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Variational wave equations of two fermions interacting via scalar, pseudoscalar, vector, pseudovector and tensor fields

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Варіяційні хвильові рівняння двох фермійонів, що взаємодіють через скалярне, псевдоскалярне, векторне, псевдовекторне і тензорне поле

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Анотація. Ми розглядаємо метод виведення релятивістичних двочастинкових хвильових рівнянь для фермійонів у координатному представленні. Лаґранжіян теорії переформульовано шляхом виключення полів–носіїв взаємодії з допомогою коваріянтних функцій Ґріна. Тоді нелокальні члени взаємодії у лаґранжіяні зводяться до локальних виразів, які наближено враховують ефекти запізнення. Використовуючи нестандартний "порожній" вакуумний стан ми будуємо гамільтоніян і двофермійонні стани квантованої теорії, та отримуємо релятивістичні двофермійонні хвильові рівняння. Ці рівняння є узагальненням рівнянь Брайта на випадки систем із скалярною, псевдоскалярною, векторною, псевдовекторною і тензорною взаємодією.

Variational wave equations of two fermions interacting via scalar, pseudoscalar, vector, pseudovector and tensor fields

A. Duviryak and J. W. Darewych

Abstract. We consider a method for deriving relativistic two-body wave equations for fermions in the coordinate representation. The Lagrangian of the theory is reformulated by eliminating the mediating fields by means of covariant Green's functions. Then, the nonlocal interaction terms in the Lagrangian are reduced to local expressions which take into account retardation effects approximately. We construct the Hamiltonian and two-fermion states of the quantized theory, employing an unconventional "empty" vacuum state, and derive relativistic two-fermion wave equations. These equations are a generalization of the Breit equation for systems with scalar, pseudoscalar, vector, pseudovector and tensor coupling.

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1. Introduction

Relativistic few-body systems can often be treated in a manner similar to nonrelativistic ones in that the number of particles, i.e., quanta of the matter fields, is fixed and there are no free quanta of fields mediating the interaction. Examples of such systems are systems of nucleons below the threshold for meson production, and quark-hadronic systems (where free gluons are not observable). In such cases the use of full-scale quantum field theory (QFT) is not necessary. Instead, some simplified or effective versions of QFT can be employed.

In this paper we use a modification of the variational method in Hamiltonian QFT for the description of two-fermion relativistic systems with a superposition of scalar, vector, pseudo-scalar, pseudo-vector and tensor interactions. We follow the general approach which was presented in previous works [1,3,2,4]. The starting point of the approach is the classical Lagrangian which describes a system of fermionic fields interacting via bosonic fields. The precise nature of the fields mediating the interaction is not specified in the present paper; the fields can originate from standard or effective field theories. We express the potentials of the mediators in terms of covariant Green functions and fermionic currents. The resulting reduced Lagrangian is a nonlocal one. In order to avoid difficulties presented by the the nonlocal nature of the field theory [5] we employ an approximation procedure which reduces the Lagrangian to local form. As a consequence the transition to the Hamiltonian formalism is straightforward. We then perform the canonical quantization.

We employ an unconventional definition of the vacuum state [1–4] and neglect terms of the Hamiltonian corresponding to the appearance of free physical quanta of thw mediating field, as a result of which the reduced Hamiltonian preserves the particle number. Thus the eigenenergy problem for two-fermion states is self-consistent and leads to a Breit-like equation for the 16-component wave function. The equation includes free-fermion terms, static potentials with appropriate spin-tensor structure and retardation terms. We concentrate on the general structure of the potentials for cases of scalar, vector, pseudo-scalar, pseudo-vector and tensor interactions and their superpositions. Finally, we compare the present results to those known in the literature, and discuss ambiguities in the retardation terms of the interaction potentials.

2. Lagrangian and field equations

We proceed from the following classical Lagrangian density:

$$\mathcal{L} = \sum_{a=1}^{2} \mathcal{L}_a + \mathcal{L}_\chi + \mathcal{L}_A + \mathcal{L}_{\tilde\chi} + \mathcal{L}_{\tilde A} + \mathcal{L}_F, \tag{1}$$

where

$$\begin{aligned} \mathcal{L}_a = \mathcal{L}_a^{\rm free} + \mathcal{L}_a^{\rm int} \;=\; & \bar\psi_a(x)\left\{\mathrm{i}\,\gamma^\mu\partial_\mu - m_a\right\}\psi_a(x) \\ & - \bar\psi_a(x)\left\{g_a\chi(x) + q_a\gamma^\mu A_\mu(x) + \tilde g_a\gamma^5\tilde\chi(x)\right. \\ & \left. + \tilde q_a\gamma^\mu\gamma^5\tilde A_\mu(x) + \frac{\varkappa_a}{2}\sigma^{\mu\nu}F_{\mu\nu}(x)\right\}\psi_a(x) \end{aligned} \tag{2}$$

is the Lagrangian density of the ath fermion field of mass m_a with scalar, vector, pseudo-scalar, pseudo-vector (i.e., axial) and tensor mediating fields, $\chi(x)$, $A_\mu(x)$, $\tilde\chi(x)$, $\tilde A_\mu(x)$ and $F_{\mu\nu}(x)$, respectively (g_a, q_a, $\tilde g_a$, $\tilde q_a$ and $\varkappa_a$ are the corresponding coupling constants). We assume that all the free Lagrangians $\mathcal{L}_\chi$, $\mathcal{L}_A$, $\mathcal{L}_{\tilde\chi}$, $\mathcal{L}_{\tilde A}$ and $\mathcal{L}_F$ of the mediating fields are bi-linear in the fields, and that A_μ, $\tilde A_\mu$ satisfy the Lorentz condition[1]:

$$\partial_\mu A^\mu(x) = 0, \qquad \partial_\mu\tilde A^\mu(x) = 0. \tag{3}$$

Varying $\mathcal{L}$, Eq. (1), with respect to the mediating fields $\chi(x)$, $A_\mu(x)$, $\tilde\chi(x)$, $\tilde A_\mu(x)$ and $F_{\mu\nu}(x)$, we obtain the following linear field equations:

$$\mathcal{E}_\chi\chi = \rho, \tag{4}$$
$$\mathcal{E}_A A^\mu = J^\mu, \tag{5}$$
$$\mathcal{E}_{\tilde\chi}\tilde\chi = \tilde\rho, \tag{6}$$
$$\mathcal{E}_{\tilde A}\tilde A^\mu = \tilde J^\mu, \tag{7}$$
$$\mathcal{E}_F F^{\mu\nu} = J^{\mu\nu}, \tag{8}$$

where $\mathcal{E}_\chi$, $\mathcal{E}_A$ etc. are the corresponding Euler-Lagrange operators. For example, $\mathcal{E}_\chi$ can be the usual form, $\partial_\mu\partial^\mu - m_\chi^2$, or it might be $\partial_\mu\partial^\mu(\partial_\nu\partial^\nu - m_\chi^2)$, $(\partial_\mu\partial^\mu - m_\chi^2)^2$, etc. as in some nonstandard and effective theories [6–8]. The quantities

$$\rho(x) \;=\; \sum_a g_a\rho_a(x) = \sum_a g_a\bar\psi_a(x)\psi_a(x), \tag{9}$$

