf{v}_1\mathbf{q}), \qquad \mathsf{r}_2^0 = k_2^0 - \alpha(\mathbf{v}_2\mathbf{q}), \qquad \mathsf{r}_a^j = Rv_a{}^i\omega_i{}^j; \qquad (C.5)$$
$$\mathsf{c}_1^0 = -\gamma_1^{-2} + \beta(\dot{\mathbf{v}}_1\mathbf{q}), \qquad \mathsf{c}_2^0 = -\gamma_2^{-2} - \alpha(\dot{\mathbf{v}}_2\mathbf{q}), \qquad \mathsf{c}_a^j = R\dot{v}_a{}^i\omega_i{}^j$$

Jacobian (4.2.5) becomes the scalar product $(\mathsf{J}\cdot n) = \mathsf{J}_0 + \mathsf{J}_1\cos\varphi + \mathsf{J}_2\sin\varphi$ where

$$\mathsf{J}_0 = \beta\mathbf{q}^2 + (1/2)\left[(k_2^0)^2 - (k_1^0)^2 - \mathbf{q}^2\right], \quad \mathsf{J}_1 = -qR, \quad \mathsf{J}_2 = 0. \quad (C.6)$$

We introduce the dual space of one-forms, say W, with basis $\hat{\omega}^0, \hat{\omega}^1$ and $\hat{\omega}^2$ such that $\hat{\omega}^\mu(\mathbf{i}_\nu) = \delta_\nu^\mu$ where $\mathbf{i}_0, \mathbf{i}_1, \mathbf{i}_2$ constitute the basis of V. Wedge product $\hat{L} = \hat{a} \wedge \hat{b}$ of two one forms $\hat{a}$ and $\hat{b}$ constitutes two-form

$$\hat{L} = (a_0b_1 - a_1b_0)\,\hat{\omega}^0 \wedge \hat{\omega}^1 + (a_0b_2 - a_2b_0)\,\hat{\omega}^0 \wedge \hat{\omega}^2 + (a_1b_2 - a_2b_1)\,\hat{\omega}^1 \wedge \hat{\omega}^2. \qquad (C.7)$$

We introduce dual three-vector $\mathsf{L} = {}^*\hat{L}$ with components

$$\mathsf{L}^\alpha = \frac{1}{2!}\varepsilon^{\alpha\beta\gamma}L_{\beta\gamma} \qquad (C.8)$$
$$= \varepsilon^{\alpha\beta\gamma}a_\beta b_\gamma.$$

$\varepsilon^{\alpha\beta\gamma}$ denotes the Ricci symbol in three dimensions:

$$\varepsilon^{\alpha\beta\gamma} = \begin{cases} 1 & \text{when} & \alpha\beta\gamma \text{ even permutation } 0, 1, 2 \\ -1 & \text{when} & \alpha\beta\gamma \text{ odd permutation } 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases} \qquad (C.9)$$

We raise indices in eq.(C.8) and define the vector product of two vectors, a and b:

$$\mathsf{L}^\alpha = \varepsilon^\alpha{}_{\mu\nu}\mathsf{a}^\mu\mathsf{b}^\nu. \qquad (C.10)$$

Tensor

$$\varepsilon^\alpha{}_{\mu\nu} = \varepsilon^{\alpha\beta\gamma}\eta_{\beta\mu}\eta_{\gamma\nu} \qquad (C.11)$$

has the components

$$\varepsilon^0{}_{\mu\nu} = \varepsilon^{0\mu\nu}, \qquad \varepsilon^1{}_{\mu\nu} = -\varepsilon^{1\mu\nu}, \qquad \varepsilon^2{}_{\mu\nu} = -\varepsilon^{2\mu\nu}. \qquad (C.12)$$

It is interesting that tensor

$$\varepsilon_{\lambda\mu\nu} = \varepsilon^{\alpha\beta\gamma}\eta_{\alpha\lambda}\eta_{\beta\mu}\eta_{\gamma\nu} \qquad (C.13)$$

is equal to $\varepsilon^{\lambda\mu\nu}$ taken with opposite sign.

Now we calculate the double vector product

$$[\mathsf{A}[\mathsf{BC}]]^\alpha = \varepsilon^\alpha{}_{\beta\gamma}\mathsf{A}^\beta \varepsilon^\gamma{}_{\mu\nu}\mathsf{B}^\mu \mathsf{C}^\nu. \qquad (C.14)$$

Since

$$\varepsilon^\alpha{}_{\beta\gamma}\varepsilon^\gamma{}_{\mu\nu} = -\delta^\alpha{}_\mu \eta_{\beta\nu} + \delta^\alpha{}_\nu \eta_{\beta\mu} \qquad (C.15)$$

we arrive to the unusual rule

$$[\mathsf{A}[\mathsf{BC}]] = -\mathsf{B}(\mathsf{A}\cdot\mathsf{C}) + \mathsf{C}(\mathsf{A}\cdot\mathsf{B}) \qquad (C.16)$$

instead of the well-known law acting in space with Euclidean metric.

To simplify denominator $r_1 r_2$ in the integrand of eq.(C.1) as much as possible we rewrite 2π-periodic functions (C.4) as follows:

$$r_a = -\mathsf{r}_{a,0} - \rho_a \sin(\varphi + \phi_a), \qquad \rho_a = \sqrt{\mathsf{r}_{a,1}^2 + \mathsf{r}_{a,2}^2}. \qquad (C.17)$$

(We recall that r_a is the scalar products $(\mathsf{r}_a \cdot n)$ taken with opposite sign, components r_a^μ are given by eqs.(C.5).) Shift in argument of a-th function is determined by the relations

$$\mathsf{r}_{a,1} = \rho_a \sin\phi_a, \qquad \mathsf{r}_{a,2} = \rho_a \cos\phi_a. \qquad (C.18)$$

After some algebra we rewrite the integrand of eq.(C.1) as the following sum

$$\frac{a}{r_1 r_2} = \frac{A_{12}^a - C_{12}^a \rho_1 \cos(\varphi + \phi_1)}{r_1} + \frac{A_{21}^a + C_{12}^a \rho_2 \cos(\varphi + \phi_2)}{r_2}. \qquad (C.19)$$

Coefficients A_{12}^a, A_{21}^a and C_{12}^a satisfy the vector equation

$$-A_{12}^a \mathsf{r}_2 - A_{21}^a \mathsf{r}_1 + C_{12}^a \mathsf{L}_{12} = \mathsf{a} \qquad (C.20)$$

where by r_1 and r_2 we mean three-vectors with components (C.5) and $\mathsf{L}_{12} = [\mathsf{r}_1 \mathsf{r}_2]$.

To solve the equation (C.20) we postmultiply it on the vector product $[\mathsf{r}_1 \mathsf{L}_{12}]$, then on the vector product $[\mathsf{r}_2 \mathsf{L}_{21}]$ and, finally, on L_{12}. After some algebra we obtain

$$A_{12}^a = \frac{([\mathsf{a}\mathsf{r}_1]\cdot\mathsf{L}_{12})}{D_{12}}, \qquad A_{21}^a = \frac{([\mathsf{a}\mathsf{r}_2]\cdot\mathsf{L}_{21})}{D_{21}}, \qquad C_{12}^a = \frac{(\mathsf{a}\cdot\mathsf{L}_{12})}{D_{12}} \quad (C.21)$$

where denominator $D_{12} = (\mathsf{L}_{12}\cdot\mathsf{L}_{12})$ is symmetric in its indices.

Substituting (C.19) into eq.(C.1) and using the identities

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\varphi}{\mathsf{r}_a^0 - \rho_a \sin(\varphi + \phi_a)} = \frac{1}{\|\mathsf{r}_a\|} \qquad (C.22)$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{\cos(\varphi + \phi_a)}{\mathsf{r}_a^0 - \rho_a \sin(\varphi + \phi_a)} = 0$$

yields

$$\mathcal{D}^a = \frac{A_{12}^a}{\|\mathsf{r}_1\|} + \frac{A_{21}^a}{\|\mathsf{r}_2\|} \qquad (C.23)$$

after integration over φ.

Now we turn to the calculation of the coefficient

$$\mathcal{B}^a = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{ac_2}{r_1(r_2)^2}. \qquad (C.24)$$

Equipped with the relations (C.19) we rewrite the integrand as a sum of terms which are proportional to the $1/r_1$, $1/r_2$ and $1/(r_2)^2$, respectively. Using the identities

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\varphi}{[\mathsf{r}_a^0 - \rho_a \sin(\varphi + \phi_a)]^2} = \frac{\mathsf{r}_a^0}{\|\mathsf{r}_a\|^3} \qquad (C.25)$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{\cos(\varphi + \phi_a)}{[\mathsf{r}_a^0 - \rho_a \sin(\varphi + \phi_a)]^2} = 0$$

and taking into account the relations (C.22) gives

$$\mathcal{B}^a = -\frac{1}{\|\mathsf{r}_2\|^3}\frac{(\mathsf{a}\cdot\mathsf{r}_2)(\mathsf{c}_2\cdot\mathsf{r}_2)}{D_{21}}(\mathsf{r}_2\cdot\mathsf{r}_1) \qquad (C.26)$$

$$+ \frac{1}{\|\mathsf{r}_2\|}\left[A_{12}^{c_2}A_{21}^a + A_{12}^a A_{21}^{c_2} - \frac{(\mathsf{a}\cdot\mathsf{c}_2)(\mathsf{r}_1\cdot\mathsf{r}_2)}{D_{21}}\right]$$

$$+ \frac{1}{\|\mathsf{r}_1\|}\left[2A_{12}^a A_{12}^{c_2} - \frac{([\mathsf{a}\mathsf{r}_1]\cdot[\mathsf{c}_2\mathsf{r}_1])}{D_{12}}\right].$$

The resulting expression for the term

$$\mathcal{C}^a = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{ac_1}{(r_1)^2 r_2} \qquad (C.27)$$

can be obtained by interchanging of indices the "first" and the "second" in the right-hand side of eq.(C.26).

After a routine computation based on the repeated usage of the relation (C.19) we find out the last term

$$\mathcal{A}^a = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{ac_1 c_2}{(r_1)^2 (r_2)^2} \tag{C.28}$$

$$= \frac{B_{12}}{\|\mathsf{r}_1\|} + \frac{B_{21}}{\|\mathsf{r}_2\|} + J_1 \frac{(\mathsf{a} \cdot \mathsf{r}_1)}{\|\mathsf{r}_1\|^3} + J_2 \frac{(\mathsf{a} \cdot \mathsf{r}_2)}{\|\mathsf{r}_2\|^3}$$

where

$$J_1 = 2A_{12}^{c_1} A_{12}^{c_2} - \frac{([\mathsf{c}_1 \mathsf{r}_1] \cdot [\mathsf{c}_2 \mathsf{r}_2])}{D_{12}} \tag{C.29}$$

$$B_{12} = 3A_{12}^a A_{12}^{c_1} A_{21}^{c_2} + 3A_{12}^a A_{12}^{c_2} A_{21}^{c_1} + 2A_{12}^{c_1} A_{12}^{c_2} A_{21}^a$$
$$+ A_{12}^a \left\{ \frac{([\mathsf{c}_1 \mathsf{r}_1] \cdot [\mathsf{c}_2 \mathsf{r}_2])}{D_{12}} + \frac{([\mathsf{c}_1 \mathsf{r}_2] \cdot [\mathsf{c}_2 \mathsf{r}_1])}{D_{12}} \right\}$$
$$- A_{21}^{c_1} \frac{([\mathsf{c}_2 \mathsf{r}_1] \cdot [\mathsf{a} \mathsf{r}_1])}{D_{12}} - A_{21}^{c_2} \frac{([\mathsf{c}_1 \mathsf{r}_1] \cdot [\mathsf{a} \mathsf{r}_1])}{D_{12}}$$
$$+ A_{12}^{c_1} \frac{([\mathsf{c}_2 \mathsf{r}_1] \cdot [\mathsf{a} \mathsf{r}_2])}{D_{12}} + A_{12}^{c_2} \frac{([\mathsf{c}_1 \mathsf{r}_1] \cdot [\mathsf{a} \mathsf{r}_2])}{D_{12}}$$

and the others, B_{21} and J_2, can be obtained by interchanging of indices 1 and 2.

