



ІНСТИТУТ
ФІЗИКИ
КОНДЕНСОВАНИХ
СИСТЕМ

ICMP-02-10E

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HEURISTIC MODELS OF TWO-FERMION RELATIVISTIC
SYSTEMS WITH FIELD-TYPE INTERACTION

УДК: 531/533; 530.145; 531.18; 539.12

PACS: 03.65.Pm, 03.65.Ge, 12.39.Pn

Евристичні моделі дво-ферміонних релятивістичних систем із взаємодією польового типу

А.Дувіряк

Анотація. Використовується ланцюжок простих евристичних прийомів для отримання пертурбативних та точно розв'язуваних спектрів для сім'ї дво-ферміонних зв'язаних систем з кулоно-подібною взаємодією. У випадку електромагнетної взаємодії отриманий спектр збігається з точністю до другого порядку за константою взаємодії із спектром, що впливає з квантової електродинаміки. Розбіжність виникає лише для S-станів, що є відомою трудностю у проблемі зв'язаних станів. Розглядається також утримуюча взаємодія.

Heuristic Models of Two-Fermion Relativistic Systems with Field-Type Interaction

A.Duviryak

Abstract. We use the chain of simple heuristic expedients to obtain perturbative and exactly solvable relativistic spectra for a family of two-fermionic bound systems with Coulomb-like interaction. In the case of electromagnetic interaction the spectrum coincides up to the second order in a coupling constant with that following from the quantum electrodynamics. Discrepancy occurs only for S-states which is the well-known difficulty in the bound-state problem. The confinement interaction is considered too.

Подається в Journal of Physics G

Submitted to Journal of Physics G

1. Introduction

A nonrelativistic two-body problem reduces completely to the single-body problem with a central potential. The single-body Schrödinger equation becomes the two-body one (for the centre-of-mass frame of reference) if to replace the particle mass by the reduced mass, and to understand the radius-vector \mathbf{r} as the relative position vector.

In the relativistic case the relation between the single- and two-particle problems is not so transparent. There are few reasons for that. First, a spin appears on a physical stage. The existence of spin diversifies properties of both interacting particles of matter and fields mediating this interaction. Thus even in the single-particle problem various relativistic wave equations such as the Klein-Gordon, Dirac, Duffin-Kemmer and other equations exist, and they can involve scalar, vector, tensor and other interaction potentials. Second, there exist different approaches to the relativistic two- and few- body problem. The most profound approaches are based on the quantum field theory (QFT), especially on the perturbative QFT. They lead to complicated integral equations (such as the Bethe-Salpeter equation [1,2]), coupled sets of differential equations (such as the Breit equation [3,2]), or their higher order differential reductions (such as the Fermi-Pauli equation [4,2]). Other approaches such as quasipotential [5]–[7] or ones based on the relativistic direct interaction theory (RDIT) [8]–[9] are semi- or purely phenomenological. They manifest a general structure of relativistic potentials and wave equations while features of concrete interactions must be brought from other sources (for example, from the classical or quantum field theory).

I. Todorov has observed a simple way how the Klein-Gordon equation can be transformed into the quasipotential equation describing the scalar and vector interaction of two spinless particles [5,7]. In the case of Coulomb-like interaction this equation gives the spectrum which agrees with QFT result up to α^4 -terms of coupling constant expansion. The generalization for an arbitrary field-type interaction (including higher rank tensor interactions) arises naturally from the Schwinger source theory [10] and the Fokker formalism [8], and results in some RDIT models [11]–[13]. In the present paper we construct relativistic wave equations appropriate for the description of the field-type interaction of two particles with spin $\frac{1}{2}$.

At the beginning we summarize results concerning spinless system. Namely, in Section 2 we formulate the family of relativistic wave equations which describe the scalar, vector and gravitational (i.e., tensor) interaction of two scalar particles. These equations have a common ef-

fective single-particle structure. In the case of Coulomb-like interaction they are exactly solvable. The mass spectra coincide with that obtained algebraically in Ref. [13], and agree up to α^4 with known QFT results. The only disagreement exists for S-states.