[1] The Lorentz condition is a consequence of the field equations for massive vector field (such as the Proca equation), but is a gauge fixing condition for massless field.

$$J^{\mu}(x) = \sum_a q_a J^{\mu}_a(x) = \sum_a q_a \bar{\psi}_a(x)\gamma^{\mu}\psi_a(x), \quad (10)$$

$$\tilde{\rho}(x) = \sum_a \tilde{g}_a \tilde{\rho}_a(x) = \sum_a \tilde{g}_a \bar{\psi}_a(x)\gamma^5\psi_a(x), \quad (11)$$

$$\tilde{J}^{\mu}(x) = \sum_a \tilde{q}_a \tilde{J}^{\mu}_a(x) = \sum_a \tilde{q}_a \bar{\psi}_a(x)\gamma^{\mu}\gamma^5\psi_a(x), \quad (12)$$

$$J^{\mu\nu}(x) = \sum_a \varkappa_a J^{\mu\nu}_a(x) = \tfrac{1}{2}\sum_a \varkappa_a \bar{\psi}_a(x)\sigma^{\mu\nu}\psi_a(x), \quad (13)$$

are the fermionic scalar, vector, pseudo-scalar, pseudo-vector (axial) and tensor source densities.

The solutions of Eqs. (4)–(8),

$$\chi(x) = \int d^4x' D_{\rm s}(x-x')\rho(x'), \quad (14)$$

$$A_{\mu}(x) = \int d^4x' D_{\rm v}(x-x')J_{\mu}(x'), \quad (15)$$

$$\tilde{\chi}(x) = \int d^4x' D_{\rm p}(x-x')\tilde{\rho}(x'), \quad (16)$$

$$\tilde{A}_{\mu}(x) = \int d^4x' D_{\rm a}(x-x')\tilde{J}_{\mu}(x'), \quad (17)$$

$$F_{\mu\nu}(x) = \int d^4x' D_{\rm t}(x-x')J_{\mu\nu}(x'), \quad (18)$$

where $D_{\rm s}(x)$, $D_{\rm v}(x)$ etc. are corresponding symmetrical Green functions, can be used in the Lagrangian (1) to obtain the reduced Lagrangian,

$$\bar{\mathcal{L}} = \sum_{a=1}^{2} \mathcal{L}^{\rm free}_a - \mathcal{U}_{\rm s} - \mathcal{U}_{\rm v} - \mathcal{U}_{\rm p} - \mathcal{U}_{\rm a} - \mathcal{U}_{\rm t}, \quad (19)$$

where

$$\mathcal{U}_{\rm s}(x) = \tfrac{1}{2}\int d^4x' \rho(x) D_{\rm s}(x-x')\rho(x'), \quad (20)$$

$$\mathcal{U}_{\rm v}(x) = \tfrac{1}{2}\int d^4x' J_{\mu}(x) D_{\rm v}(x-x')J^{\mu}(x'). \quad (21)$$

$$\mathcal{U}_{\rm p}(x) = \tfrac{1}{2}\int d^4x' \tilde{\rho}(x) D_{\rm p}(x-x')\tilde{\rho}(x'), \quad (22)$$

$$\mathcal{U}_{\rm a}(x) = \tfrac{1}{2}\int d^4x' \tilde{J}_{\mu}(x) D_{\rm a}(x-x')\tilde{J}^{\mu}(x'), \quad (23)$$

$$\mathcal{U}_{\rm t}(x) = \tfrac{1}{2}\int d^4x' J_{\mu\nu}(x) D_{\rm v}(x-x')J^{\mu\nu}(x'). \quad (24)$$

Note that we have suppressed the source-free solutions $\chi_0(x)$, $A^{\mu}_0(x)$ etc., which can be added to the right-hand-sides of Eqs. (14)–(18). This excludes processes involving creation and destruction of physical quanta of the mediating fields.

The reduced Lagrangian (19) leads to the fermion field equations,

$$\Big\{\gamma^{\mu}\Big(\mathrm{i}\,\partial_{\mu} - q_a A_{\mu}(x) - \tilde{q}_a\gamma^5\tilde{A}_{\mu}(x)\Big) - m_a - g_a\chi(x) - \tilde{g}_a\tilde{\chi}(x) - \frac{\varkappa_a}{2}\sigma^{\mu\nu}F_{\mu\nu}\Big\}\psi_a(x) = 0, \quad (25)$$

$$\bar{\psi}_a(x)\Big\{\gamma^{\mu}\Big(\mathrm{i}\,\overleftarrow{\partial}_{\mu} + q_a A_{\mu}(x) + \tilde{q}_a\gamma^5\tilde{A}_{\mu}(x)\Big) + m_a + g_a\chi(x) + \tilde{g}_a\tilde{\chi}(x) + \frac{\varkappa_a}{2}\sigma^{\mu\nu}F_{\mu\nu}\Big\} = 0, \quad (26)$$

provided that the mediating fields $\chi(x)$, $A^{\mu}(x)$ etc. are as given by Eqs. (14), (15), (16), (17), (18), with the free fields $\chi_0(x)$, $A^{\mu}_0(x)$ etc. left out.

3. Approximate accounting for time non-locality

The action integral,

$$\bar{I} = \int d^4x\,\bar{\mathcal{L}} = \sum_{a=1}^{2} I^{\rm free}_a + I_{\rm s} + I_{\rm v} + I_{\rm p} + I_{\rm a} + I_{\rm t}, \quad (27)$$

where $I_{\rm s} = -\int \mathcal{U}_{\rm s} d^4x$, etc., has a structure similar to that of the Fokker integral in direct interaction theory [9–11]. The scalar, vector and other potentials (14)–(18) (and thus the interaction parts of Lagrangians (20)–(24)) are non-local in the time variable $t = x^0$. Because of this the standard hamiltonization procedure is not applicable.

In order to employ the canonical Hamiltonian formalism it is necessary to convert the interaction Lagrangians to single-time form. The simplest way is to set $t' = t$ for all field variables involved in the Lagrangian. This results in the neglect of retardation of interactions, which may lead to the loss of essential relativistic effects, as was shown previously in the case of QED [4]. Therefore, we shall use the procedure employed in [4], which takes retardation effects into account approximately.