We now turn to the differentiation of coefficient (C.23) with respect to time variables t_1 and t_2. We will use Latin indices a and b which run from 1 to 2 ($a \neq b$). We introduce new denotations $\kappa_1 = \alpha$ and $\kappa_2 = \beta$ for time-independent variables β and $\alpha = 1 - \beta$. Differentiation of $\mathcal{D}^a$ is based on the relations obtained via differentiation of zeroth components (C.5), (C.6) and square of radius (4.2.2):

$$\frac{\partial \mathsf{r}_a^0}{\partial t_a} = \mathsf{c}_a^0 - \kappa_a \mathbf{v}_a^2, \qquad \frac{\partial \mathsf{r}_a^0}{\partial t_b} = -\kappa_b (\mathbf{v}_a \mathbf{v}_b)$$

$$\frac{\partial \mathsf{J}^0}{\partial t_a} = -(-1)^a \mathsf{r}_a^0 - \kappa_a (\mathbf{v}_a \mathbf{q}), \qquad \frac{\partial R^2}{\partial t_a} = -2\kappa_a \mathsf{r}_a^0.$$

They immediately give

$$\frac{\partial (\mathsf{r}_a \cdot \mathsf{r}_a)}{\partial t_a} = 2(\mathsf{r}_a \cdot \mathsf{c}_a), \quad \frac{\partial (\mathsf{r}_a \cdot \mathsf{r}_a)}{\partial t_b} = \frac{2\kappa_b}{R^2} \left[\mathsf{r}_a^0 (\mathsf{r}_a \cdot \mathsf{r}_b) - \mathsf{r}_b^0 (\mathsf{r}_a \cdot \mathsf{r}_b) \right]$$

$$\frac{\partial (\mathsf{r}_b \cdot \mathsf{r}_a)}{\partial t_b} = (\mathsf{r}_a \cdot \mathsf{c}_b) + \frac{\kappa_b}{R^2} \left[\mathsf{r}_a^0 (\mathsf{r}_b \cdot \mathsf{r}_b) - \mathsf{r}_b^0 (\mathsf{r}_a \cdot \mathsf{r}_b) \right] \tag{C.30}$$

and eventually gives

$$\frac{\partial}{\partial t_a} \left(\frac{1}{\|\mathsf{r}_a\|} \right) = \frac{(\mathsf{c}_a \cdot \mathsf{r}_a)}{\|\mathsf{r}_a\|^3}, \quad \frac{\partial}{\partial t_b} \left(\frac{1}{\|\mathsf{r}_a\|} \right) = \frac{\kappa_b}{\|\mathsf{r}_a\|^3} \frac{\mathsf{r}_a^0 (\mathsf{r}_a \cdot \mathsf{r}_b) - \mathsf{r}_b^0 (\mathsf{r}_a \cdot \mathsf{r}_a)}{R^2}. \tag{C.31}$$

After some algebra we also obtain

$$\frac{\partial (\mathsf{J} \cdot \mathsf{r}_a)}{\partial t_a} = (-1)^a (\mathsf{r}_a \cdot \mathsf{r}_a) + (\mathsf{J} \cdot \mathsf{c}_a) + \frac{\kappa_a}{R^2} \left[\mathsf{J}^0 (\mathsf{r}_a \cdot \mathsf{r}_a) - \mathsf{r}_a^0 (\mathsf{J} \cdot \mathsf{r}_a) \right]$$

$$\frac{\partial (\mathsf{J} \cdot \mathsf{r}_a)}{\partial t_b} = (-1)^b (\mathsf{r}_b \cdot \mathsf{r}_a) + \frac{\kappa_b}{R^2} \left[\mathsf{J}^0 (\mathsf{r}_a \cdot \mathsf{r}_b) + \mathsf{r}_a^0 (\mathsf{J} \cdot \mathsf{r}_b) - 2\mathsf{r}_b^0 (\mathsf{J} \cdot \mathsf{r}_a) \right].$$

Usage of these relations allows us to calculate the following derivatives

$$\frac{\partial A_{ab}^J}{\partial t_a} = A_{ab}^J A_{ba}^{c_a} + A_{ab}^{c_a} A_{ba}^J + \frac{([\mathsf{J} \mathsf{r}_a] \cdot [\mathsf{c}_a \mathsf{r}_b])}{D_{ab}} \tag{C.32}$$

$$\frac{\partial A_{ab}^J}{\partial t_b} = a - b + 2A_{ab}^{c_b} A_{ab}^J - \frac{([\mathsf{J} \mathsf{r}_a] \cdot [\mathsf{c}_b \mathsf{r}_a])}{D_{ab}} - \frac{\kappa_b}{R^2} \left(\mathsf{J}^0 + \mathsf{r}_2^0 A_{12}^J + \mathsf{r}_1^0 A_{21}^J \right)$$

where latin indices a and b run from 1 to 2, $a \neq b$.

Having differentiated eq.(C.23), after a straightforward calculations we derive the following relations

$$\frac{\partial \mathcal{D}^J}{\partial t_1} = \mathcal{C}^J - \frac{\partial}{\partial \beta} \left(\frac{\alpha}{\|\mathsf{r}_2\|} \right), \qquad \frac{\partial \mathcal{D}^J}{\partial t_2} = \mathcal{B}^J - \frac{\partial}{\partial \beta} \left(\frac{\beta}{\|\mathsf{r}_1\|} \right). \tag{C.33}$$

Further we find out the expression $\partial \mathcal{C}^J / \partial t_2$ and prove the identity

$$\mathcal{A}^J - \frac{\partial \mathcal{C}^J}{\partial t_2} = \frac{\partial}{\partial t_1} \left(\mathcal{B}^J - \frac{\partial \mathcal{D}^J}{\partial t_2} \right) \quad \text{i.e.} \quad \pi^J = 0. \tag{C.34}$$

(One can derive $\partial \mathcal{B}^J / \partial t_1$, subtract it from $\mathcal{A}^J$ and compare the result with $\partial / \partial t_2 (\mathcal{C}^J - \partial \mathcal{D}^J / \partial t_1)$.)

Similarly we derive analogous equality where Jacobian J with components (C.6) is replaced by unit three-vector $\mathsf{o} = (-1, 0, 0)$. Having substituted o for a in the expressions (C.23), (C.26) and (C.28) we obtain the terms $\mathcal{D}^0$, $\mathcal{B}^0$ and $\mathcal{A}^0$, respectively. The remaining term, $\mathcal{C}^0$, can be obtained from $\mathcal{B}^0$ via reciprocity. The derivatives of coefficients A_{ab}^0 are as follows:

$$\frac{\partial A_{ab}^0}{\partial t_a} = A_{ab}^0 A_{ba}^{c_a} + A_{ab}^{c_a} A_{ba}^0 + \frac{([\mathsf{o} \mathsf{r}_a] \cdot [\mathsf{c}_a \mathsf{r}_b])}{D_{ab}} \tag{C.35}$$

$$\frac{\partial A_{ab}^0}{\partial t_b} = 2A_{ab}^{c_b} A_{ab}^0 - \frac{([\mathsf{o} \mathsf{r}_a] \cdot [\mathsf{c}_b \mathsf{r}_a])}{D_{ab}} - \frac{\kappa_b}{R^2} \left(\mathsf{o}^0 + \mathsf{r}_2^0 A_{12}^0 + \mathsf{r}_1^0 A_{21}^0 \right).$$

Substituting these into equality

$$\frac{\partial \mathcal{D}^0}{\partial t_a} = \frac{\partial}{\partial t_a} \left(\frac{A_{12}^0}{\|r_1\|} + \frac{A_{21}^0}{\|r_2\|} \right) \tag{C.36}$$

and using the identities (C.31) yields

$$\frac{\partial \mathcal{D}^0}{\partial t_1} = \mathcal{C}^0 - \frac{\alpha \mathbf{v}_2^2}{\|r_2\|^3}, \qquad \frac{\partial \mathcal{D}^0}{\partial t_2} = \mathcal{B}^0 - \frac{\beta \mathbf{v}_1^2}{\|r_1\|^3}. \tag{C.37}$$

And, finally, we calculate the partial derivative $\partial \mathcal{C}^0/\partial t_2$, subtract it from $\mathcal{A}^0$ and compare the result with $\partial/\partial t_1(\mathcal{B}^0 - \partial \mathcal{D}^0/\partial t_2)$. We obtain

$$\mathcal{A}^0 - \frac{\partial \mathcal{B}^0}{\partial t_1} - \frac{\partial \mathcal{C}^0}{\partial t_2} + \frac{\partial^2 \mathcal{D}^0}{\partial t_1 \partial t_2} = 0 \quad \text{i.e.} \quad \pi^0 = 0. \tag{C.38}$$

Now, we calculate the "tail"

$$\pi_a^\alpha = \mathcal{A}_a^\alpha - \frac{\partial \mathcal{B}_a^\alpha}{\partial t_1} - \frac{\partial \mathcal{C}_a^\alpha}{\partial t_2} + \frac{\partial^2 \mathcal{D}_a^\alpha}{\partial t_1 \partial t_2} \tag{C.39}$$

where

$$\mathcal{D}_a^\alpha = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{K_a^\alpha}{r_1 r_2}, \qquad \mathcal{B}_a^\alpha = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{K_a^\alpha c_2}{r_1 (r_2)^2} \tag{C.40}$$

$$\mathcal{C}_a^\alpha = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{K_a^\alpha c_1}{(r_1)^2 r_2}, \qquad \mathcal{A}_a^\alpha = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{K_a^\alpha c_1 c_2}{(r_1)^2 (r_2)^2}.$$

Zeroth component, $K_a^0 = k_a^0$, does not depend on φ. Inserting relations $\mathcal{D}_a^0 = k_a^0 \mathcal{D}^0$, $\mathcal{B}_a^0 = k_a^0 \mathcal{B}^0$, $\mathcal{C}_a^0 = k_a^0 \mathcal{C}^0$, and $\mathcal{A}_a^0 = k_a^0 \mathcal{A}^0$ into eq.(C.39) and taking into account identity (C.38) yields

$$\pi_1^0 = \mathcal{B}^0 - \frac{\partial \mathcal{D}^0}{\partial t_2}, \quad \pi_2^0 = \mathcal{C}^0 - \frac{\partial \mathcal{D}^0}{\partial t_1}. \tag{C.41}$$

Space components, K_a^i, depend on φ. They can be expressed as the scalar product $(K_a^i \cdot n)$ where components of three-vectors $K_a^i \in V$ are as follows:

$$K_{1,0}^i = -\beta q^i, \quad K_{2,0}^i = \alpha q^i, \quad K_{a,1}^i = R\omega^i{}_1, \quad K_{a,2}^i = R\omega^i{}_2, \tag{C.42}$$

where $\omega^i{}_j$ are components of orthogonal matrix (B.2). Having substituted K_a^i for a in the expressions (C.23), (C.26) and (C.28) we obtain the

terms $\mathcal{D}_a^i$, $\mathcal{B}_a^i$ and $\mathcal{A}_a^i$, respectively. The last term, $\mathcal{C}_a^i$, can be obtained from $\mathcal{B}_a^i$ via reciprocity. To differentiate them we need the equalities

$$\frac{\partial (K_a^i \cdot r_a)}{\partial t_a} = (K_a^i \cdot c_a) - v_a^i r_a^0 - \frac{\kappa_a}{R^2} \left[K_{a,0}^i (r_a \cdot r_a) + r_a^0 (K_a^i \cdot r_a) \right]$$

$$\frac{\partial (K_a^i \cdot r_b)}{\partial t_a} = -v_a^i r_b^0 + \frac{\kappa_a}{R^2} \left[r_b^0 (K_a^i \cdot r_a) - K_{a,0}^i (r_b \cdot r_a) - 2r_a^0 (K_a^i \cdot r_b) \right]$$

$$\frac{\partial (K_a^i \cdot r_b)}{\partial t_b} = (K_a^i \cdot c_b) - \frac{\kappa_b}{R^2} \left[K_{a,0}^i (r_b \cdot r_b) + r_b^0 (K_a^i \cdot r_b) \right]$$

$$\frac{\partial (K_a^i \cdot r_a)}{\partial t_b} = \frac{\kappa_b}{R^2} \left[r_a^0 (K_a^i \cdot r_b) - K_{a,0}^i (r_b \cdot r_a) - 2r_b^0 (K_a^i \cdot r_a) \right]$$

in addition to eqs.(C.30) and (C.31).