Then we modify the wave equations in order to describe two-fermion systems. The spin-orbital, spin-spin and tensor corrections to the scalar, vector and gravitational interactions are brought from QFT¹ (Section 3). In Section 4 by means of an appropriate rescaling of \mathbf{r} spin interaction is included into the eigenstate problem as small correction to the effective Coulomb Hamiltonian. Calculations with the first order perturbation theory (Section 5) reproduce the QED muonium spectrum up to α^4 [15] and give a generalization for the cases of scalar and gravitational interactions.

Spin potential corrections depend on $r = |\mathbf{r}|$ as $1/r^3$. Due to properties of matrix elements with Coulomb bound states we transform spin corrections into $1/r^2$ terms in such a way that perturbative spectrum (up to α^4) does not change. In this form the problem appears exactly solvable, which is shown explicitly in Section 6.

The Todorov construction was proposed in the case of Coulomb-like interactions. Here (in Section 7) we approve this recipe for a system with confining interaction, and make an appropriate modification to account spin effects.

2. Spectra of systems of two spinless particles

Let us consider the stationary Klein-Gordon equation for particle of the rest mass m in the scalar potential $V_s(r)$ and the vector one $V_v(r)$ depending on $r = |\mathbf{r}|$:

$$\Delta \Psi(\mathbf{r}) + ([E - V_v(r)]^2 - [m + V_s(r)]^2) \Psi(\mathbf{r}) = 0. \quad (1)$$

Following the Todorov's observation in the quasipotential approach [5,7] one can construct the appropriate two-particle wave equation by the following substitution:

$$E \rightarrow E_M = \frac{M^2 - m_1^2 - m_2^2}{2M}, \quad m \rightarrow m_M = \frac{m_1 m_2}{M}, \quad (2)$$

where m_a is the rest mass of a th particle, and M is the total mass of the system, i.e., the energy in the centre-of-mass frame of reference. One

¹We note that two-particle spin-orbital interaction can be recovered completely from the single-particle one [14] but this is not concerned with spin-spin and tensor terms.

obtains the Schrödinger-like equation:

$$\Delta\Psi(\mathbf{r}) + [Q_M - U_M(r)]\Psi(\mathbf{r}) = 0, \quad (3)$$

where

$$Q_M \equiv E_M^2 - m_M^2 = \frac{1}{4M^2}[M^2 - m_+^2][M^2 - m_-^2], \quad (4)$$

is the on-shell value of the relative momentum squared as a function of M [7],

$$U_M(r) = 2[m_M V_s(r) + E_M V_v(r)] + V_s^2(r) - V_v^2(r) \quad (5)$$

is the effective potential, and $m_{\pm} = m_1 \pm m_2$. In the nonrelativistic limit the equation (3) becomes the usual Schrödinger equation with the nonrelativistic potential $V(r) = V_s(r) + V_v(r)$ where r is the distance between particles.

The effective potential $U_M(r)$ depends of the total mass M . Thus the equation (3) is of the quasipotential type, and problems can occur with the consistent quantum-mechanical treatment [7].

In the case of Coulomb-like interaction

$$V_s(r) = -\alpha_s/r, \quad (6)$$

$$V_v(r) = -\alpha_v/r \quad (7)$$

with the coupling constants $\alpha_s > 0$ and $\alpha_v > 0$ the equation (3) is exactly solvable. Moreover, for the case of electromagnetic (vector) interaction the corresponding spectra coincide (except for the ground S-states) with those following from the scalar quantum electrodynamics in the second order of a coupling constant [6,5].