The symmetric Green functions $D_{\rm s}(x)$, $D_{\rm v}(x)$ etc. on the r.h.s. of Eqs. (14)–(18) are (whatever equation they satisfy) Lorentz-invariant functions of a single argument only– the interval[2] $x^2 = t^2 - \boldsymbol{x}^2$. Then,

[2] In fact, these functions may be any elementary solutions of equations (4)–(8) (i.e., solutions of the same equations, but with δ-functions on the r.h.s.) depending on x^2 only.

using the substitution $t' = t + \lambda$ and the Taylor series expansion $f(t + \lambda, \boldsymbol{x}') = f(t, \boldsymbol{x}') + \lambda \dot{f}(t, \boldsymbol{x}') + \cdots$, the common structure of the potentials (14)–(18) can be expressed as follows:

$$\begin{aligned}
&\int d^4x' D\left[(t-t')^2 - r^2\right] f(t', \boldsymbol{x}') = \\
&= \int\limits_{-\infty}^{\infty} d\lambda \int d^3x' D(\lambda^2 - r^2) \left\{ f(t, \boldsymbol{x}') + \lambda \dot{f}(t, \boldsymbol{x}') + \tfrac{1}{2}\lambda^2 \ddot{f}(t, \boldsymbol{x}') + \cdots \right\} \\
&= \int d^3x' \left\{ G(r) f(t, \boldsymbol{x}') + \tfrac{1}{2} Q(r) \ddot{f}(t, \boldsymbol{x}') + \cdots \right\}, \qquad (28)
\end{aligned}$$

where $\dot{f} = \partial f / \partial t$, $r = |\boldsymbol{r}| = |\boldsymbol{x} - \boldsymbol{x}'|$, the functions

$$G(r) = \int\limits_{-\infty}^{\infty} d\lambda D(\lambda^2 - r^2), \qquad (29)$$

$$Q(r) = \int\limits_{-\infty}^{\infty} d\lambda D(\lambda^2 - r^2) \lambda^2 \qquad (30)$$

satisfy the relation

$$Q'(r) = rG(r), \qquad (31)$$

and all odd-order derivative terms vanish because D is an even function of λ. Thus we express the interaction integrals $I_{\rm s}$, $I_{\rm v}$ etc. in the single-time form:

$$\begin{aligned}
I_{\rm s} &\approx I_{\rm s}^{(0)} + I_{\rm s}^{(1)} = -\int d^4x \left\{ \mathcal{U}_{\rm s}^{(0)} + \mathcal{U}_{\rm s}^{(1)} \right\}, \\
I_{\rm v} &\approx I_{\rm v}^{(0)} + I_{\rm v}^{(1)} = -\int d^4x \left\{ \mathcal{U}_{\rm v}^{(0)} + \mathcal{U}_{\rm v}^{(1)} \right\} \qquad (32)
\end{aligned}$$

etc., where

$$\mathcal{U}_{\rm s}^{(0)} = \tfrac{1}{2} \int d^3x' \rho(\boldsymbol{x}) G_{\rm s}(r) \rho(\boldsymbol{x}'), \qquad (33)$$

$$\mathcal{U}_{\rm s}^{(1)} = -\tfrac{1}{4} \int d^3x' \dot{\rho}(\boldsymbol{x}) Q_{\rm s}(r) \dot{\rho}(\boldsymbol{x}'), \qquad (34)$$

$$\mathcal{U}_{\rm v}^{(0)} = \tfrac{1}{2} \int d^3x' J^\mu(\boldsymbol{x}) G_{\rm v}(r) J_\mu(\boldsymbol{x}'), \qquad (35)$$

$$\mathcal{U}_{\rm v}^{(1)} = -\tfrac{1}{4} \int d^3x' \dot{J}^\mu(\boldsymbol{x}) Q_{\rm v}(r) \dot{J}_\mu(\boldsymbol{x}') \qquad (36)$$

etc., and the common time argument t is suppressed.

From now on we restrict ourselves to the description of weakly relativistic systems. Thus we can consider the terms $\mathcal{U}_{\rm s}^{(1)}$, $\mathcal{U}_{\rm v}^{(1)}$ ... as small as compared to $\mathcal{U}_{\rm s}^{(0)}$, $\mathcal{U}_{\rm v}^{(0)}$ .... Indeed, if the speed of light c is used explicitly, the expansion (28) appears as a series in powers of $1/c$. Then $\mathcal{U}_{\rm s}^{(1)}$, $\mathcal{U}_{\rm v}^{(1)} \ldots \sim 1/c^2$ and $\mathcal{U}_{\rm s}^{(0)}$, $\mathcal{U}_{\rm v}^{(0)} \ldots \sim 1$, while the terms $\mathcal{U}_{\rm s}^{(2)}$, $\mathcal{U}_{\rm v}^{(2)} \ldots \sim 1/c^4$ etc. are negligible in the present approximation.

The resulting Lagrangian,

$$\begin{aligned}
\bar{\mathcal{L}}^{(1)} &= \sum_{a=1}^{2} \mathcal{L}_a^{\rm free} - \mathcal{U}_{\rm s}^{(0)} - \mathcal{U}_{\rm v}^{(0)} - \mathcal{U}_{\rm p}^{(0)} - \mathcal{U}_{\rm a}^{(0)} - \mathcal{U}_{\rm t}^{(0)} \\
&\qquad - \mathcal{U}_{\rm s}^{(1)} - \mathcal{U}_{\rm v}^{(1)} - \mathcal{U}_{\rm p}^{(1)} - \mathcal{U}_{\rm a}^{(1)} - \mathcal{U}_{\rm t}^{(1)} \\
&= \mathcal{L}^{(0)} - \mathcal{U}_{\rm s}^{(1)} - \mathcal{U}_{\rm v}^{(1)} - \mathcal{U}_{\rm p}^{(1)} - \mathcal{U}_{\rm a}^{(1)} - \mathcal{U}_{\rm t}^{(1)}, \qquad (37)
\end{aligned}$$

leads to Euler-Lagrange equations which are second-order in time derivatives, because of the terms $\mathcal{U}_{\rm s}^{(1)}$, $\mathcal{U}_{\rm v}^{(1)}$ .... Thus, it describes the system as having twice as many degrees of freedom as the truncated Lagrangian $\mathcal{L}^{(0)}$ does, because $\psi^\dagger$ are no longer the conjugate momenta of ψ. This changes completely the dynamical content of the fields ψ_a. Since the second-order time derivatives occur in small terms only, they should be eliminated by means of the Euler-Lagrange equations of a lower-order approximation. But the resulting field equations are then not necessarily the Euler-Lagrange equations of a known Lagrangian. Thus the transition to the Hamiltonian and hence to a canonical quantum description becomes unclear.

To avoid this difficulty we can eliminate the time derivative in the small terms by means of the field equations or conservation laws. Though this can be done in general, it is more illuminating to do this separately, for example, for the cases of purely scalar and purely vector interactions.

3.1. Vector interaction only

The quantum-mechanical relation $\boldsymbol{\alpha} = \boldsymbol{v}/c$, where $\boldsymbol{v}$ is the free-particle velocity operator and c is the speed of light (here $\hbar = 1$), suggests the use of the estimate $\boldsymbol{\alpha} \sim \boldsymbol{\gamma} \sim 1/c$, at least in the small terms of the Lagrangian (37). Thus $\boldsymbol{J} \sim 1/c$, hence the $\mathcal{U}_{\rm v}^{(1)}$ term containing $\dot{\boldsymbol{J}}$ is negligible compared to the term (36) with $\dot{J}^0$. We use this estimate initially for the sake of simplicity. The full expression for $\mathcal{U}_{\rm v}^{(1)}$ including terms of order $1/c^4$ will be calculated later.