The derivation of equalities

$$\mathcal{C}_1^i - \frac{\partial \mathcal{D}_1^i}{\partial t_1} = v_1^i \mathcal{D}^0 + \frac{\alpha}{\|r_2\|^3} \left(-\beta q^i \mathbf{v}_2^2 + r_2^0 v_2^i \right) \tag{C.43}$$

$$\mathcal{B}_1^i - \frac{\partial \mathcal{D}_1^i}{\partial t_2} = \frac{\beta}{\|r_1\|^3} \left(-\beta q^i \mathbf{v}_1^2 + r_1^0 v_1^i \right)$$

$$\mathcal{C}_2^i - \frac{\partial \mathcal{D}_2^i}{\partial t_1} = \frac{\alpha}{\|r_2\|^3} \left(\alpha q^i \mathbf{v}_2^2 + r_2^0 v_2^i \right)$$

$$\mathcal{B}_2^i - \frac{\partial \mathcal{D}_2^i}{\partial t_2} = v_2^i \mathcal{D}^0 + \frac{\beta}{\|r_1\|^3} \left(\alpha q^i \mathbf{v}_1^2 + r_1^0 v_1^i \right)$$

is virtually identical to that presented above, and we shall not bother with the detail. Finally, after a straightforward (but fairly lengthy) calculations we derive the following relations

$$\pi_1^i = v_1^i \left(\mathcal{B}^0 - \frac{\partial \mathcal{D}^0}{\partial t_2} \right), \qquad \pi_2^i = v_2^i \left(\mathcal{C}^0 - \frac{\partial \mathcal{D}^0}{\partial t_1} \right) \tag{C.44}$$

which generalize eqs.(C.41).

In analogous way one can derive the equalities

$$\pi_1^{iJ} = v_1^i \left(\mathcal{B}^J - \frac{\partial \mathcal{D}^J}{\partial t_2} \right), \qquad \pi_2^{iJ} = v_2^i \left(\mathcal{C}^J - \frac{\partial \mathcal{D}^J}{\partial t_1} \right) \tag{C.45}$$

which arise in $\varphi-$integration of angular momentum carried by the electromagnetic field.

We will need also the "tail"

$$\pi_{12}^{\alpha\beta} = \frac{\partial^2 \mathcal{D}_{12}^{\alpha\beta}}{\partial t_1 \partial t_2} - \frac{\partial \mathcal{B}_{12}^{\alpha\beta}}{\partial t_1} - \frac{\partial \mathcal{C}_{12}^{\alpha\beta}}{\partial t_2} + \mathcal{A}_{12}^{\alpha\beta} \tag{C.46}$$

where

$$\mathcal{D}_{12}^{\alpha\beta} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{K_1^\alpha K_2^\beta}{r_1 r_2}, \qquad \mathcal{B}_{12}^{\alpha\beta} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{K_1^\alpha K_2^\beta c_2}{r_1 (r_2)^2} \quad (C.47)$$

$$\mathcal{C}_{12}^{\alpha\beta} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{K_1^\alpha K_2^\beta c_1}{(r_1)^2 r_2}, \qquad \mathcal{A}_{12}^{\alpha\beta} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{K_1^\alpha K_2^\beta c_1 c_2}{(r_1)^2 (r_2)^2}.$$

It can be obtained by means of covariant generalization of previous relations. Putting $\alpha = 0$ and $\beta = 0$ and taking into account eq.(C.38) we obtain

$$\pi_{12}^{00} = k_1^0\left(\mathcal{C}^0 - \frac{\partial\mathcal{D}^0}{\partial t_1}\right) + k_2^0\left(\mathcal{B}^0 - \frac{\partial\mathcal{D}^0}{\partial t_2}\right) + \mathcal{D}^0$$

$$= \mathcal{C}_1^0 - \frac{\partial\mathcal{D}_1^0}{\partial t_1} + \mathcal{B}_2^0 - \frac{\partial\mathcal{D}_2^0}{\partial t_2} - \mathcal{D}^0 \qquad (C.48)$$

where relations $\mathcal{D}_a^0 = k_a^0\mathcal{D}^0$, $\mathcal{B}_a^0 = k_a^0\mathcal{B}^0$, and $\mathcal{C}_a^0 = k_a^0\mathcal{C}^0$ are taken into account. If $\alpha = i$ and $\beta = 0$ we have

$$\pi_{12}^{i0} = \mathcal{C}_1^i - \frac{\partial\mathcal{D}_1^i}{\partial t_1} + k_2^0 v_1^i\left(\mathcal{B}^0 - \frac{\partial\mathcal{D}^0}{\partial t_2}\right)$$

$$= \mathcal{C}_1^i - \frac{\partial\mathcal{D}_1^i}{\partial t_1} + v_1^i\left(\mathcal{B}_2^0 - \frac{\partial\mathcal{D}_2^0}{\partial t_2}\right) - v_1^i\mathcal{D}^0. \qquad (C.49)$$

If $\alpha = 0$ and $\beta = j$ we arrive at

$$\pi_{12}^{0j} = k_1^0 v_2^j\left(\mathcal{C}^0 - \frac{\partial\mathcal{D}^0}{\partial t_1}\right) + \mathcal{B}_2^j - \frac{\partial\mathcal{D}_2^j}{\partial t_2}$$

$$= v_2^j\left(\mathcal{C}_1^0 - \frac{\partial\mathcal{D}_1^0}{\partial t_1}\right) + \mathcal{B}_2^j - \frac{\partial\mathcal{D}_2^j}{\partial t_2} - v_2^j\mathcal{D}^0. \qquad (C.50)$$

An obvious generalization of expressions (C.48)-(C.50) is

$$\pi_{12}^{\alpha\beta} = v_1^\alpha\left(\mathcal{B}_2^\beta - \frac{\partial\mathcal{D}_2^\beta}{\partial t_2}\right) + v_2^\beta\left(\mathcal{C}_1^\alpha - \frac{\partial\mathcal{D}_1^\alpha}{\partial t_1}\right) - v_1^\alpha v_2^\beta\mathcal{D}^0. \qquad (C.51)$$

Appendix D. Calculation of integrals where $t_1 \to t$

In this Appendix we compute the integrals in eq.(4.3.3) where time parameter t_1 tends to the observation time t:

$$p_t^\alpha = e^2\left[\lim_{k_1^0\to 0}\int_{\beta_0}^0 d\beta G_{12}^\alpha\right]_{t_2\to-\infty}^{t_2=t} + e^2\int_{-\infty}^t dt_2 \lim_{k_1^0\to 0}\left[G_{12}^\alpha\big|_{\beta=\beta_0}\frac{\partial\beta_0}{\partial t_2}\right]$$

$$+ e^2\int_{-\infty}^t dt_2 \lim_{k_1^0\to 0}\left[\int_{\beta_0}^0 d\beta G_1^\alpha\right] \qquad (D.1)$$

Equality (B.10) implies that the lower limit β_0 tends to 0 if $k_1^0 = t - t_1$ vanishes. With a degree of accuracy sufficient for our purpose

$$\beta_0 = -\frac{(k_1^0)^2}{(k_2^0)^2 - \mathbf{q}^2(t, t_2)}. \qquad (D.2)$$

Integrals over parameter β vanishes whenever an expression under integral sign is smooth. So, we must limit our computations to the singular terms only. We expand expressions in powers of the small parameter β and then integrate them.

Let us consider contribution p_t^0 to the energy p_{em}^0 carried by electromagnetic field. Putting $k_1^0 = 0$ in two-point functions (4.2.20) we obtain

$$\kappa(t, t_2) = \nu(t, t_2)\frac{\partial\sigma}{\partial t_2} + \sigma\mu(t, t_2) \qquad (D.3)$$

$$\mu(t, t_2) = -\frac{\partial\nu(t, t_2)}{\partial t_2}$$

where

$$\nu(t, t_2) = \frac{1}{2}\left[k_2^0 + (\mathbf{q}\mathbf{v}_t)\right]. \qquad (D.4)$$

Integration of singular part of G_{12}^0 given by eq.(4.3.4) gives regular expression:

$$\lim_{t_1\to t}\int_{\beta_0}^0 d\beta G_{12}^0 = -\frac{1}{|\mathbf{v}_t|}\ln\frac{1 + |\mathbf{v}_t|}{1 - |\mathbf{v}_t|}\frac{\nu(t, t_2)}{\sqrt{2\sigma(t, t_2)}} \qquad (D.5)$$

where $\mathbf{v}_t$ denotes the particle's velocity referred to the time of observation. According to (D.1), we should take the function (D.5) at the end points, i.e. its value at the remote past $t_2 \to -\infty$ should be subtracted from its value near the observation instant t.

Similarly we calculate the integral

$$\lim_{t_1\to t}\int_{\beta_0}^0 d\beta G_1^0 = \left(2 - \frac{1}{|\mathbf{v}_t|}\ln\frac{1 + |\mathbf{v}_t|}{1 - |\mathbf{v}_t|}\right)\frac{\kappa(t, t_2)}{[2\sigma(t, t_2)]^{3/2}}$$

$$- \left(1 + \frac{1}{2|\mathbf{v}_t|}\ln\frac{1 + |\mathbf{v}_t|}{1 - |\mathbf{v}_t|}\right)\frac{\mu(t, t_2)}{\sqrt{2\sigma(t, t_2)}}. \qquad (D.6)$$

According to eq.(D.1), the result should be added to the limit

$$\lim_{t_1\to t}\left[G_{12}^0\big|_{\beta=\beta_0}\frac{\partial\beta_0}{\partial t_2}\right] = \frac{-2\kappa(t, t_2)}{[2\sigma(t, t_2)]^{3/2}} + \frac{\mu(t, t_2)}{\sqrt{2\sigma(t, t_2)}} \qquad (D.7)$$

and the sum should be integrated over t_2. After a straightforward calculations we obtain the expression (D.5), taken with opposite sign. Their

sum vanishes. Therefore p_t^0 does not contribute in the energy carried by electromagnetic field.