The simple Todorov recipe embraces the cases of scalar and vector interactions (and their superposition). The generalization to the cases of higher-rank tensor interactions can be built on the base of the family of RDIT models (known as time-asymmetric) [11,12]. This leads to the Schrödinger-like (quasipotential) equation (3) with

$$U_M(r) = -2m_M f(\lambda) \frac{\alpha}{r} + \gamma \frac{\alpha^2}{r^2}, \quad (8)$$

where $\lambda = E_M/m_M$, and the function $f(\lambda)$ (such that $f(1) = 1$) as well as the constant γ depend on the tensor nature of interaction [12,13]. For example, for the scalar, vector and gravitational interaction (or another second-rank tensor interaction, for example, the strong gravitation) we

have:

$$f_s(\lambda) = 1, \quad \gamma_s = 1, \quad (9)$$

$$f_v(\lambda) = \lambda, \quad \gamma_v = -1, \quad (10)$$

$$f_g(\lambda) = 2\lambda^2 - 1, \quad \gamma_g = -6. \quad (11)$$

The superposition of these interactions can be considered by means of the superposition of the functions and constants (9)-(11). Also we note that for scalar and vector interactions (and their superposition) the effective potentials (8) (from the time-asymmetric models) is identical to (5) with (6) or/and (7) (from the Todorov recipe).

The mass spectrum following from the equation (3), (4), (8) can be presented in the implicit form:

$$M^2 = m_1^2 + m_2^2 + 2m_1 m_2 \lambda, \quad (12)$$

where λ is a positive solution of the equation:

$$\frac{1 - \lambda^2}{f^2(\lambda)} = \frac{\alpha^2}{\nu^2}, \quad (13)$$

and

$$\nu = n_r + \frac{1}{2} + \sqrt{(\ell + \frac{1}{2})^2 + \alpha^2 \gamma} \quad (14)$$

is effective ‘‘principal quantum number’’; here $n_r = 0, 1, \dots$ is the radial quantum number and $\ell = 0, 1, \dots$ is the angular quantum number.

For the scalar, vector and gravitational interactions the equation (13) is solvable:

$$\lambda_s = \sqrt{1 - \alpha^2/\nu^2}, \quad (15)$$

$$\lambda_v = 1 / \sqrt{1 + \alpha^2/\nu^2}, \quad (16)$$

$$\lambda_g = \frac{1}{2\sqrt{2}} \sqrt{4 - \frac{\nu^2}{\alpha^2} + \frac{\nu}{\alpha} \sqrt{8 + \nu^2/\alpha^2}}. \quad (17)$$

Approximately, with accuracy up to α^4 , we have :

$$M \approx m_+ - \frac{m_r \alpha^2}{2n^2} + \frac{m_r \alpha^4}{2n^4} \left[f'(1) - \frac{1}{4} - \frac{m_r}{4m_+} \right] + \frac{m_r \alpha^4}{n^3} \frac{\gamma}{2\ell + 1}, \quad (18)$$

where $f' = df/d\lambda$, $m_r = m_1 m_2 / m_+$ is the reduced mass, and $n = n_r + \ell + 1$ is the principal quantum number.

3. Two-fermion systems: including spin effects

The weakly relativistic system of two fermions interacting via scalar or/and vector field can be described by the generalized Breit-Fermi Hamiltonian [16,17]. Besides the nonrelativistic Coulomb Hamiltonian, it includes relativistic kinematic terms, spin-independent and spin-dependent corrections to the interaction. Some of these terms are singular and can be taken into account as perturbations only.