To eliminate the time derivative of the charge density $\dot{J}^0$ one can use the conservation law

$$\partial_\mu J^\mu = 0, \qquad \text{i.e.} \qquad \dot{J}^0 + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0. \tag{38}$$

This result is a consequence of the Euler-Lagrange equations. It follows from the reduced Lagrangian $\bar{\mathcal{L}}$ as well as from the truncated one $\mathcal{L}^{(0)}$. However, the direct use of the equations of motion (or their consequences) in the Lagrangian is not a correct procedure: it changes the equations of motion themselves. This fact was first emphasized in the case of the Golubenkov-Smorodinskii Lagrangian [12,13], and subsequently discussed in the literature [14]- [18]. Instead, one can use the method of "double zero", employed in Refs. [14,17]. In our case this consists of the following modification of the Lagrangian:

$$\bar{\mathcal{L}}^{(1)} \to \tilde{\mathcal{L}}^{(1)} = \bar{\mathcal{L}}^{(1)} - \mathcal{Z}, \tag{39}$$

where, in the present case of purely vector interactions,

$$\mathcal{Z}_{\rm v} = \tfrac{1}{4}\int d^3x' Q_{\rm v}(r)\left\{\dot{J}^0(\boldsymbol{x}) + \boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{x})\right\}\left\{\dot{J}^0(\boldsymbol{x}') + \boldsymbol{\nabla}'\cdot\boldsymbol{J}(\boldsymbol{x}')\right\}. \tag{40}$$

It is easy to see that the term $\mathcal{Z}$ possesses the property:

$$\delta\int d^3x\,\mathcal{Z}\Big|_{\delta\int d^3x\,\mathcal{L}^{(0)}=0} = 0, \tag{41}$$

so that it does not change the variational problem to the accuracy desired. On the other hand, it cancels those terms of $\bar{\mathcal{L}}^{(1)}$ which are quadratic in time derivatives of the fields. Thus the modified Lagrangian $\tilde{\mathcal{L}}^{(1)}$ yields equations of motion, which are first order in the time derivatives of the particle fields ψ_a.

Next, we perform the following transformation of the field variables:

$$\begin{aligned}
\psi_a &\to \xi_a = (1-{\rm i}\,q_a W_{\rm v})\psi_a \approx e^{-{\rm i}\,q_a W_{\rm v}}\psi_a,\\
\overline{\psi}_a &\to \overline{\xi}_a = (1+{\rm i}\,q_a W_{\rm v})\overline{\psi}_a \approx e^{{\rm i}\,q_a W_{\rm v}}\overline{\psi}_a,
\end{aligned} \tag{42}$$

where

$$W_{\rm v}(\boldsymbol{x}) = \tfrac{1}{2}\int d^3x' Q_{\rm v}(r)\boldsymbol{\nabla}'\cdot\boldsymbol{J}(\boldsymbol{x}'). \tag{43}$$

The transformation (42) removes time derivatives from the interaction part of $\tilde{\mathcal{L}}^{(1)}$. To the accuracy desired, and using the equality (31), the Lagrangian $\tilde{\mathcal{L}}^{(1)}$ can be written in the form

$$\tilde{\mathcal{L}}^{(1)} = \mathcal{L}^{(0)} - \mathcal{V}, \tag{44}$$

with $\mathcal{V} = \mathcal{V}_{\rm v}^{\parallel}$ (for the case of truncated vector interactions), where

$$\begin{aligned}
\mathcal{V}_{\rm v}^{\parallel} &= \tfrac{1}{4}\int d^3x' Q_{\rm v}(r)(\boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{x}))(\boldsymbol{\nabla}'\cdot\boldsymbol{J}(\boldsymbol{x}'))\\
&\simeq \tfrac{1}{4}\int d^3x'\left\{G_{\rm v}(r)\boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{J}(\boldsymbol{x}')) + rG'_{\rm v}(r)(\boldsymbol{n}\cdot\boldsymbol{J}(\boldsymbol{x}))(\boldsymbol{n}\cdot\boldsymbol{J}(\boldsymbol{x}'))\right\},
\end{aligned} \tag{45}$$

$\boldsymbol{n} = \boldsymbol{r}/r$, the symbol $\simeq$ denotes equality modulo surface terms, and the symbols ψ_a in the expressions $\mathcal{L}^{(0)}$, $\mathcal{V}$, *etc.* are replaced by ξ_a.

We note that time derivatives are now present in the Lagrangian $\tilde{\mathcal{L}}^{(1)}$ via its standard free-field part only. Moreover, the structure of $\tilde{\mathcal{L}}^{(1)}$ looks to be incorrect, i.e., as if the conservation law (38) is used directly in the small term $\mathcal{U}_{\rm v}^{(1)}$ of the Lagrangian $\bar{\mathcal{L}}^{(1)}$. The only correction resulting from the use of the double-zero method is that the Lagrangian $\tilde{\mathcal{L}}^{(1)}$ describes the dynamics of new fields ξ_a, not the old ones ψ_a.

Due to this structure of the Lagrangian, and since only the new ξ_a fields are used in the subsequent quantization procedure, we will use, in the final expressions for Lagrangian, the old symbols ψ_a to represent the new fields.

We now calculate $\mathcal{V}_{\rm v}$ using the total expression for $\mathcal{U}_{\rm v}^{(1)}$ (i.e., including those terms which contain $\dot{\boldsymbol{J}}$). For this purpose, from the equations (25) and (26), we deduce the equality:

$$\begin{aligned}
\dot{\boldsymbol{J}} \;=\; -\sum_{a=1}^{2} q_a\big\{&(\boldsymbol{\nabla}\psi_a^\dagger\cdot\boldsymbol{\alpha})\boldsymbol{\alpha}\psi_a + \psi_a^\dagger\boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi_a)\\
&- 2{\rm i}\,m_a\psi_a^\dagger\boldsymbol{\gamma}\psi_a + 2q_a\psi_a^\dagger\boldsymbol{A}\times\boldsymbol{\sigma}\psi_a\big\},
\end{aligned} \tag{46}$$

and then eliminate the time derivative $\dot{\boldsymbol{J}}$ from the interaction Lagrangian (36). Using the procedure described above, one obtains the Lagrangian (44) with $\mathcal{V}_{\rm v} = \mathcal{V}_{\rm v}^{\parallel} + \mathcal{V}_{\rm v}^{\perp}$, where

$$\begin{aligned}
\mathcal{V}_{\rm v}^{\perp} \;=\; &\tfrac{1}{4}\sum_a\sum_b q_a q_b\int d^3x' Q_{\rm v}(r)\\
&\times\big\{(\boldsymbol{\nabla}\psi_a^\dagger(\boldsymbol{x})\cdot\boldsymbol{\alpha})\boldsymbol{\alpha}\psi_a(\boldsymbol{x}) + \psi_a^\dagger(\boldsymbol{x})\boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi_a(\boldsymbol{x}))\\
&\qquad - 2{\rm i}\,m_a\psi_a^\dagger(\boldsymbol{x})\boldsymbol{\gamma}\psi_a(\boldsymbol{x})\big\}\\
&\times\Big\{(\boldsymbol{\nabla}'\psi_b^\dagger(\boldsymbol{x}')\cdot\boldsymbol{\alpha})\boldsymbol{\alpha}\psi_b(\boldsymbol{x}') + \psi_b^\dagger(\boldsymbol{x}')\boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}'\psi_b(\boldsymbol{x}'))\\
&\qquad - 2{\rm i}\,m_b\psi_b^\dagger(\boldsymbol{x}')\boldsymbol{\gamma}\psi_b(\boldsymbol{x}')\Big\}.
\end{aligned} \tag{47}$$

In deriving this expression we neglected the last term in the r.h.s. of Eq. (46) since it is nonlinear in the coupling constant and thus is not important in the first approximation.