In analogous way we calculate contribution p_t^i to the momentum of electromagnetic field. Integration of the singular part of function (4.3.5) over β gives

$$\left[\lim_{t_1 \to t} \int_{\beta_0}^0 d\beta G_{12}^i\right]_{t_2 \to -\infty}^{t_2=t} = -\frac{1}{2|\mathbf{v}_t|} \ln \frac{1+|\mathbf{v}_t|}{1-|\mathbf{v}_t|} \left[\frac{q^i + v_t^i k_2^0}{\sqrt{2\sigma(t,t_2)}}\right]_{t_2 \to -\infty}^{t_2=t}. \tag{D.8}$$

It is the first term in eq.(D.1). The limit under the integral sign (second term) is as follows:

$$\lim_{t_1 \to t} \left[G_{12}^i\big|_{\beta=\beta_0} \frac{\partial\beta_0}{\partial t_2}\right] = -\frac{q^i + v_t^i k_2^0}{[2\sigma(t,t_2)]^{3/2}} \frac{\partial\sigma}{\partial t_2}. \tag{D.9}$$

We add it to the integral

$$\lim_{t_1 \to t} \int_{\beta_0}^0 d\beta G_1^i = \left(1 - \frac{1}{2|\mathbf{v}_t|} \ln \frac{1+|\mathbf{v}_t|}{1-|\mathbf{v}_t|}\right) \frac{q^i + v_t^i k_2^0}{[2\sigma(t,t_2)]^{3/2}} \frac{\partial\sigma}{\partial t_2}$$
$$- \frac{1}{2|\mathbf{v}_t|} \ln \frac{1+|\mathbf{v}_t|}{1-|\mathbf{v}_t|} \frac{v_2^i + v_t^i}{\sqrt{2\sigma(t,t_2)}} \tag{D.10}$$

and integrate over t_2. We arrive at the function of the end points only which annuls eq.(D.8).

Appendix E. Calculation of integrals where $t_2 \to t_1$ and $t_2 \to -\infty$

In this Appendix we compute the integrals in eq.(4.3.3) where time parameters t_1 and t_2 are equal to each other. We add also the integral evaluated at the remote past:

$$p_\triangle^\alpha(t) = -e^2 \int_{-\infty}^t dt_2 \lim_{\triangle t \to 0} \left[\int_{\beta_0}^0 d\beta \left(\frac{\partial G_{12}^\alpha}{\partial t_2} + G_1^\alpha\right)\right]_{t_1=t_2+\triangle t} \tag{E.1}$$
$$+ e^2 \int_{-\infty}^t dt_1 \lim_{\triangle t \to 0} \left[\int_{\beta_0}^0 d\beta G_2^\alpha\right]_{t_2=t_1-\triangle t}$$
$$- e^2 \int_{-\infty}^t dt_1 \lim_{t_2 \to -\infty} \int_{\beta_0}^0 d\beta G_2^\alpha.$$

For fixed instant t_1 we assume that the limit

$$A^i = \lim_{t_2 \to -\infty} \frac{q^i(t_1, t_2)}{t_1 - t_2} \tag{E.2}$$

is finite if the motion is infinite. (In specific case of finite motion $A^i=0$.) If $k_2^0 \to +\infty$ the lower limit of β-integrals

$$\beta_0 = -\left(\frac{k_1^0}{k_2^0}\right)^2 \frac{1}{1-\mathbf{A}^2} \tag{E.3}$$

tends to the upper limit (zero) and we must limit our computations to the singular terms only.

According to eq.(B.10), the equality $t_1 = t_2$ yields $\sin\vartheta_0 = 1$ and lower limit, $\beta_0 = -\tan\vartheta_0$, tends to $-\infty$. In this case the small parameter is the difference $\triangle t = t_1 - t_2$. If the instants t_1 and t_2 are close to each other, function $\beta_0(t, t_1, t_2)$ raises as $(\triangle t)^{-1}$: the product $\beta_0 \triangle t$ possesses finite limit. We expand expressions under integral sign in powers of $\triangle t$ and thereafter we integrate series. The integration over β can be handled via the relations

$$\int_{\beta_0}^0 \frac{d\beta}{\sqrt{-\beta\alpha}} = 2\ln\left(\sqrt{-\beta_0} + \sqrt{\alpha_0}\right) \tag{E.4}$$
$$\int_{\beta_0}^0 \frac{d\beta}{\sqrt{-\beta\alpha}}\beta = -\alpha_0\sqrt{\frac{-\beta_0}{\alpha_0}} + \ln\left(\sqrt{-\beta_0} + \sqrt{\alpha_0}\right)$$
$$\int_{\beta_0}^0 \frac{d\beta}{\sqrt{-\beta\alpha}}\beta^2 = -\frac{1}{2}\beta_0\alpha_0\sqrt{\frac{-\beta_0}{\alpha_0}} - \frac{3}{4}\alpha_0\sqrt{\frac{-\beta_0}{\alpha_0}} + \frac{3}{4}\ln\left(\sqrt{-\beta_0} + \sqrt{\alpha_0}\right).$$

Next we take the limit $\triangle t \to 0$. Suffice it to know that

$$\lim_{\triangle t \to 0} \triangle t \int_{\beta_0}^0 \frac{d\beta}{\sqrt{-\beta\alpha}}\beta = -\frac{k_2^0}{1+\sqrt{1-\mathbf{v}_2^2}} \tag{E.5}$$
$$\lim_{\triangle t \to 0} (\triangle t)^2 \int_{\beta_0}^0 \frac{d\beta}{\sqrt{-\beta\alpha}}\beta^2 = \frac{1}{2}\frac{(k_2^0)^2}{\left[1+\sqrt{1-\mathbf{v}_2^2}\right]^2}.$$

We begin with zeroth component. Putting eqs.(4.3.4) in eq.(E.1), after some algebra we arrive at

$$p_\triangle^0(t) = -e^2 \int_{-\infty}^t dt_2 \lim_{\triangle t \to 0} \left[\int_{\beta_0}^0 \frac{d\beta}{\sqrt{-\beta\alpha}} \left(\frac{\partial\kappa}{\partial t_2}\mathcal{D}^0 + \kappa\mathcal{B}^0\right)\right. \tag{E.6}$$
$$- \int_{\beta_0}^0 d\beta\frac{\alpha-\beta}{\sqrt{-\beta\alpha}} \left(\frac{\partial\mu}{\partial t_2}\mathcal{D}^J + \mu\mathcal{B}^J\right)\Big]_{t_1=t_2+\triangle t}$$
$$+ e^2 \int_{-\infty}^t dt_1 \left[\kappa \int_{\beta_0}^0 d\beta\sqrt{\frac{\alpha}{-\beta}}\frac{\mathbf{v}_2^2}{\|r_2\|^3}\right.$$

$$- \mu \int_{\beta_0}^0 d\beta I' \frac{\partial}{\partial \beta} \left(\frac{\alpha}{\|r_2\|} \right) \Big]_{t_2 \to -\infty}^{t_2 = t_1 - \triangle t} .$$

Recall that

$$\mathcal{D}^a = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{a}{r_1 r_2}, \qquad \mathcal{B}^a = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{a c_2}{r_1 (r_2)^2} \qquad \text{(E.7)}$$

where $a = 1$ for $\mathcal{D}^0, \mathcal{B}^0$ and a is equal to Jacobian (4.2.5) for coefficients labelled by J.

Quantity $r_2 = -(\mathsf{r}_2 \cdot n)$ can be related to the quantity $r_1 = -(\mathsf{r}_1 \cdot n)$ by Taylor expansion in powers of $\triangle t$. With a degree of accuracy sufficient for our purposes we obtain

$$r_2 = r_1 + \triangle t c_2 \qquad \text{(E.8)}$$

where $c_2 = (\mathsf{c}_2 \cdot n)$. Integration of (E.7) over the angular variable gives

$$\mathcal{D}^a = \frac{(\mathsf{a} \cdot \mathsf{r}_2)}{\|\mathsf{r}_2\|^3} + \triangle t \left[\frac{3}{2} \frac{(\mathsf{a} \cdot \mathsf{r}_2)(\mathsf{c}_2 \cdot \mathsf{r}_2)}{\|\mathsf{r}_2\|^5} + \frac{1}{2} \frac{(\mathsf{a} \cdot \mathsf{c}_2)}{\|\mathsf{r}_2\|^3} \right] \qquad \text{(E.9)}$$

$$\mathcal{B}^a = \frac{3}{2} \frac{(\mathsf{a} \cdot \mathsf{r}_2)(\mathsf{c}_2 \cdot \mathsf{r}_2)}{\|\mathsf{r}_2\|^5} + \frac{1}{2} \frac{(\mathsf{a} \cdot \mathsf{c}_2)}{\|\mathsf{r}_2\|^3}$$

$$+ \triangle t \left[\frac{5}{2} \frac{(\mathsf{a} \cdot \mathsf{r}_2)(\mathsf{c}_2 \cdot \mathsf{r}_2)^2}{\|\mathsf{r}_2\|^7} + \frac{1}{2} \frac{(\mathsf{a} \cdot \mathsf{r}_2)(\mathsf{c}_2 \cdot \mathsf{c}_2) + 2(\mathsf{a} \cdot \mathsf{c}_2)(\mathsf{c}_2 \cdot \mathsf{r}_2)}{\|\mathsf{r}_2\|^5} \right] .$$

Expanding the integrands in eq.(E.6) in powers of $\triangle t$ and using the relations (E.5) we finally obtain

$$p_\triangle^0(t) = e^2 \int_{-\infty}^t dt_2 \frac{(\mathbf{v}_2 \dot{\mathbf{v}}_2)}{[1 - \mathbf{v}_2^2]^{3/2}} \lim_{\triangle t \to 0} \ln \left(\sqrt{\alpha_0} + \sqrt{-\beta_0} \right) \Big|_{t_1 = t_2 + \triangle t}$$

$$- e^2 \int_{-\infty}^t dt_1 \frac{\mathbf{v}_1^2}{k_1^0 \sqrt{1 - \mathbf{v}_1^2}} \lim_{\triangle t \to 0} \sqrt{-\beta_0 \alpha_0} \Big|_{t_2 = t_1 - \triangle t} \qquad \text{(E.10)}$$

$$+ e^2 \int_{-\infty}^t dt_2 \frac{(\mathbf{v}_2 \dot{\mathbf{v}}_2)}{[1 - \mathbf{v}_2^2]^{3/2}} \frac{1}{1 + \sqrt{1 - \mathbf{v}_2^2}} \left[2 - \frac{1}{2} \frac{1 + \mathbf{v}_2^2}{1 + \sqrt{1 - \mathbf{v}_2^2}} \right]$$

$$+ e^2 \int_{-\infty}^t dt_1 \left[\mu(t_1, t_2) I_0' \frac{\alpha_0}{\mathsf{r}_2^0} \right]_{t_2 \to -\infty}^{t_2 = t_1 - \triangle t}$$

$$+ e^2 \int_{-\infty}^t dt_1 \lim_{k_2^0 \to \infty} \frac{\mu(t_1, t_2) \sqrt{1 - \mathbf{A}^2}}{k_1^0 [1 - (\mathbf{A} \mathbf{v}_2)]}$$

after integration by parts of the last term in eq.(E.6).