Here we do not consider the Breit-Fermi Hamiltonian. Instead, we modify the Todorov recipe in order to describe the spin effects in two-fermionic systems. For this purpose we need only a spin-dependent part W of the Breit-Fermi Hamiltonian. For the scalar and vector interaction it is [16,17]:

$$\begin{aligned} W_s &= -\frac{1}{4} \mathbf{L} \cdot \left(\frac{\boldsymbol{\sigma}_1}{m_1^2} + \frac{\boldsymbol{\sigma}_2}{m_2^2} \right) \frac{V_s'(r)}{r}, \\ W_v &= \frac{1}{4} \mathbf{L} \cdot \left(\left[\frac{1}{m_1^2} + \frac{2}{m_1 m_2} \right] \boldsymbol{\sigma}_1 + \left[\frac{1}{m_2^2} + \frac{2}{m_1 m_2} \right] \boldsymbol{\sigma}_2 \right) \frac{V_v'(r)}{r} \\ &\quad + \frac{1}{12m_1 m_2} \left(\frac{1}{r} V_v'(r) - V_v''(r) \right) T \\ &\quad + \frac{1}{6m_1 m_2} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \Delta V_v(r), \end{aligned} \quad (19)$$

where $\mathbf{L} = -i \mathbf{r} \times \boldsymbol{\nabla}$ is the orbital momentum operator, $\boldsymbol{\sigma}_a$ is the spin operator acting on the a th particle spin variable, and $T = 3(\boldsymbol{\sigma}_1 \cdot \mathbf{n})(\boldsymbol{\sigma}_2 \cdot \mathbf{n}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ is the tensor operator. In the case of gravitational interaction (with the nonrelativistic potential $V_g(r) = -\alpha_g/r$, where $\alpha_g = Gm_1 m_2$, and G is the gravitational constant) we have [18]:

$$\begin{aligned} W_g &= \frac{1}{4} \mathbf{L} \cdot \left(\left[\frac{3}{m_1^2} + \frac{4}{m_1 m_2} \right] \boldsymbol{\sigma}_1 + \left[\frac{3}{m_2^2} + \frac{4}{m_1 m_2} \right] \boldsymbol{\sigma}_2 \right) \frac{\alpha_g}{r^3} \\ &\quad + \frac{\alpha_g}{4m_1 m_2 r^3} T + \frac{2\pi\alpha_g}{3m_1 m_2} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \delta(\mathbf{r}). \end{aligned} \quad (21)$$

Now in order to construct a two-fermion equation we replace a non-relativistic potential V by $\tilde{V} = V + W$ in the effective potential U_M (Eq. (5) or (8)). The resulting quasipotential equation is not solvable, and we should apply some approximate method.

In the case of Coulomb-like interaction the spin term W is meant to be small as to compare to the nonrelativistic potential $V(r) = -\alpha/r$. Thus we can modify approximately the effective potential (8) as follows:

$$U_M(r) \rightarrow \tilde{U}_M(r) \approx U_M(r) + 2m_M f(\lambda) W. \quad (22)$$

Now one can account the spin correction by means of the perturbation method. In so doing we note the following. First, the original (non-perturbed) equation (3) fails to describe correctly S-states. Thus we will neglect δ -functional terms in W (i.e., the last term in r.h.s. of Eq. (21), and the last term in r.h.s. of Eq. (20) in the case (7)) as they contribute in S-states only. Second, the modified equation (as well as the unperturbed one (3)) is quasipotential but not the true Schrödinger equation. Thus it needs some minor reformulation to be tractable within the perturbation method.

4. Spin corrections to the Coulomb-like interaction

In the case of Coulomb-like interaction the two-fermion wave equation reads:

$$\left[\mathbf{p}^2 - 2m_M f(\lambda) \left(\frac{\alpha}{r} - W \right) + \gamma \frac{\alpha^2}{r^2} - Q_M \right] \Psi = 0, \quad (23)$$

where $\mathbf{p} = -i \boldsymbol{\nabla}$. Using the substitution

$$\mathbf{r} = \boldsymbol{\rho}/R_M, \quad \mathbf{p} = R_M \boldsymbol{\pi} \quad \text{with} \quad R_M = \alpha m_M f(\lambda) \quad (24)$$

we present the equation (23) in the dimensionless Hamiltonian form:

$$H \Psi = \varepsilon \Psi. \quad (25)$$

Here

$$H = H^{(0)} + H^{(1)} \quad (26)$$

is the total Hamiltonian,

$$H^{(0)} = \frac{1}{2} \boldsymbol{\pi}^2 - \frac{1}{\rho} \quad (27)$$

is the basic Coulomb Hamiltonian,

$$H^{(1)} = \alpha^2 \Gamma, \quad \Gamma = \frac{\gamma}{2\rho^2} + \frac{1}{\rho^3} \Sigma(\mathbf{n}) \quad (28)$$

is a perturbation since α is considered as a small parameter, and

$$\varepsilon = \frac{Q_M}{2R_M^2} = \frac{\lambda^2 - 1}{2\alpha^2 f^2(\lambda)} \quad (29)$$

is a dimensionless energy (i.e., a spectral parameter).

With sufficient accuracy (i.e., up to terms $\sim O(\alpha)$) the last term of Γ is equal to $W/(\alpha^3 R_M)$ but does not depend on M . This is provided by use of approximate equality $M \approx m_+$ in small terms. The general form of the operator Σ acting on angular and spin variables is:

$$\Sigma = [(\xi - \delta^2)\mathbf{L} \cdot \boldsymbol{\sigma}_+ + 2\eta\delta\mathbf{L} \cdot \boldsymbol{\sigma}_- + \zeta(1 - \delta^2)T] / 16, \quad (30)$$

where $\boldsymbol{\sigma}_\pm = \boldsymbol{\sigma}_1 \pm \boldsymbol{\sigma}_2$, $\delta = m_-/m_+$, and the constants ξ , η and ζ for scalar, vector and gravitational interactions are defined as follows:

$$\xi_s = -1, \quad \eta_s = 1, \quad \zeta_s = 0, \quad (31)$$

$$\xi_v = 3, \quad \eta_v = -1, \quad \zeta_v = 1, \quad (32)$$

$$\xi_g = 7, \quad \eta_v = -3, \quad \zeta_v = 1. \quad (33)$$

5. Basic states and first-order perturbation theory

The basic Hamiltonian (27) commutes with operators of orbital angular momentum \mathbf{L} , total spin $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}_+$, total angular momentum $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\sigma}_+$ and parity P . In order to write down the basic eigenfunctions $\Psi^{(0)}(\boldsymbol{\rho})$ we use the angular ‘‘bispinor harmonics’’ $\varphi^i(\mathbf{n})$ ($i = A, 0, -, +$). In 2×2 matrix representation they are [19]:

$$\varphi^A(\mathbf{n}) = \frac{1}{\sqrt{2}} Y_j^\mu(\mathbf{n}) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad (34)$$

$$\begin{aligned} \varphi^0(\mathbf{n}) &= \frac{1}{\sqrt{2j(j+1)}} \times \\ &\times \begin{bmatrix} -\sqrt{(j-\mu+1)(j+\mu)} Y_j^{\mu-1} & \mu Y_j^\mu \\ \mu Y_j^\mu & \sqrt{(j+\mu+1)(j-\mu)} Y_j^{\mu+1} \end{bmatrix}, \end{aligned} \quad (35)$$

$$\begin{aligned} \varphi^-(\mathbf{n}) &= \frac{1}{\sqrt{2(j+1)(2j+3)}} \times \\ &\times \begin{bmatrix} \sqrt{(j-\mu+1)(j-\mu+2)} Y_{j+1}^{\mu-1} & -\sqrt{(j+\mu+1)(j-\mu+1)} Y_{j+1}^\mu \\ -\sqrt{(j+\mu+1)(j-\mu+1)} Y_{j+1}^\mu & \sqrt{(j+\mu+1)(j+\mu+2)} Y_{j+1}^{\mu+1} \end{bmatrix}, \end{aligned} \quad (36)$$