3.2. Scalar interaction only

Using the field equations (25) and (26) with $\chi \neq 0$ leads to the result,

$$\dot{\rho} = -\sum_{a=1}^{2} g_a \bar{\psi}_a \boldsymbol{\alpha} \cdot \overleftrightarrow{\boldsymbol{\nabla}} \psi_a, \tag{48}$$

where $a\overleftrightarrow{\boldsymbol{\nabla}} b = a(\boldsymbol{\nabla} b) - (\boldsymbol{\nabla} a)b$. We then apply the "double zero" method (39) with

$$\begin{aligned} \mathcal{Z}_{\rm s} &= \tfrac{1}{4}\int d^3x' Q_{\rm s}(r) \left\{ \dot{\rho}(\boldsymbol{x}) + \sum_a g_a \bar{\psi}_a(\boldsymbol{x}) \boldsymbol{\alpha} \cdot \overleftrightarrow{\boldsymbol{\nabla}} \psi_a(\boldsymbol{x}) \right\} \\ &\quad \times \left\{ \dot{\rho}(\boldsymbol{x}') + \sum_b g_b \bar{\psi}_b(\boldsymbol{x}') \boldsymbol{\alpha} \cdot \overleftrightarrow{\boldsymbol{\nabla}}' \psi_b(\boldsymbol{x}') \right\} \end{aligned} \tag{49}$$

and use the transformation:

$$\begin{aligned} \psi_a &\;\to\; \xi_a = (1 - \mathrm{i}\, g_a W_{\rm s})\psi_a \approx e^{-\mathrm{i}\, g_a W_{\rm s}} \psi_a, \\ \overline{\psi}_a &\;\to\; \overline{\xi}_a = (1 + \mathrm{i}\, g_a W_{\rm s})\overline{\psi}_a \approx e^{\mathrm{i}\, g_a W_{\rm s}} \overline{\psi}_a, \end{aligned} \tag{50}$$

where

$$W_{\rm s}(\boldsymbol{x}) = \tfrac{1}{2}\sum_b g_b \int d^3x' Q_{\rm s}(r) \bar{\psi}_b(\boldsymbol{x}') \boldsymbol{\alpha} \cdot \overleftrightarrow{\boldsymbol{\nabla}}' \psi_b(\boldsymbol{x}'). \tag{51}$$

The resulting Lagrangian has the form (44) with

$$\begin{aligned} \mathcal{V}_{\rm s}(\boldsymbol{x}) &= -\tfrac{1}{4}\sum_a \sum_b g_a g_b \int d^3x' Q_{\rm s}(r) \left\{ \bar{\psi}_a(\boldsymbol{x}) \boldsymbol{\alpha} \cdot \overleftrightarrow{\boldsymbol{\nabla}} \psi_a(\boldsymbol{x}) \right\} \\ &\quad \times \left\{ \bar{\psi}_b(\boldsymbol{x}') \boldsymbol{\alpha} \cdot \overleftrightarrow{\boldsymbol{\nabla}}' \psi_b(\boldsymbol{x}') \right\}. \end{aligned} \tag{52}$$

3.3. Scalar-vector interaction

In this case we use the "double zero" method (39) with $\mathcal{Z} = \mathcal{Z}_{\rm s} + \mathcal{Z}_{\rm v}$. Of course, the equality (48) no longer holds exactly but does hold approximately as a linear approximation in the coupling constant. This accuracy is sufficient for obtaining results, which are accurate at least to lowest order relativistic corrections. In place of equations (42) or (50), the transformation of the field variables is now

$$\begin{aligned} \psi_a &\;\to\; \xi_a = (1 - \mathrm{i}\, g_a W_{\rm s} - \mathrm{i}\, q_a W_{\rm v})\psi_a \approx e^{-\mathrm{i}\, g_a W_{\rm s}} e^{-\mathrm{i}\, q_a W_{\rm v}} \psi_a, \\ \overline{\psi}_a &\;\to\; \overline{\xi}_a = (1 + \cdots)\overline{\psi}_a. \end{aligned} \tag{53}$$

The final Lagrangian can be expressed in the following canonical form,

$$\tilde{\mathcal{L}}^{(1)} = \sum_a \mathrm{i}\, \xi_a^\dagger \dot{\xi}_a - \mathcal{H}[\xi], \tag{54}$$

where the term $\mathcal{H}$ is free of time derivatives.

Thus, the total Hamiltonian is

$$\mathcal{H} = \sum_a \mathcal{H}_a^{\rm free} + \mathcal{U}_{\rm s}^{(0)} + \mathcal{U}_{\rm v}^{(0)} + \mathcal{V}_{\rm s} + \mathcal{V}_{\rm v} = \mathcal{H}^{(0)} + \mathcal{V}_{\rm s} + \mathcal{V}_{\rm v}, \tag{55}$$

where $\mathcal{H}^{(0)}$ is the truncated Hamiltonian (with retardation effects neglected),

$$\mathcal{H}_a^{\rm free} = \psi_a^\dagger(\boldsymbol{x})(-\mathrm{i}\, \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + m_a \beta)\psi_a(\boldsymbol{x}) \tag{56}$$

is the free Hamiltonian of the ψ_a field, and the remaining terms on the r.h.s. of Eq. (55) are given in equations (33), (35), (52), (45) and (47).

It follows from Eq. (54) that the variables ξ_a (but not ψ_a) are canonical ones, that is $\xi_a^\dagger$ are canonically conjugate to ξ_a (while ψ_a and $\psi_a^\dagger$ are not conjugates), and $\mathcal{H}[\xi]$ (not $\mathcal{H}[\psi]$) is a proper Hamiltonian. The original variables ψ_a are, therefore, no longer of concern for us. That is why we can (and do) use in final expressions the notation ψ_a for the new variables, ξ_a.

3.4. Pseudo-scalar, pseudo-vector and tensor interactions

The procedure is, in all respects, similar to the previous cases. Thus we shall write down only the essential input and output formulas.