The first and the second terms are singular. To deal with divergences it is efficient to introduce the hyperbolic angles Ψ and ψ:

$$\cosh \Psi = \frac{k_2^0 + k_1^0}{\mathbf{q}}, \qquad \sinh \Psi = \frac{\sqrt{2\Sigma}}{\mathbf{q}} \qquad \text{(E.11)}$$

$$\cosh \psi = \frac{k_2^0 - k_1^0}{\mathbf{q}}, \qquad \sinh \psi = \frac{\sqrt{2\sigma}}{\mathbf{q}} .$$

(Function $\Sigma(t, t_1, t_2)$ is introduced in Appendix B.) In these notations

$$\beta_0 = -\frac{1}{2} [\cosh(\Psi - \psi) - 1] = -\sinh^2 \frac{\Psi - \psi}{2} \qquad \text{(E.12)}$$

$$\alpha_0 = \frac{1}{2} [\cosh(\Psi - \psi) + 1] = \cosh^2 \frac{\Psi - \psi}{2} .$$

so that

$$\ln \left(\sqrt{\alpha_0} + \sqrt{-\beta_0} \right) = \frac{\Psi - \psi}{2} \qquad \text{(E.13)}$$

$$\sqrt{-\beta_0 \alpha_0} = \frac{1}{2} \sinh(\Psi - \psi) .$$

Since the factor before the sign of limit is the total time derivative, the logarithmic divergence in eq.(E.10) can be integrated by parts:

$$\int_{-\infty}^t dt_2 \frac{d}{dt_2} \left(\frac{1}{\sqrt{1 - \mathbf{v}_2^2}} \right) \lim_{\triangle t \to 0} \frac{\Psi - \psi}{2} \Big|_{t_1 = t_2 + \triangle t} \qquad \text{(E.14)}$$

$$= \frac{1}{\sqrt{1 - \mathbf{v}_2^2}} \lim_{\triangle t \to 0} \frac{\Psi - \psi}{2} \Big|_{t_2 \to -\infty}^{t_2 = t}$$

$$- \frac{1}{2} \int_{-\infty}^t \frac{dt_2}{\sqrt{1 - \mathbf{v}_2^2}} \lim_{\triangle t \to 0} \left[\frac{\partial (\Psi - \psi)}{\partial t_1} + \frac{\partial (\Psi - \psi)}{\partial t_2} \right]_{t_1 = t_2 + \triangle t} .$$

Taking into account that at the end points hyperbolic angles vanish, we finally obtain

$$e^2 \int_{-\infty}^t dt_2 \frac{(\mathbf{v}_2 \dot{\mathbf{v}}_2)}{[1 - \mathbf{v}_2^2]^{3/2}} \lim_{\triangle t \to 0} \ln \left(\sqrt{\alpha_0} + \sqrt{-\beta_0} \right) \Big|_{t_1 = t_2 + \triangle t}$$

$$= \frac{e^2}{2} \int_{-\infty}^t dt_2 \left[\frac{1}{k_2^0 \sqrt{1 - \mathbf{v}_2^2}} - \frac{(\mathbf{v}_2 \dot{\mathbf{v}}_2)}{(1 - \mathbf{v}_2^2)(1 + \sqrt{1 - \mathbf{v}_2^2})} \right] . \qquad \text{(E.15)}$$

Now we rewrite the second divergent term involved in eq.(E.10). We expand the integrand in powers of $\triangle t$. Passing to the limit $\triangle t \to 0$, we

arrive at:

$$\int_{-\infty}^{t} dt_1 \frac{\mathbf{v}_1^2}{k_1^0 \sqrt{1-\mathbf{v}_1^2}} \lim_{\triangle t \to 0} \left. \sqrt{-\beta_0 \alpha_0} \right|_{t_2 = t_1 - \triangle t} \quad (E.16)$$

$$= \int_{-\infty}^{t} dt_1 \lim_{\triangle t \to 0} \frac{1 - \sqrt{1-\mathbf{v}_1^2}}{\triangle t \sqrt{1-\mathbf{v}_1^2}}$$

$$+ \int_{-\infty}^{t} dt_1 \left[\frac{1 - \sqrt{1-\mathbf{v}_1^2}}{2k_1^0 \sqrt{1-\mathbf{v}_1^2}} + \frac{(\mathbf{v}_1 \dot{\mathbf{v}}_1)}{\sqrt{1-\mathbf{v}_1^2}(1+\sqrt{1-\mathbf{v}_1^2})} - \frac{(\mathbf{v}_1 \dot{\mathbf{v}}_1)}{2(1-\mathbf{v}_1^2)} \right].$$

Inserting these expressions in eq.(E.10) we finally obtain

$$p_\triangle^0(t) = \frac{e^2}{2} \int_{-\infty}^{t} dt_2 \left[\frac{1}{k_2^0 \sqrt{1-\mathbf{v}_2^2}} - \frac{(\mathbf{v}_2 \dot{\mathbf{v}}_2)}{(1-\mathbf{v}_2^2)(1+\sqrt{1-\mathbf{v}_2^2})} \right] \quad (E.17)$$

$$- e^2 \int_{-\infty}^{t} dt_1 \lim_{\triangle t \to 0} \frac{1 - \sqrt{1-\mathbf{v}_1^2}}{\triangle t \sqrt{1-\mathbf{v}_1^2}}$$

$$- e^2 \int_{-\infty}^{t} dt_1 \left[\frac{1 - \sqrt{1-\mathbf{v}_1^2}}{2k_1^0 \sqrt{1-\mathbf{v}_1^2}} + \frac{(\mathbf{v}_1 \dot{\mathbf{v}}_1)}{\sqrt{1-\mathbf{v}_1^2}(1+\sqrt{1-\mathbf{v}_1^2})} \right.$$

$$\left. - \frac{(\mathbf{v}_1 \dot{\mathbf{v}}_1)}{2(1-\mathbf{v}_1^2)} \right]$$

$$+ e^2 \int_{-\infty}^{t} dt_2 \frac{(\mathbf{v}_2 \dot{\mathbf{v}}_2)}{[1-\mathbf{v}_2^2]^{3/2}} \frac{1}{1+\sqrt{1-\mathbf{v}_2^2}} \left[2 - \frac{1}{2} \frac{1+\mathbf{v}_2^2}{1+\sqrt{1-\mathbf{v}_2^2}} \right]$$

$$+ e^2 \int_{-\infty}^{t} dt_1 \left[\mu(t_1, t_2) I_0' \frac{\alpha_0}{\mathsf{r}_2^0} \right]_{t_2 \to -\infty}^{t_2 = t_1 - \triangle t}$$

$$+ e^2 \int_{-\infty}^{t} dt_1 \lim_{k_2^0 \to \infty} \frac{\mu(t_1, t_2) \sqrt{1-\mathbf{A}^2}}{k_1^0 [1 - (\mathbf{A}\mathbf{v}_2)]}.$$

The calculation of momentum corrections is virtually identical to what is presented here, and we shall not worry with the details. It suffices to present the resulting expression

$$p_\triangle^i(t) = \frac{e^2}{2} \int_{-\infty}^{t} dt_2 \left[\frac{v_2^i}{k_2^0 \sqrt{1-\mathbf{v}_2^2}} - \frac{v_2^i (\mathbf{v}_2 \dot{\mathbf{v}}_2)}{(1-\mathbf{v}_2^2)(1+\sqrt{1-\mathbf{v}_2^2})} \right] \quad (E.18)$$

$$- e^2 \int_{-\infty}^{t} dt_1 \lim_{\triangle t \to 0} \frac{v_1^i}{\triangle t \sqrt{1-\mathbf{v}_1^2}(1+\sqrt{1+\mathbf{v}_1^2})}$$

$$- \frac{e^2}{2} \int_{-\infty}^{t} \frac{dt_1}{1+\sqrt{1-\mathbf{v}_1^2}} \left[\frac{v_1^i}{k_1^0 \sqrt{1-\mathbf{v}_1^2}} - \frac{v_1^i (\mathbf{v}_1 \dot{\mathbf{v}}_1)}{(1-\mathbf{v}_1^2)(1+\sqrt{1-\mathbf{v}_1^2})} \right]$$

$$+ e^2 \int_{-\infty}^{t} dt_2 \frac{1}{1+\sqrt{1-\mathbf{v}_2^2}} \left\{ \frac{(\mathbf{v}_2 \dot{\mathbf{v}}_2) v_2^i}{[1-\mathbf{v}_2^2]^{3/2}} \left[2 - \frac{1}{2} \frac{1+\mathbf{v}_2^2}{1+\sqrt{1-\mathbf{v}_2^2}} \right] \right.$$

$$\left. + \frac{\dot{v}_2^i}{\sqrt{1-\mathbf{v}_2^2}} \right\}$$

$$+ \frac{e^2}{2} \int_{-\infty}^{t} dt_1 \left[(v_1^i + v_2^i) I_0' \frac{\alpha_0}{\mathsf{r}_2^0} \right]_{t_2 \to -\infty}^{t_2 = t_1 - \triangle t}$$

$$+ \frac{e^2}{2} \int_{-\infty}^{t} dt_1 \lim_{k_2^0 \to \infty} \frac{(v_1^i + v_2^i) \sqrt{1-\mathbf{A}^2}}{k_1^0 [1 - (\mathbf{A}\mathbf{v}_2)]}.$$

In the next Appendix the contribution from integrals at point $\beta = \beta_0$ is found. Its bound part contains the terms which annihilate the divergent terms in expressions (E.17) and (E.18).

Appendix F. Calculation of integrals at point where $\beta = \beta_0$

We now would like to extract the partial derivatives with respect to time variables from the integrand of the following double integral

$$p_0^\alpha(t) = e^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t} dt_2 \int_{t_2}^{t} dt_1 \left(\left[\frac{\partial G_{12}^\alpha}{\partial t_2} + G_1^\alpha \right]_{\beta_0} \frac{\partial \beta_0}{\partial t_1} + G_2^\alpha |_{\beta_0} \frac{\partial \beta_0}{\partial t_2} \right)$$

$$(F.1)$$

Note that

$$\frac{\partial \beta_0}{\partial t_1} = \alpha_0 \frac{\mathsf{r}_1^0}{\mathsf{J}_0}, \qquad \frac{\partial \beta_0}{\partial t_2} = \beta_0 \frac{\mathsf{r}_2^0}{\mathsf{J}_0}. \quad (F.2)$$

and functions $G_{12}^\alpha, G_1^\alpha, G_2^\alpha$ are given by eqs.(4.3.4) and (4.3.5).

If $\beta = \beta_0$ the radius R of the smallest circle pictured at figure 5 vanishes and it reduces to the point A. Norms $\|r_a\|$ and $\|c_a\|$ become zeroth component r_a^0 and c_a^0, respectively. Hence coefficients (4.2.10) get simplified, e.g.

$$\mathcal{D}^0 = \frac{1}{\mathsf{r}_1^0 \mathsf{r}_2^0}, \qquad \mathcal{D}^J = \frac{\mathsf{J}_0}{\mathsf{r}_1^0 \mathsf{r}_2^0} \qquad \mathcal{D}_1^i = \frac{-\beta_0 q^i}{\mathsf{r}_1^0 \mathsf{r}_2^0}, \qquad \mathcal{D}_2^i = \frac{\alpha_0 q^i}{\mathsf{r}_1^0 \mathsf{r}_2^0}. \quad (F.3)$$

In terms of two-point functions $\sigma = -1/2(q \cdot q)$ and $\Sigma = \sigma + k_1^0 k_2^0$ the angle-free functions κ, μ and λ_a involved in p_{em}^α are as follows:

$$\kappa = \frac{1}{2} \left(\Sigma \frac{\partial^2 \Sigma}{\partial t_1 \partial t_2} - \frac{\partial \Sigma}{\partial t_1} \frac{\partial \Sigma}{\partial t_2} \right), \qquad \mu = \frac{1}{2} \frac{\partial^2 \Sigma}{\partial t_1 \partial t_2}$$

$$\lambda_a = k_a^0 \frac{\partial \Sigma}{\partial t_a} + \Sigma$$
$$= k_a^0 \frac{\partial \sigma}{\partial t_a} + \sigma. \tag{F.4}$$

Firstly we put $\alpha = 0$. Routine scrupulous calculations allow us to rewrite the expression under the integral signs in eq.(F.1) as follows:

$$\left[\frac{\partial G_{12}^0}{\partial t_2} + G_1^0 \right]_{\beta_0} \frac{\partial \beta_0}{\partial t_1} + G_2^0 \big|_{\beta_0} \frac{\partial \beta_0}{\partial t_2} = \frac{\partial}{\partial t_2} \left(\sqrt{\frac{\alpha_0}{-\beta_0}} \frac{\kappa}{\mathsf{r}_2^0 \mathsf{J}_0} - I_0' \mu \frac{\alpha_0}{\mathsf{r}_2^0} \right)$$
$$+ \frac{1}{2} \frac{\partial}{\partial t_2} \left[\frac{k_2^0 + (\mathbf{q}\mathbf{v}_1)}{k_1^0} \left(\frac{1}{\sqrt{2\Sigma}} + \frac{1}{\sqrt{2\sigma}} \right) + \frac{1}{\sqrt{2\Sigma}} \right] \tag{F.5}$$
$$+ \frac{1}{2} \frac{\partial}{\partial t_1} \left[\frac{k_1^0 - (\mathbf{q}\mathbf{v}_2)}{k_2^0} \left(\frac{1}{\sqrt{2\Sigma}} - \frac{1}{\sqrt{2\sigma}} \right) + \frac{1}{\sqrt{2\Sigma}} \right]$$
$$- \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \frac{q^0}{(2\sigma)^{3/2}} + \frac{1}{2} \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \right) \frac{1}{\sqrt{2\sigma}}.$$

This expression contains the term which is proportional to the mixed second order partial derivative of σ which can not be rewritten as a derivative with respect to t_1 or t_2.