$$\begin{aligned} \varphi^+(\mathbf{n}) &= \frac{1}{\sqrt{2j(2j-1)}} \times \\ &\times \begin{bmatrix} \sqrt{(j+\mu-1)(j+\mu)} Y_{j-1}^{\mu-1} & \sqrt{(j+\mu)(j-\mu)} Y_{j-1}^\mu \\ \sqrt{(j+\mu)(j-\mu)} Y_{j-1}^\mu & \sqrt{(j-\mu-1)(j-\mu)} Y_{j-1}^{\mu+1} \end{bmatrix}, \end{aligned} \quad (37)$$

where $Y_\ell^\mu(\mathbf{n})$ ($\mu = -\ell, \dots, \ell$) are the spherical harmonics depending on the direction $\mathbf{n} = \mathbf{r}/r$. The bispinor harmonics form an orthonormal set, in the sense that $\langle i|k \rangle = \int d\mathbf{n} \text{Tr}(\varphi_i^\dagger \varphi_k) = \delta_{ik}$, where the integrations are taken over the entire solid angle.

The bispinor harmonics possess the following properties (besides those due to properties of the spherical harmonics):

$$\begin{aligned} \mathbf{L} \cdot \boldsymbol{\sigma} \varphi^A &= \sqrt{j(j+1)} \varphi^0, \\ \mathbf{L} \cdot \boldsymbol{\sigma} \varphi^0 &= \sqrt{j(j+1)} \varphi^A - \varphi^0, \\ \mathbf{L} \cdot \boldsymbol{\sigma} \varphi^- &= -(j+2) \varphi^-, \\ \mathbf{L} \cdot \boldsymbol{\sigma} \varphi^+ &= (j-1) \varphi^+, \\ \mathbf{n} \cdot \boldsymbol{\sigma} \varphi^{A,0} &= -\sqrt{\frac{j+1}{2j+1}} \varphi^\mp \pm \sqrt{\frac{j}{2j+1}} \varphi^\pm, \\ \mathbf{n} \cdot \boldsymbol{\sigma} \varphi^\mp &= -\sqrt{\frac{j+1}{2j+1}} \varphi^{A,0} \mp \sqrt{\frac{j}{2j+1}} \varphi^{0,A}, \end{aligned} \quad (38)$$

where the components of the vector operator $\boldsymbol{\sigma}$ are the Pauli matrices.

The action of spin operators on the bispinor harmonics is as follows: $\boldsymbol{\sigma}_1 \varphi = \boldsymbol{\sigma} \varphi$, $\boldsymbol{\sigma}_2 \varphi = \varphi \boldsymbol{\sigma}^T$. We note that φ^A is antisymmetric and $\varphi^{0,\pm}$ are symmetric matrices. Then it follows from this and Eqs. (38) that φ^A

and $\varphi^{0,\pm}$ satisfy the following equalities:

$$\begin{aligned}
\mathbf{J}^2\varphi &= j(j+1)\varphi, & j &= 0, 1, \dots, \\
J_3\varphi &= \mu\varphi, & \mu &= -j, \dots, j, \\
\mathbf{L}^2\varphi^i &= \ell(\ell+1)\varphi^i, & \ell &= \begin{cases} j, & i = A, 0, \\ j \pm 1, & i = \mp, \end{cases} \\
\mathbf{S}^2\varphi^i &= s(s+1)\varphi^i, & s &= \begin{cases} 0, & i = A, \\ 1, & i = 0, \mp, \end{cases} \\
P\varphi^{A,0} &= (-)^j\varphi^{A,0}, \\
P\varphi^\mp &= (-)^{j+1}\varphi^\mp, \\
\frac{1}{2}\mathbf{L} \cdot \boldsymbol{\sigma}_+\varphi^A &= 0, \\
\frac{1}{2}\mathbf{L} \cdot \boldsymbol{\sigma}_+\varphi^0 &= -\varphi^0, \\
\frac{1}{2}\mathbf{L} \cdot \boldsymbol{\sigma}_+\varphi^- &= -(j+2)\varphi^-, \\
\frac{1}{2}\mathbf{L} \cdot \boldsymbol{\sigma}_+\varphi^+ &= (j-1)\varphi^+, \\
\frac{1}{2}\mathbf{L} \cdot \boldsymbol{\sigma}_-\varphi^{A,0} &= \sqrt{j(j+1)}\varphi^{0,A}, \\
\frac{1}{2}\mathbf{L} \cdot \boldsymbol{\sigma}_-\varphi^\mp &= 0, \\
\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\varphi^A &= -3\varphi^A, \\
\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\varphi^{0,\mp} &= \varphi^{0,\mp}, \\
\frac{1}{2}T\varphi^A &= 0, \\
\frac{1}{2}T\varphi^0 &= \varphi^0, \\
\frac{1}{2}T\varphi^- &= 3\frac{\sqrt{j(j+1)}}{2j+1}\varphi^+ - \frac{j+2}{2j+1}\varphi^-, \\
\frac{1}{2}T\varphi^+ &= 3\frac{\sqrt{j(j+1)}}{2j+1}\varphi^- - \frac{j-1}{2j+1}\varphi^+.
\end{aligned} \tag{39}$$

Now one can choose four independent basic eigenfunctions $\Psi_i^{(0)}(\boldsymbol{\rho})$ ($i = A, 0, -, +$) of $H^{(0)}$ as follows:

$$\Psi_{A,0}^{(0)}(\boldsymbol{\rho}) = \frac{1}{\rho}u_{n,j}(\rho)\varphi^{A,0}(\mathbf{n}), \quad \Psi_{\mp}^{(0)}(\boldsymbol{\rho}) = \frac{1}{\rho}u_{n,j\pm 1}(\rho)\varphi^\mp(\mathbf{n}), \tag{40}$$

where $u_{n,\ell}(\rho)$ is a solution of the radial Coulomb problem

$$H_\ell u_{n,\ell}(\rho) = \varepsilon^{(0)}u_{n,\ell}(\rho) \tag{41}$$

with the effective Hamiltonian

$$H_\ell = -\frac{1}{2} \left\{ \frac{d}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} \right\} - \frac{1}{\rho} \tag{42}$$

and the dimensionless eigenenergy

$$\varepsilon^{(0)} = -1/(2n^2), \quad n = 1, 2, \dots \tag{43}$$

We note that the basic eigenfunctions $\Psi_{A,0}^{(0)}(\boldsymbol{\rho})$ have the parity $P = (-)^j$, and $\Psi_{\mp}^{(0)}(\boldsymbol{\rho})$ have the parity $P = (-)^{j+1}$. The function $\Psi_A^{(0)}(\boldsymbol{\rho})$ describes the singlet ($s = 0, \ell = j$) state while $\Psi_{0,\mp}^{(0)}(\boldsymbol{\rho})$ correspond to triplet ($s = 1, \ell = j, j \pm 1$) states.

Let us calculate the first-order correction $\varepsilon^{(1)}$ to the dimensionless energy $\varepsilon \approx \varepsilon^{(0)} + \alpha^2\varepsilon^{(1)}$. The total Hamiltonian $H = H^{(0)} + \alpha^2\Gamma$ commutes with operators of parity P and total angular momentum $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\sigma}_+$. One can choose the wave functions $\Psi(\boldsymbol{\rho})$ as the eigenfunctions of \mathbf{J}^2 , J_3 and P . Thus they can be spanned onto states $\Psi_{A,0}^{(0)}$ if $P = (-)^j$, or onto $\Psi_{\mp}^{(0)}$ if $P = (-)^{j\mp 1}$. In the each parity case zero-order eigenvalues $\varepsilon^{(0)}$ are twice degenerated. Thus in the first order of perturbation theory we have