The field equations (25), (26) yield the following expressions:

$$\dot{\tilde{\rho}} = -\sum_{a=1}^{2} \tilde{g}_a \left\{ \bar{\psi}_a \gamma^5 \boldsymbol{\alpha} \cdot \overleftrightarrow{\boldsymbol{\nabla}} \psi_a - 2\mathrm{i}\, m_a \psi_a^\dagger \gamma^5 \psi_a \right\}, \tag{57}$$

$$\dot{\tilde{J}}^0 = -\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{J}} + 2\mathrm{i} \sum_a \tilde{q}_a m_a \bar{\psi}_a \gamma^5 \psi_a, \tag{58}$$

$$\begin{aligned} \dot{\tilde{\boldsymbol{J}}} &= -\sum_{a=1}^{2} \tilde{q}_a \left\{ (\boldsymbol{\nabla}\psi_a^\dagger \cdot \boldsymbol{\alpha})\boldsymbol{\alpha}\gamma^5\psi_a + \psi_a^\dagger \boldsymbol{\alpha}\gamma^5(\boldsymbol{\alpha} \cdot \boldsymbol{\nabla}\psi_a) \right. \\ &\qquad \left. + 2\tilde{q}_a \psi_a^\dagger \tilde{\boldsymbol{A}} \times \boldsymbol{\sigma}\psi_a \right\}, \end{aligned} \tag{59}$$

$$\begin{aligned} \dot{J}^{\mu\nu} &= \tfrac{1}{2}\sum_{a=1}^{2} \varkappa_a \left\{ (\boldsymbol{\nabla}\bar{\psi}_a \cdot \boldsymbol{\alpha})\sigma^{\mu\nu}\psi_a - \bar{\psi}_a \sigma^{\mu\nu}(\boldsymbol{\alpha} \cdot \boldsymbol{\nabla}\psi_a) \right. \\ &\qquad \left. + \mathrm{i}\, m_a \bar{\psi}_a [\beta, \sigma^{\mu\nu}]\psi_a \right\} + \ldots \end{aligned} \tag{60}$$

The resulting Lagrangian has the form (44) where terms $\mathcal{V}_{\rm p}$, $\mathcal{V}_{\rm a}$ or $\mathcal{V}_{\rm t}$ are equal to $\mathcal{U}_{\rm p}^{(1)}$, $\mathcal{U}_{\rm a}^{(1)}$ or $\mathcal{U}_{\rm t}^{(1)}$, respectively, with the time derivatives of the currents eliminated by means of Eqs. (57)–(60).

4. Quantization, empty vacuum and two-fermion state

Equal-time quantization corresponds to the imposition of anticommutation rules for the fermion fields, namely

$$\{\psi_{a\alpha}(\boldsymbol{x}),\psi^\dagger_{b\beta}(\boldsymbol{y})\}=\delta_{ab}\delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}),\quad a,b=1,2,\ \alpha,\beta=1,...,4, \tag{61}$$

and all others vanish. We shall work in the Schrödinger picture thus we shall omit the time argument t in the expressions for the field and other operators, that is $\psi(\boldsymbol{x},t)=\psi(\boldsymbol{x})$, etc.

Next, we define the unconventional ("empty") vacuum state $|\tilde{0}\rangle$ by [2–4]

$$\psi_{a\alpha}(\boldsymbol{x})|\tilde{0}\rangle=0, \tag{62}$$

and write the normal-ordered Hamiltonian

$$;H;\ =\ \int d^3x\,;\mathcal{H}; \tag{63}$$

where $\mathcal{H}$ is defined by Eqs. (55), (56), (33), (35), (52), (45), (47) etc. The normal ordering is achieved by using the anticommutation rules (61) as usual; but note that it is not identical to the conventional normal ordering because of the unconventional empty vacuum that is being used, and the unconventional definition of ψ_a as annihilation operators and of $\psi^\dagger_a$ as creation operators. To underscore this unconventional procedure we use the notation $;H;$ rather than $:H:$.

With these stipulations, it follows that one–, two–, three–, ..., N-Dirac-fermion states form sectors of the Fock space which are closed (*i.e.* not coupled) under the action of the Hamiltonian (63). Thus each of these sectors can serve as a space for the eigenstate problem of (63).

Our first interest is the two-fermion state:

$$|2\rangle=\int d^3x\,d^3y\,F_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})\psi^\dagger_{1\alpha}(\boldsymbol{x})\psi^\dagger_{2\beta}(\boldsymbol{y})|\tilde{0}\rangle, \tag{64}$$

(summation on $\alpha,\beta=1,...,4$ is implied). This is an eigenstate of $;H;$ (Eq.(63)), that is

$$;H;|2\rangle=E|2\rangle, \tag{65}$$

provided that the 4×4 eigenmatrix F satisfies the wave equation which we deduce below.

The action of the free-field part of $;H;$ on $|2\rangle$ is evident:

$$;H_1^{\rm free};|2\rangle=\int d^3x\,d^3y\,\psi^\dagger_{1\alpha}(\boldsymbol{x})\psi^\dagger_{2\beta}(\boldsymbol{y})\left\{-{\rm i}\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_x+m_1\beta\right\}_{\alpha\gamma}F_{\gamma\beta}(\boldsymbol{x},\boldsymbol{y})|\tilde{0}\rangle, \tag{66}$$

and similarly for $;H_2^{\rm free};|2\rangle$.

Let us represent the interaction part of $;H;$ in integral form. It is easy to do so by inserting an appropriate number of δ–functions and their derivatives. Then any operator constituting the interaction part of $;H;$ can be expressed in the following general form:

$$\begin{aligned};O;\ &=\ \sum_{a=1}^{2}\sum_{b=1}^{2};O_{ab};=\sum_{a=1}^{2}\sum_{b=1}^{2}\int d^3z\,d^3u\,d^3v\,d^3w\\ &\quad\times\psi^\dagger_{a\alpha}(\boldsymbol{z})\psi^\dagger_{b\beta}(\boldsymbol{u})\psi_{a\gamma}(\boldsymbol{v})\psi_{b\delta}(\boldsymbol{w})\mathcal{O}_{ab\alpha\beta\gamma\delta}(\boldsymbol{z},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\end{aligned} \tag{67}$$

(summation on Greek indices implied). It is easy to show, using the anticommutation relations (61) and the definitions of the vacuum and two-fermion states, (62) and (64), that $;O_{11};|2\rangle=;O_{22};|2\rangle=0$ while

$$\begin{aligned};O_{12};|2\rangle\ &=\ \int d^3z\,d^3u\,d^3v\,d^3w\\ &\quad\times\psi^\dagger_{1\alpha}(\boldsymbol{z})\psi^\dagger_{2\beta}(\boldsymbol{u})\mathcal{O}_{12\alpha\beta\gamma\delta}(\boldsymbol{z},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})F_{\gamma\delta}(\boldsymbol{v},\boldsymbol{w})|\tilde{0}\rangle,\end{aligned} \tag{68}$$

and a similar result for $;O_{21};|2\rangle$. Using Eqs. (63), (55), (56), (33), (35), (52), (45), (66) and (68) we obtain the following wave equation for the eigenfunction matrix $F(\boldsymbol{x},\boldsymbol{y})$:

$$\begin{aligned}\{h_1(\boldsymbol{x})+h_2(\boldsymbol{y})+U_{\rm v}(\boldsymbol{x},\boldsymbol{y})+V_{\rm v}(\boldsymbol{x},\boldsymbol{y})+U_{\rm s}(\boldsymbol{x},\boldsymbol{y})\\ +V_{\rm s}(\boldsymbol{x},\boldsymbol{y})+\ldots-E\}\,F(\boldsymbol{x},\boldsymbol{y})=0,\end{aligned} \tag{69}$$

where

$$h_a(\boldsymbol{x})\ =\ -\,{\rm i}\,\boldsymbol{\alpha}_a\cdot\boldsymbol{\nabla}_x+m_a\beta_a,\qquad a=1,2, \tag{70}$$