In analogous way we rewrite the β_0-part of space components p_{em}^i of the momentum carried by the electromagnetic field:

$$\left[\frac{\partial G_{12}^i}{\partial t_2} + G_1^i \right]_{\beta_0} \frac{\partial \beta_0}{\partial t_1} + G_2^i \big|_{\beta_0} \frac{\partial \beta_0}{\partial t_2} = \tag{F.6}$$
$$= \frac{1}{2} \frac{\partial}{\partial t_2} \left[\sqrt{\frac{\alpha_0}{-\beta_0}} \frac{1}{\mathsf{r}_2^0 \mathsf{J}_0} \left(\frac{\partial \lambda_1}{\partial t_2} \alpha_0 q^i - \frac{\partial \lambda_2}{\partial t_1} \beta_0 q^i + v_1^i \lambda_2 + v_2^i \lambda_1 \right) \right.$$
$$\left. - I_0' \frac{\alpha_0}{\mathsf{r}_2^0} \left(v_1^i + v_2^i \right) \right]$$
$$+ \frac{1}{2} \frac{\partial}{\partial t_2} \left[\frac{v_1^i k_2^0 + q^i}{k_1^0} \left(\frac{1}{\sqrt{2\Sigma}} + \frac{1}{\sqrt{2\sigma}} \right) + \frac{v_1^i}{\sqrt{2\Sigma}} \right]$$
$$+ \frac{1}{2} \frac{\partial}{\partial t_1} \left[\frac{v_2^i k_1^0 - q^i}{k_2^0} \left(\frac{1}{\sqrt{2\Sigma}} - \frac{1}{\sqrt{2\sigma}} \right) + \frac{v_2^i}{\sqrt{2\Sigma}} \right]$$
$$- \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \frac{q^i}{(2\sigma)^{3/2}} + \frac{1}{2} \frac{\partial}{\partial t_1} \left(\frac{v_2^i}{\sqrt{2\sigma}} \right) - \frac{1}{2} \frac{\partial}{\partial t_2} \left(\frac{v_1^i}{\sqrt{2\sigma}} \right).$$

Having integrated the expressions (F.5) and (F.6) over time variables t_1 and t_2 according to the rule (F.1) we obtain

$$p_0^0(t) = e^2 \int_{-\infty}^{t} dt_1 \left[\sqrt{\frac{\alpha_0}{-\beta_0}} \frac{\kappa}{\mathsf{r}_2^0 \mathsf{J}_0} - I_0' \mu \frac{\alpha_0}{\mathsf{r}_2^0} \right]_{t_2 \to -\infty}^{t_2 = t_1} \tag{F.7}$$

$$+ \frac{e^2}{2} \int_{-\infty}^{t} dt_1 \left[\frac{k_2^0 + (\mathbf{q}\mathbf{v}_1)}{k_1^0} \left(\frac{1}{\sqrt{2\Sigma}} + \frac{1}{\sqrt{2\sigma}} \right) + \frac{1}{\sqrt{2\Sigma}} \right]_{t_2 \to -\infty}^{t_2 = t_1}$$
$$+ \frac{e^2}{2} \int_{-\infty}^{t} dt_2 \left[\frac{k_1^0 - (\mathbf{q}\mathbf{v}_2)}{k_2^0} \left(\frac{1}{\sqrt{2\Sigma}} - \frac{1}{\sqrt{2\sigma}} \right) + \frac{1}{\sqrt{2\Sigma}} \right]_{t_1 = t_2}^{t_1 = t}$$
$$+ e^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \left[-\frac{q^0 (v_1 \cdot v_2)}{(2\sigma)^{3/2}} + \frac{1}{2} \frac{(q \cdot v_1)}{(2\sigma)^{3/2}} + \frac{1}{2} \frac{(q \cdot v_2)}{(2\sigma)^{3/2}} \right]$$
$$p_0^i(t) = \frac{e^2}{2} \int_{-\infty}^{t} dt_1 \left[\sqrt{\frac{\alpha_0}{-\beta_0}} \frac{1}{\mathsf{r}_2^0 \mathsf{J}_0} \left(\frac{\partial \lambda_1}{\partial t_2} \alpha_0 q^i - \frac{\partial \lambda_2}{\partial t_1} \beta_0 q^i \right. \right.$$
$$\left. + v_1^i \lambda_2 + v_2^i \lambda_1 \right) - I_0' \frac{\alpha_0}{\mathsf{r}_2^0} \left(v_1^i + v_2^i \right) \bigg]_{t_2 \to -\infty}^{t_2 = t_1}$$
$$+ \frac{e^2}{2} \int_{-\infty}^{t} dt_1 \left[\frac{v_1^i k_2^0 + q^i}{k_1^0} \left(\frac{1}{\sqrt{2\Sigma}} + \frac{1}{\sqrt{2\sigma}} \right) + \frac{v_1^i}{\sqrt{2\Sigma}} \right]_{t_2 \to -\infty}^{t_2 = t_1}$$
$$+ \frac{e^2}{2} \int_{-\infty}^{t} dt_2 \left[\frac{v_2^i k_1^0 - q^i}{k_2^0} \left(\frac{1}{\sqrt{2\Sigma}} - \frac{1}{\sqrt{2\sigma}} \right) + \frac{v_2^i}{\sqrt{2\Sigma}} \right]_{t_1 = t_2}^{t_1 = t}$$
$$+ e^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \left[-\frac{q^i (v_1 \cdot v_2)}{(2\sigma)^{3/2}} + \frac{1}{2} \frac{v_2^i (q \cdot v_1)}{(2\sigma)^{3/2}} + \frac{1}{2} \frac{v_1^i (q \cdot v_2)}{(2\sigma)^{3/2}} \right]$$

So, besides the double integral which describes the self-action which depends not only on the current state of motion of the particle, but also on its past history, we have the integrals of functions of the end points only.

According to eqs.(B.11), function $\Sigma(t, t_1, t_2)|_{t_1 = t}$ is equal to the function $\sigma(t, t_2)$. This circumstance simplifies evaluation of the terms referred to this end point. We expand the terms near $t_2 = t_1$ in powers of $\triangle t = t_1 - t_2$ and take the limit $\triangle t \to 0$. We use the assumption (E.2) when $t_2 \to -\infty$. After some algebra we finally obtain:

$$p_0^0(t) = -e^2 \int_{-\infty}^{t} dt_1 \left[\lim_{\triangle t \to 0} \frac{1}{\triangle t} + \frac{1}{2k_1^0} - \frac{(\mathbf{v}_1 \dot{\mathbf{v}}_1)}{2\sqrt{1 - \mathbf{v}_1^2}(1 + \sqrt{1 - \mathbf{v}_1^2})} \right]$$
$$- e^2 \int_{-\infty}^{t} dt_1 \lim_{t_2 \to -\infty} \frac{\mu(t_1, t_2)\sqrt{1 - \mathbf{A}^2}}{k_1^0 [1 - (\mathbf{A}\mathbf{v}_2)]} \tag{F.8}$$
$$- e^2 \int_{-\infty}^{t} dt_1 \left[\mu(t_1, t_2) I_0' \frac{\alpha_0}{\mathsf{r}_2^0} \right]_{t_2 \to -\infty}^{t_2 = t_1}$$
$$+ \frac{e^2}{2} \int_{-\infty}^{t} dt_1 \lim_{\triangle t \to 0} \frac{1}{\triangle t \sqrt{1 - \mathbf{v}_1^2}}$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_2 \lim_{\triangle t \to 0} \frac{1}{\triangle t \sqrt{1 - \mathbf{v}_2^2}} + \frac{e^2}{2} \int_{-\infty}^t dt_2 \frac{1}{\sqrt{2\sigma(t, t_2)}}$$

$$+ e^2 \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \left[-\frac{q^0(v_1 \cdot v_2)}{(2\sigma)^{3/2}} + \frac{1}{2} \frac{(q \cdot v_1)}{(2\sigma)^{3/2}} + \frac{1}{2} \frac{(q \cdot v_2)}{(2\sigma)^{3/2}} \right]$$

$$p_0^i(t) = e^2 \int_{-\infty}^t \frac{dt_1}{1 + \sqrt{1 - \mathbf{v}_1^2}} \left[-\lim_{\triangle t \to 0} \frac{v_1^i}{\triangle t} - \frac{v_1^i}{2k_1^0} \right.$$

$$+ \left. \frac{v_1^i(\mathbf{v}_1 \dot{\mathbf{v}}_1)}{\sqrt{1 - \mathbf{v}_1^2}(1 + \sqrt{1 - \mathbf{v}_1^2})} + \frac{\dot{v}_1^i}{2} \right] \tag{F.9}$$

$$- \frac{e^2}{2} \int_{-\infty}^t dt_1 \lim_{t_2 \to -\infty} \frac{(v_1^i + v_2^i)\sqrt{1 - \mathbf{A}^2}}{k_1^0 [1 - (\mathbf{A}\mathbf{v}_2)]}$$

$$- \frac{e^2}{2} \int_{-\infty}^t dt_1 \left[(v_1^i + v_2^i) I_0' \frac{\alpha_0}{\mathsf{r}_2^0} \right]_{t_2 \to -\infty}^{t_2 = t_1}$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_1 \lim_{\triangle t \to 0} \frac{v_1^i}{\triangle t \sqrt{1 - \mathbf{v}_1^2}} + \frac{e^2}{2} \int_{-\infty}^t dt_2 \lim_{\triangle t \to 0} \frac{v_2^i}{\triangle t \sqrt{1 - \mathbf{v}_2^2}}$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_2 \frac{v_2^i}{\sqrt{2\sigma(t, t_2)}}$$

$$+ e^2 \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \left[-\frac{q^i(v_1 \cdot v_2)}{(2\sigma)^{3/2}} + \frac{1}{2} \frac{v_2^i(q \cdot v_1)}{(2\sigma)^{3/2}} + \frac{1}{2} \frac{v_1^i(q \cdot v_2)}{(2\sigma)^{3/2}} \right]$$

Divergent terms annul their counterparts aroused in eqs.(E.17) and (E.18). After that only one term remains:

$$\frac{e^2}{2} \int_{-\infty}^t dt_2 \frac{v_2^\mu}{\sqrt{2\sigma(t, t_2)}} = \frac{e^2}{2} \int_{-\infty}^t dt_2 \left. \frac{v_2^\mu}{\sqrt{2\Sigma}} \right|_{t_1 = t_2}^{t_1 = t} \tag{F.10}$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_1 \left. \frac{v_1^\mu}{\sqrt{2\Sigma}} \right|_{t_2 \to -\infty}^{t_2 = t_1}.$$

It is the singular part of energy-momentum carried by electromagnetic field. (It is worth noting that inverse square root $[2\Sigma(t, t_1, t_2)|_{t_2 \to -\infty}]^{-1/2}$ vanishes even if $t_1 \to -\infty$, see eqs.(B.11).)