$$U_{\rm v}(\boldsymbol{x},\boldsymbol{y})\ =\ q_1q_2G_{\rm v}(r)\left\{1-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2\right\}, \tag{71}$$

$$V_{\rm v}(\boldsymbol{x},\boldsymbol{y})\ =\ V_{\rm v}^{\parallel}(\boldsymbol{x},\boldsymbol{y})+V_{\rm v}^{\perp}(\boldsymbol{x},\boldsymbol{y}), \tag{72}$$

$$V_{\rm v}^{\parallel}(\boldsymbol{x},\boldsymbol{y})\ =\ \tfrac{1}{2}q_1q_2\left\{G_{\rm v}(r)\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2+rG'_{\rm v}(r)(\boldsymbol{n}\cdot\boldsymbol{\alpha}_1)(\boldsymbol{n}\cdot\boldsymbol{\alpha}_2)\right\}, \tag{73}$$

$$\begin{aligned}V_{\rm v}^{\perp}(\boldsymbol{x},\boldsymbol{y})\ &=\ \tfrac{1}{2}q_1q_2\left\{(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)Q_{\rm v}(r)\right.\\ &\quad\left.-(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)Q_{\rm v}(r)(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)\right.\end{aligned}$$

$$
\begin{aligned}
&- (\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)Q_{\rm v}(r)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)\\
&+ (\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)Q_{\rm v}(r)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)\}\,, \qquad (74)\\
U_{\rm s}(\boldsymbol{x},\boldsymbol{y}) &= g_1g_2G_{\rm s}(r)\beta_1\beta_2, \qquad (75)\\
V_{\rm s}(\boldsymbol{x},\boldsymbol{y}) &= -\tfrac{1}{2}g_1g_2\,\{Q_{\rm s}(r)(\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_y)\\
&+ (\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}_x)Q_{\rm s}(r)(\boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_y)\\
&+ (\boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_y)Q_{\rm s}(r)(\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}_x)\\
&+ (\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_y)Q_{\rm s}(r)\}\\
&= \tfrac{1}{2}g_1g_2\beta_1\beta_2\,\{Q_{\rm s}(r)h_1(\boldsymbol{x})h_2(\boldsymbol{y}) + h_1(\boldsymbol{x})h_2(\boldsymbol{y})Q_{\rm s}(r)\\
&+ h_1(\boldsymbol{x})Q_{\rm s}(r)h_2(\boldsymbol{y}) + h_2(\boldsymbol{y})Q_{\rm s}(r)h_1(\boldsymbol{x})\}\,, \qquad (76)\\
U_{\rm p}(\boldsymbol{x},\boldsymbol{y}) &= \tilde{g}_1\tilde{g}_2G_{\rm p}(r)\beta_1\gamma_1^5\beta_2\gamma_2^5, \qquad (77)\\
V_{\rm p}(\boldsymbol{x},\boldsymbol{y}) &= -\tfrac{1}{2}\tilde{g}_1\tilde{g}_2\gamma_1^5\gamma_2^5\,\{Q_{\rm p}(r)(\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_y)\\
&+ (\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}_x)Q_{\rm p}(r)(\boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_y)\\
&+ (\boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_y)Q_{\rm p}(r)(\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}_x)\\
&+ (\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_y)Q_{\rm p}(r)\\
&+ 2{\rm i}\,m_1\,(Q_{\rm p}(r)\boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_y + \boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_yQ_{\rm p}(r))\\
&+ 2{\rm i}\,m_2\,(Q_{\rm p}(r)\boldsymbol{\gamma}_1\cdot\boldsymbol{\nabla}_x + \boldsymbol{\gamma}_2\cdot\boldsymbol{\nabla}_xQ_{\rm p}(r))\\
&- 4m_1m_2Q_{\rm p}(r)\}\\
&= \tfrac{1}{2}\tilde{g}_1\tilde{g}_2\gamma_1^5\gamma_2^5\beta_1\beta_2\,\{Q_{\rm p}(r)h_1(\boldsymbol{x})h_2(\boldsymbol{y}) + h_1(\boldsymbol{x})h_2(\boldsymbol{y})Q_{\rm p}(r)\\
&+ h_1(\boldsymbol{x})Q_{\rm p}(r)h_2(\boldsymbol{y}) + h_2(\boldsymbol{y})Q_{\rm p}(r)h_1(\boldsymbol{x})\}\,, \qquad (78)\\
U_{\rm a}(\boldsymbol{x},\boldsymbol{y}) &= \tilde{q}_1\tilde{q}_2G_{\rm a}(r)\gamma_1^5\gamma_2^5\,\{1-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2\}\,, \qquad (79)\\
V_{\rm a}(\boldsymbol{x},\boldsymbol{y}) &= \tfrac{1}{2}\tilde{q}_1\tilde{q}_2\gamma_1^5\gamma_2^5\,\{G_{\rm a}(r)\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2 + rG'_{\rm a}(r)(\boldsymbol{n}\cdot\boldsymbol{\alpha}_1)(\boldsymbol{n}\cdot\boldsymbol{\alpha}_2)\\
&- 2{\rm i}\,m_1\beta_1(\boldsymbol{\alpha}_2\cdot\boldsymbol{n})rG_{\rm p} + 2{\rm i}\,m_2\beta_2(\boldsymbol{\alpha}_1\cdot\boldsymbol{n})rG_{\rm p}\\
&- 4m_1m_2\beta_1\beta_2Q_{\rm p}(r)\\
&+ (\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)Q_{\rm a}(r)\\
&- (\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)Q_{\rm a}(r)(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)\\
&- (\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)Q_{\rm a}(r)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)\\
&+ (\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)Q_{\rm a}(r)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)\}, \qquad (80)\\
U_{\rm t}(\boldsymbol{x},\boldsymbol{y}) &= \tfrac{1}{4}\varkappa_1\varkappa_2G_{\rm t}(r)\beta_1\beta_2\sigma_1^{\mu\nu}\sigma_{2\mu\nu}\\
&= \tfrac{1}{2}\varkappa_1\varkappa_2G_{\rm t}(r)\beta_2\beta_2\{\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2 + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\}, \qquad (81)\\
V_{\rm t}(\boldsymbol{x},\boldsymbol{y}) &= -\tfrac{1}{8}\varkappa_1\varkappa_2\,\{(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)\sigma_1^{\mu\nu}\sigma_{2\mu\nu}Q_{\rm t}(r)\\
&+ (\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)\sigma_1^{\mu\nu}\sigma_{2\mu\nu}Q_{\rm v}(r)(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)\\
&+ (\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)\sigma_1^{\mu\nu}\sigma_{2\mu\nu}Q_{\rm v}(r)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)\\
&+ \sigma_1^{\mu\nu}\sigma_{2\mu\nu}Q_{\rm t}(r)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)
\end{aligned}
$$

$$
\begin{aligned}
&- {\rm i}\,m_1[\beta_1,\sigma_1^{\mu\nu}]\,((\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y)\sigma_{2\mu\nu}Q_{\rm t}(r)\\
&+ \sigma_{2\mu\nu}Q_{\rm t}(r)(\boldsymbol{\alpha}_2\cdot\boldsymbol{\nabla}_y))\\
&- {\rm i}\,m_2[\beta_2,\sigma_2^{\mu\nu}]\,((\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x)\sigma_{1\mu\nu}Q_{\rm t}(r)\\
&+ \sigma_{1\mu\nu}Q_{\rm t}(r)(\boldsymbol{\alpha}_1\cdot\boldsymbol{\nabla}_x))\\
&- m_1m_2[\beta_2,\sigma_1^{\mu\nu}][\beta_2,\sigma_{2\mu\nu}]Q_{\rm t}(r)\}\,, \qquad (82)
\end{aligned}
$$