Final expressions are presented in Section 4 (see eqs.(4.3.6) and (4.3.7)).

Appendix G. Angular momentum in (2+1)-electrodynamics

We now turn to the calculation of the angular momentum tensor

$$M_{\text{em}}^{\mu\nu}(t) = \int_{\Sigma_t} d\sigma_0 \left(y^\mu T^{0\nu} - y^\nu T^{0\mu} \right) \tag{G.1}$$

carried by the electromagnetic field due to a point-like charge. We apply the convenient coordinate system introduced in Section 4 and detailed in Appendix B.

We present the integrand of eq.(G.1) in the following form

$$m_{\text{em}}^{\mu\nu} = m_{12}^{\mu\nu} + m_{21}^{\mu\nu} - m_{12}^{\nu\mu} - m_{21}^{\nu\mu} \tag{G.2}$$

where

$$m_{12}^{\mu\nu} = (z_1^\mu + K_1^\mu) \frac{1}{2\pi} \left[f_{(1)}^{0\lambda} f_{(2)\lambda}^\nu - \frac{1}{4} \eta^{0\nu} f_{(1)}^{\alpha\beta} f_{\alpha\beta}^{(2)} \right]. \tag{G.3}$$

It is straightforward to substitute the fields (4.1.7) into this expression to calculate the first term of the integrand (G.2). The others can be obtained by interchanging of the pair of indices $1, 2$ and μ, ν.

Having integrated expression $Jm_{12}^{\mu\nu}$ over φ we obtain

$$\mathcal{M}_{12}^{\mu\nu} = \frac{1}{2} I \left\{ \hat{\mathcal{T}}_{12}^{\mu\nu} \left(\frac{\partial \lambda_1}{\partial t_2} \right) + \hat{\mathcal{T}}_1^\mu (v_2^\nu \lambda_1) - v_2^\nu \frac{\partial \lambda_1}{\partial t_2} \mathcal{C}_1^\mu - v_2^\nu \frac{\partial^2 \lambda_1}{\partial t_1 \partial t_2} \mathcal{D}_1^\mu \right\}$$

$$- \frac{1}{2} I' \hat{\mathcal{T}}_1^{\mu J} (v_2^\nu) \tag{G.4}$$

$$+ \frac{z_1^\mu}{2} I \left\{ \hat{\mathcal{T}}_2^\nu \left(\frac{\partial \lambda_1}{\partial t_2} \right) + \hat{\mathcal{T}}^0 (v_2^\nu \lambda_1) - v_2^\nu \frac{\partial \lambda_1}{\partial t_2} \mathcal{C}^0 - v_2^\nu \frac{\partial^2 \lambda_1}{\partial t_1 \partial t_2} \mathcal{D}^0 \right\}$$

$$- \frac{z_1^\mu}{2} I' \hat{\mathcal{T}}^J (v_2^\nu)$$

$$- \frac{\eta^{0\nu}}{4} \left\{ I \left[\hat{\mathcal{T}}_1^\mu (\lambda) + z_1^\mu \hat{\mathcal{T}}^0 (\lambda) \right] - I' \left[\hat{\mathcal{T}}_1^{\mu J} (\lambda_0) + z_1^\mu \hat{\mathcal{T}}^J (\lambda_0) \right] \right\}$$

where functions λ and λ_0 are given by eqs.(4.2.7).

Usage of the equalities (4.2.15) derived in Appendix C allow us to rewrite the integrand (G.4) as the following sum:

$$\mathcal{M}_{12}^{\mu\nu} = \frac{1}{2} I \left\{ \hat{\Pi}_{12}^{\mu\nu} \left(\frac{\partial \lambda_1}{\partial t_2} \right) + \hat{\Pi}_1^\mu (v_2^\nu \lambda_1) - \frac{\partial}{\partial t_1} \left(v_2^\nu \frac{\partial \lambda_1}{\partial t_2} \mathcal{D}_1^\mu \right) \tag{G.5}$$

$$+ \hat{\Pi}_2^\nu \left(z_1^\mu \frac{\partial \lambda_1}{\partial t_2} \right) - \frac{\partial}{\partial t_2} \left(v_1^\mu \frac{\partial \lambda_1}{\partial t_2} \mathcal{D}_2^\nu \right)$$

$$+ \hat{\Pi}^0 (z_1^\mu v_2^\nu \lambda_1) - \frac{\partial}{\partial t_1} \left(z_1^\mu v_2^\nu \frac{\partial \lambda_1}{\partial t_2} \mathcal{D}^0 \right) - \frac{\partial}{\partial t_2} \left(v_1^\mu v_2^\nu \lambda_1 \mathcal{D}^0 \right) \right\}$$

$$- \frac{1}{2} I' \left\{ \hat{\Pi}_1^{\mu J} (v_2^\nu) + \hat{\Pi}^J (z_1^\mu v_2^\nu) - \frac{\partial}{\partial t_2} \left(v_1^\mu v_2^\nu \mathcal{D}^J \right) \right\}$$

$$- \frac{\eta^{0\nu}}{4} \left\{ I \left[\hat{\Pi}_1^\mu (\lambda) + \hat{\Pi}^0 (z_1^\mu \lambda) - \frac{\partial}{\partial t_2} \left(v_1^\mu \lambda \mathcal{D}^0 \right) \right] \right.$$

$$- \left. I' \left[\hat{\Pi}_1^{\mu J} (\lambda_0) + \hat{\Pi}^J (z_1^\mu \lambda_0) - \frac{\partial}{\partial t_2} \left(v_1^\mu \lambda_0 \mathcal{D}^J \right) \right] \right\}.$$

Operators Π^a are combinations of partial derivatives (4.2.12).

Further we perform the integration over the time variables and β according to the rule (4.3.3). It results functions of the end points only. We deal with four types of integrals described in subsections 4.1^o-4.4^o. All of them possess specific small parameter. Near the observation time t the small parameter is β as well as when $t_2 \to -\infty$. If t_2 tends to t_1 (or vice versa), their difference $t_1 - t_2$ tends to zero. These circumstances simplify the computation of integrals of types 4.1^o-4.3^o which is virtually identical to that presented in Appendix D and Appendix E, and we shall not bother the detail. We obtain the bound terms only which should be absorbed within the renormalization procedure. Radiative terms arise from the integration near the point $\beta = \beta_0$ where radial variable $R = 0$ (see subsection 4.4^o).

So, having computed the radiative angular momentum we are not going beyond the limit $R \to 0$. The terms involved in the final expression (G.5) get simplified sufficiently:

$$\mathcal{D}_{12}^{ij} = -\frac{\alpha_0\beta_0 q^i q^j}{\mathsf{r}_1^0 \mathsf{r}_2^0} \tag{G.6}$$

$$\mathcal{C}_{12}^{ij} - \frac{\partial \mathcal{D}_{12}^{ij}}{\partial t_1} = -\frac{\alpha_0^2\beta_0 q^i q^j}{(\mathsf{r}_2^0)^3}\mathbf{v}_2^2 + \alpha_0 \frac{v_1^i q^j}{\mathsf{r}_1^0 \mathsf{r}_2^0} - \alpha_0\beta_0 \frac{q^i v_2^j}{(\mathsf{r}_2^0)^2}$$
$$+ \alpha_0^2 \frac{v_2^i q^j}{(\mathsf{r}_2^0)^2} + \alpha_0 \frac{\delta^{ij}}{\mathsf{r}_2^0}$$

$$\mathcal{B}_{12}^{ij} - \frac{\partial \mathcal{D}_{12}^{ij}}{\partial t_2} = -\frac{\alpha_0\beta_0^2 q^i q^j}{(\mathsf{r}_1^0)^3}\mathbf{v}_1^2 - \beta_0 \frac{v_2^j q^i}{\mathsf{r}_1^0 \mathsf{r}_2^0} + \alpha_0\beta_0 \frac{q^j v_1^i}{(\mathsf{r}_1^0)^2}$$
$$- \beta_0^2 \frac{v_1^j q^i}{(\mathsf{r}_1^0)^2} + \beta_0 \frac{\delta^{ij}}{\mathsf{r}_1^0};$$

$$\mathcal{D}_1^{iJ} = -\frac{\beta_0 q^i \mathsf{J}_0}{\mathsf{r}_1^0 \mathsf{r}_2^0} \qquad \mathcal{D}_2^{iJ} = \frac{\alpha_0 q^i \mathsf{J}_0}{\mathsf{r}_1^0 \mathsf{r}_2^0} \tag{G.7}$$

$$\mathcal{C}_1^{iJ} - \frac{\partial \mathcal{D}_1^{iJ}}{\partial t_1} = v_1^i \frac{\mathsf{J}_0}{\mathsf{r}_1^0 \mathsf{r}_2^0} + \alpha_0 v_2^i \frac{\mathsf{J}_0}{(\mathsf{r}_2^0)^2} - \left[\frac{\partial}{\partial\beta}\left(\frac{\alpha\beta q^i}{\|\mathsf{r}_2\|}\right)\right]_{\beta_0}$$

$$\mathcal{B}_1^{iJ} - \frac{\partial \mathcal{D}_1^{iJ}}{\partial t_2} = \beta_0 v_1^i \frac{\mathsf{J}_0}{(\mathsf{r}_1^0)^2} - \left[\frac{\partial}{\partial\beta}\left(\frac{\beta^2 q^i}{\|\mathsf{r}_1\|}\right)\right]_{\beta_0}$$

$$\mathcal{C}_2^{iJ} - \frac{\partial \mathcal{D}_2^{iJ}}{\partial t_1} = \alpha_0 v_2^i \frac{\mathsf{J}_0}{(\mathsf{r}_2^0)^2} + \left[\frac{\partial}{\partial\beta}\left(\frac{\alpha^2 q^i}{\|\mathsf{r}_2\|}\right)\right]_{\beta_0}$$

$$\mathcal{B}_2^{iJ} - \frac{\partial \mathcal{D}_2^{iJ}}{\partial t_2} = v_2^i \frac{\mathsf{J}_0}{\mathsf{r}_1^0 \mathsf{r}_2^0} + \beta_0 v_1^i \frac{\mathsf{J}_0}{(\mathsf{r}_1^0)^2} + \left[\frac{\partial}{\partial\beta}\left(\frac{\alpha\beta q^i}{\|\mathsf{r}_1\|}\right)\right]_{\beta_0}.$$

They are supplemented with the expressions (C.23), (C.37), and (C.33) taken in the point where radial variable $R = 0$.