$r=|\boldsymbol{r}|=|\boldsymbol{x}-\boldsymbol{y}|$, $\boldsymbol{n}=\boldsymbol{r}/r$, and the action of the operators $\boldsymbol{\alpha}_a$, β_a and $\boldsymbol{\gamma}_a$ $(a=1,2)$ is defined as follows: $\boldsymbol{\alpha}_1F=\boldsymbol{\alpha}F$, $\boldsymbol{\alpha}_2F=F\boldsymbol{\alpha}^T$ etc., where $\boldsymbol{\alpha}$, β and $\boldsymbol{\gamma}$ are the standard Dirac matrices.

In the $\mathbf{P}_{\rm total}=0$ ("rest") reference frame the problem simplifies since the wave function $F(\boldsymbol{r})$ depends only on the relative position vector $\boldsymbol{r}$. The corresponding wave equation for $F(\boldsymbol{r})$ follows from (69)-(82) by the formal substitution: $\boldsymbol{\nabla}_x\to\boldsymbol{\nabla}_r$, $\boldsymbol{\nabla}_y\to-\boldsymbol{\nabla}_r$.

The equation (69)-(82) is our main result. It is of sufficient generality so as to describe various systems: two nucleon systems, two quark systems, and even muonium-like atomic systems. Indeed, this last case (muonium) corresponds to a purely massless vector interaction (i.e., only $U_{\rm v}$, $V_{\rm v}^{\parallel}$ are non-zero, $G_{\rm v}=-\alpha/r$). The partial wave decomposition and solutions of this equation for the muonium system are given in Ref. [4].

The case of a purely scalar interaction has been investigated by Childers [19]. Indeed his equation (3.7) corresponds identically to the present result (76) with Coulombic purely scalar interactions (only $U_{\rm s}$ and $V_{\rm s}$ non-zero, with $G_{\rm s}\propto 1/r$ and $Q_{\rm s}\propto r$, cf. Eqs. (29)–(31)). However in this paper we give the general form of the potentials for arbitrary superposition of vector, scalar, pseudovector, pseudoscalar and tensor interaction, including retardation effects.

Several authors have considered the two-fermion problem with empirical forms of potentials. For example, Brayshaw [23] employs the so-called minimal coupling form of the scalar confining interaction $(\beta_1+\beta_2)r$ (his Eq. (7) in which the Dirac's β matrices appear summed rather than as a product in our case, cf. $U_{\rm s}$, Eq. (64)). Simenog and Turovsky [24] define the pseudoscalar potential to be of the form $\boldsymbol{\alpha}_1\beta_1\cdot\boldsymbol{\alpha}_2\beta_2V_{\rm p}(r)$, in contrast to our result $U_{\rm p}=\tilde{g}_1\tilde{g}_2\beta_1\gamma_1^5\beta_2\gamma_2^5G_{\rm p}(r)$. However, the forms given in Eqs. (69)–(82) of this paper we derived from the underlying quantum field theory.

5. Concluding remarks

Starting from the classical field model of two fermionic fields interacting via an arbitrary superposition of any of (pseudo) scalar, vector and ten-

sor interactions we eliminate the mediating fields by expressing them in terms of corresponding Green functions and the fermionic source densities and currents. This “reduced" theory is then quantized.

The characteristic feature of the present approach is that the Hamiltonian of the “reduced" theory has exact few-fermion eigenstates, if an unconventional “empty" vacuum state is used and terms of the Hamiltonian that describe the emission and absorption of physical quanta of the mediating fields are ignored (cf. below Eq. (24)). Thus the two-fermion problem set up in the present work appears as an eigenstate-eigenenergy problem. The relativistic wave equations for two fermions, interacting via an arbitrary superposition of any of (pseudo) scalar, vector and tensor interactions, are derived.

The lowest order terms describing retardation effect are determined and retained in the interaction potentials. The retardation corrections of the scalar interaction agrees with that presented in [19] for the case of standard massless mediator. This correction contains second-order derivatives of the wave function while the principal part is of first order. In general, the same applies to the other interactions. This means that the retardation corrections should be considered perturbationally, not within an exact treatment of the eigenstate problem.

It is not obvious whether all derived retardation terms contribute in the first nontrivial approximation. Indeed, the transverse retardation term $V_{\rm v}^{\perp}$ of the vector interaction is quartic in the $\boldsymbol{\alpha}$-matrices. Thus, following the estimate $\boldsymbol{\alpha} \sim 1/c$ used in the present work, this term $\sim 1/c^4$ would seem to be negligibly small. With the remaining longitudinal correction $V_{\rm v}^{\parallel}$ equation (69) coincides with the Breit equation [20,21] (but generalized for the case of arbitrary mediating fields of the vector type), and without $V_{\rm v}^{\parallel}$ it reproduces the Eddington-Gaunt equation [21].

On the other hand, the estimate $\boldsymbol{\alpha} \sim 1/c$ originates from the fact that $\boldsymbol{\alpha}$ has off-diagonal elements only, and mixes small and large components of the wave function. The matrices $\boldsymbol{\gamma}$, γ^5 etc. have the same property. But products of these matrices mix the components further and may return the large component to its original place. This suggests that products of $\boldsymbol{\alpha}$'s should be estimated as 1 rather than $1/c^2$. This ambiguity does not occur within conventional QFT, where operators in wave equations (such as the Salpeter equation), which contain the Dirac matrices, are projected onto positive-energy states. Thus powers of off-diagonal operators vanish. In the present approach (with the “empty" vacuum) positive- and negative-energy states are mixed together. Thus products of off-diagonal operators should be suppressed “by hand”, i.e., using the estimate $\boldsymbol{\alpha} \sim 1/c$.

Breit-type equations have been used recently in hadron and nuclear physics (see, for example, [19,23,22,24] and Refs. therein). Thus, taking account of interactions of various relativistic structure, especially with retardation corrections, as is done in the present paper, may refine results in these fields. The interaction in the present approach is determined by standard or some effective covariant Green function of the mediating fields (and by its spin-tensor structure), which may be phenomenologically defined. Alternatively, one can choose the function $G(r)$ to be proportional to the nonrelativistic potential. Then the relation (31) determines the function $Q(r)$ up to an additive constant which can be chosen on physical grounds.

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Аскольд Дувіряк
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Варіяційні хвильові рівняння двох фермійонів, що взаємодіють через скалярне, псевдоскалярне, векторне, псевдовекторне і тензорне поле

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