Firstly we put $\mu = 0$ and $\nu = i$ into eq.(G.5). The other terms which constitute the mixed space-time components $\mathcal{M}_{\mathrm{em}}^{0i}$ are obtained by interchanging of indices. A direct consequence of the reciprocity is the following combination of partial derivatives in t_1 and t_2:

$$\mathcal{M}_{\mathrm{em}}^{0i} = \frac{e^2}{2} I \left[\hat{\Pi}_1^i\left(t\frac{\partial\lambda_2}{\partial t_1}\right) + \hat{\Pi}_2^i\left(t\frac{\partial\lambda_1}{\partial t_2}\right) + \hat{\Pi}^0\left[t\left(v_2^i\lambda_1 + v_1^i\lambda_2\right)\right] \right. \tag{G.8}$$
$$- \frac{\partial}{\partial t_1}\left(tv_2^i\frac{\partial\lambda_1}{\partial t_2}\mathcal{D}^0\right) - \frac{\partial}{\partial t_2}\left(tv_1^i\frac{\partial\lambda_2}{\partial t_1}\mathcal{D}^0\right)\right]$$
$$- \frac{e^2}{2} I'\hat{\Pi}^J\left[t\left(v_1^i + v_2^i\right)\right]$$
$$- \frac{e^2}{2} I\left[\hat{\Pi}_1^i(\Lambda) + \hat{\Pi}_2^i(\Lambda) + \hat{\Pi}^0\left[(z_1^i + z_2^i)\Lambda\right] - \frac{\partial}{\partial t_1}\left(v_2^i\Lambda\mathcal{D}^0\right)\right.$$
$$\left. - \frac{\partial}{\partial t_2}\left(v_1^i\Lambda\mathcal{D}^0\right)\right]$$
$$+ \frac{e^2}{2} I'\left[\hat{\Pi}_1^{iJ}(1) + \hat{\Pi}_2^{iJ}(1) + \hat{\Pi}^J\left(z_1^i + z_2^i\right) - \frac{\partial}{\partial t_1}\left(v_2^i\mathcal{D}^J\right)\right.$$
$$\left. - \frac{\partial}{\partial t_2}\left(v_1^i\mathcal{D}^J\right)\right]$$
$$- \frac{e^2}{4}\left\{I\left[\hat{\Pi}_1^i(\lambda) + \hat{\Pi}_2^i(\lambda) + \hat{\Pi}^0\left[\lambda\left(z_1^i + z_2^i\right)\right] - \frac{\partial}{\partial t_1}\left(v_2^i\lambda\mathcal{D}^0\right)\right.\right.$$
$$\left. - \frac{\partial}{\partial t_2}\left(v_1^i\lambda\mathcal{D}^0\right)\right]$$
$$- I'\left[\hat{\Pi}_1^{iJ}(\lambda_0) + \hat{\Pi}_2^{iJ}(\lambda_0) + \hat{\Pi}^J\left[\lambda_0\left(z_1^i + z_2^i\right)\right] - \frac{\partial}{\partial t_1}\left(v_2^i\lambda_0\mathcal{D}^J\right)\right.$$
$$\left.\left. - \frac{\partial}{\partial t_2}\left(v_1^i\lambda_0\mathcal{D}^J\right)\right]\right\}$$

where

$$\Lambda = k_1^0 k_2^0 \frac{\partial^2\sigma}{\partial t_1 \partial t_2} + k_1^0 \frac{\partial\sigma}{\partial t_1} + k_2^0 \frac{\partial\sigma}{\partial t_2} + \sigma. \tag{G.9}$$

Now we turn to the integration over times t_1 and t_2. It is sufficient to examine the integrals near the point $R = 0$. The computation is virtually identical to that presented in Appendix F. After a tedious calculation we obtain the following cumbersome expression:

$$M_{\beta_0}^{0i} = \frac{e^2}{2}\int_{-\infty}^{t} dt_1 t\frac{\alpha_0}{r_2^0}\left[\frac{I_0}{\mathsf{J}_0}\left(\frac{\partial\lambda_1}{\partial t_2}\alpha_0 q^i - \frac{\partial\lambda_2}{\partial t_1}\beta_0 q^i + v_1^i\lambda_2 + v_2^i\lambda_1\right)\right.$$

$$- I'_0 \left(v_1^i + v_2^i \right) \Big]_{t_2 \to -\infty}^{t_2 = t_1}$$

$$- e^2 \int_{-\infty}^t dt_1 \left[\frac{\alpha_0}{r_2^0} \left(\frac{I_0}{J_0} \kappa - I'_0 \mu \right) (\alpha_0 z_1^i + \beta_0 z_2^i) \right]_{t_2 \to -\infty}^{t_2 = t_1} \quad (G.10)$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_2 \left[\left(t_2 \frac{v_2^i k_1^0 - q^i}{k_2^0} - z_2^i \frac{k_1^0 - (\mathbf{q}\mathbf{v}_2)}{k_2^0} \right) \left(\frac{1}{\sqrt{2\Sigma}} - \frac{1}{\sqrt{2\sigma}} \right) \right.$$

$$+ \frac{2 k_1^0 v_2^i}{\sqrt{2\sigma}} \Big]_{t_1 = t_2}^{t_1 = t}$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_1 \left[\left(t_1 \frac{v_1^i k_2^0 + q^i}{k_1^0} - z_1^i \frac{k_2^0 + (\mathbf{q}\mathbf{v}_1)}{k_1^0} \right) \left(\frac{1}{\sqrt{2\Sigma}} + \frac{1}{\sqrt{2\sigma}} \right) \right.$$

$$+ \frac{2 k_2^0 v_1^i}{\sqrt{2\sigma}} \Big]_{t_2 \to -\infty}^{t_2 = t_1}$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_2 \left[\frac{t_1 v_2^i - z_1^i}{\sqrt{2\Sigma}} \right]_{t_1 = t_2}^{t_1 = t} + \frac{e^2}{2} \int_{-\infty}^t dt_1 \left[\frac{t_2 v_1^i - z_2^i}{\sqrt{2\Sigma}} \right]_{t_2 \to -\infty}^{t_2 = t_1}$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \left[2 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \frac{t_1 z_2^i - t_2 z_1^i}{(2\sigma)^{3/2}} + t_1 \frac{\partial}{\partial t_1} \left(\frac{v_2^i}{\sqrt{2\sigma}} \right) \right.$$

$$- z_1^i \frac{\partial}{\partial t_1} \left(\frac{1}{\sqrt{2\sigma}} \right) - t_2 \frac{\partial}{\partial t_2} \left(\frac{v_1^i}{\sqrt{2\sigma}} \right) + z_2^i \frac{\partial}{\partial t_2} \left(\frac{1}{\sqrt{2\sigma}} \right) \Big].$$

The single integrals belong to the boundary part of angular momentum carried by the electromagnetic field. We couple them with integrals over β taken at the end points $t_1 = t_2$ and $t_2 \to -\infty$. (Such integrals are described in subsections 4.1^o-4.4^o.) The result is as follows:

$$M_S^{0i} = \frac{e^2}{2} t \int_{-\infty}^t ds \frac{v^i(s)}{\sqrt{2\sigma(t,s)}} - \frac{e^2}{2} z^i(t) \int_{-\infty}^t \frac{ds}{\sqrt{2\sigma(t,s)}}. \quad (G.11)$$

The double integral in eq.(G.10) describes the radiative part; it can be rewritten as follows:

$$M_R^{0i} = \frac{e^2}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \left[t_1 v_{1,\alpha} \frac{-v_2^\alpha q^i + v_2^i q^\alpha}{(2\sigma)^{3/2}} - z_1^i v_{1,\alpha} \frac{-v_2^\alpha q^0 + q^\alpha}{(2\sigma)^{3/2}} \right.$$

$$+ t_2 v_{2,\alpha} \frac{-v_1^\alpha q^i + v_1^i q^\alpha}{(2\sigma)^{3/2}} - z_2^i v_{2,\alpha} \frac{-v_1^\alpha q^0 + q^\alpha}{(2\sigma)^{3/2}} \Big]. \quad (G.12)$$

Taking $t_2 \to t_1$ limit reveals the proper short-distance behaviour.

Now we calculate the space component $\mathcal{M}_{\text{em}}^{ij}$. Putting $\mu = i$ and $\nu = j$ into eq.(G.5) we obtain $\mathcal{M}_{12}^{ij}$. Having interchanged upper and

lower indices we find out all the terms which constitute the expression $\mathcal{M}_{\text{em}}^{ij}$ obtained from eq.(G.2) via integration over φ. Further we integrate them over times t_1 and t_2. After tedious calculations we finally obtain:

$$M_{\beta_0}^{ij} = \frac{e^2}{2} \int_{-\infty}^t dt_1 \left\{ \frac{\alpha_0}{r_2^0} \left[\frac{I_0}{J_0} \left(\frac{\partial \lambda_1}{\partial t_2} \alpha_0 (z_1^i q^j - z_1^j q^i) - \frac{\partial \lambda_2}{\partial t_1} \beta_0 (z_2^i q^j - z_2^j q^i) \right. \right. \right.$$

$$+ \quad (\alpha_0 z_1^i + \beta_0 z_2^i)(v_1^j \lambda_2 + v_2^j \lambda_1) - (\alpha_0 z_1^j + \beta_0 z_2^j)(v_1^i \lambda_2 + v_2^i \lambda_1) \right)$$

$$- \quad I'_0 \left((\alpha_0 z_1^i + \beta_0 z_2^i)(v_1^j + v_2^j) - (\alpha_0 z_1^j + \beta_0 z_2^j)(v_1^i + v_2^i) \right) \right] \Big\}_{t_2 \to -\infty}^{t_2 = t_1}$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_2 \left[\frac{1}{k_2^0} \left(z_1^i z_2^j - z_1^j z_2^i + k_1^0 (z_2^i v_2^j - z_2^j v_2^i) \right) \right.$$

$$\times \left. \left(\frac{1}{\sqrt{2\Sigma}} - \frac{1}{\sqrt{2\sigma}} \right) \right]_{t_1 = t_2}^{t_1 = t} \quad (G.13)$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_1 \left[\frac{1}{k_1^0} \left(-z_1^i z_2^j + z_1^j z_2^i + k_2^0 (z_1^i v_1^j - z_1^j v_1^i) \right) \right.$$

$$\times \left. \left(\frac{1}{\sqrt{2\Sigma}} + \frac{1}{\sqrt{2\sigma}} \right) \right]_{t_2 \to -\infty}^{t_2 = t_1}$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_2 \left[\frac{z_1^i v_2^j - z_1^j v_2^i}{\sqrt{2\Sigma}} \right]_{t_1 = t_2}^{t_1 = t} + \frac{e^2}{2} \int_{-\infty}^t dt_1 \left[\frac{z_2^i v_1^j - z_2^j v_1^i}{\sqrt{2\Sigma}} \right]_{t_2 \to -\infty}^{t_2 = t_1}$$

$$+ \frac{e^2}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \left[2 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \frac{z_1^i z_2^j - z_1^j z_2^i}{(2\sigma)^{3/2}} + z_1^i \frac{\partial}{\partial t_1} \left(\frac{v_2^j}{\sqrt{2\sigma}} \right) \right.$$

$$- z_1^j \frac{\partial}{\partial t_1} \left(\frac{v_2^i}{\sqrt{2\sigma}} \right) - z_2^i \frac{\partial}{\partial t_2} \left(\frac{v_1^j}{\sqrt{2\sigma}} \right) + z_2^j \frac{\partial}{\partial t_2} \left(\frac{v_1^i}{\sqrt{2\sigma}} \right) \Big].$$

All the single integrals should be added to the integrals over β evaluated at limit points $t_1 = t_2$ and $t_2 \to -\infty$; the sum is the singular part of angular momentum of the electromagnetic field:

$$M_S^{ij} = \frac{e^2}{2} z^i(t) \int_{-\infty}^t ds \frac{v^j(s)}{\sqrt{2\sigma(t,s)}} - \frac{e^2}{2} z^j(t) \int_{-\infty}^t ds \frac{v^i(s)}{\sqrt{2\sigma(t,s)}}. \quad (G.14)$$

The integrand of radiative part is symmetric in indices (12) and antisymmetric in indices (ij):

$$M_R^{ij} = \frac{e^2}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \left[z_1^i v_{1,\alpha} \frac{-v_2^\alpha q^j + v_2^j q^\alpha}{(2\sigma)^{3/2}} \right. \quad (G.15)$$

$$- z_1^j v_{1,\alpha} \frac{-v_2^\alpha q^i + v_2^i q^\alpha}{(2\sigma)^{3/2}} + z_2^i v_{2,\alpha} \frac{-v_1^\alpha q^j + v_1^j q^\alpha}{(2\sigma)^{3/2}} - z_2^j v_{2,\alpha} \frac{-v_1^\alpha q^i + v_1^i q^\alpha}{(2\sigma)^{3/2}} \Bigg].$$

The resulting expressions (G.12) and (G.15) can be rewritten in a manifestly covariant fashion.

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