$$\varepsilon_{(i,k)}^{(1)} = \frac{1}{2} \left[\Gamma_{ii} + \Gamma_{kk} \pm \sqrt{(\Gamma_{ii} - \Gamma_{kk})^2 + 4\Gamma_{ik}^2} \right] \quad (i \neq k) \tag{44}$$

with $i = A, k = 0$ if $P = (-)^j$ and $i = -, k = +$ if $P = (-)^{j+1}$, where the matrix $\Gamma = [\Gamma_{ik}]$ is defined as follows:

$$\Gamma = [\Gamma_{ik}] = [\langle i|\Gamma|k \rangle] = \left[\int d\mathbf{r} \text{Tr} \left(\Psi_i^\dagger(\mathbf{r})\Gamma\Psi_k(\mathbf{r}) \right) \right]. \tag{45}$$

Taking (28) and (40) into account we have:

$$\Gamma = \frac{\gamma}{2} \langle j|\rho^{-2}|j \rangle 1 + \langle j|\rho^{-3}|j \rangle \begin{bmatrix} \langle A|\Sigma|A \rangle & \langle A|\Sigma|0 \rangle \\ \langle 0|\Sigma|A \rangle & \langle 0|\Sigma|0 \rangle \end{bmatrix} \tag{46}$$

if $P = (-)^j$, and

$$\begin{aligned}
\Gamma &= \frac{\gamma}{2} \begin{bmatrix} \langle j+1|\rho^{-2}|j+1 \rangle & 0 \\ 0 & \langle j-1|\rho^{-2}|j-1 \rangle \end{bmatrix} \\
&+ \begin{bmatrix} \langle j+1|\rho^{-3}|j+1 \rangle \langle A|\Sigma|A \rangle & \langle j+1|\rho^{-3}|j-1 \rangle \langle A|\Sigma|0 \rangle \\ \langle j-1|\rho^{-3}|j+1 \rangle \langle 0|\Sigma|A \rangle & \langle j-1|\rho^{-3}|j-1 \rangle \langle 0|\Sigma|0 \rangle \end{bmatrix} \tag{47}
\end{aligned}$$

if $P = (-)^{j+1}$, where

$$\langle i|\Sigma|k \rangle = \int d\mathbf{n} \text{Tr} \left(\varphi_i^\dagger(\mathbf{n})\Sigma\varphi_k(\mathbf{n}) \right) \tag{48}$$

and

$$\langle \ell'|\rho^s|\ell \rangle = \int d\rho u_{n,\ell'}(\rho) \rho^s u_{n,\ell}(\rho). \tag{49}$$

In particular,

$$\langle \ell | \rho^{-2} | \ell \rangle = \frac{1}{n^3(\ell + \frac{1}{2})}, \quad \langle \ell | \rho^{-3} | \ell \rangle = \frac{\langle \ell | \rho^{-2} | \ell \rangle}{\ell(\ell + 1)}, \quad (50)$$

$$\langle \ell + 1 | \rho^{-2} | \ell - 1 \rangle = 0, \quad \langle \ell + 1 | \rho^{-3} | \ell - 1 \rangle = 0. \quad (51)$$

The relations (50) are well known in literature (see [20] or [21]), and (51) can be calculated by means of formulae given in [20, chap. *Mathematical Supplements*, § f].

Using (50), (51) and calculating the matrix elements $\langle i | \Sigma | k \rangle$ by means of Eqs. (30)–(33), (39) one obtains the matrix Γ and then the corrections $\varepsilon^{(0)}$ to the dimensionless energy. Then, using (12), (29) and expanding the total mass M in α one obtains the first-order mass spectra (i.e., with accuracy up to α^4).

Due to the relations (50) and (51) the matrix Γ is not diagonal if $P = (-)^j$. Thus the correspondent first-order states are the mixture of singlet ($s = 0, \ell = j$) and triplet ($s = 1, \ell = j$) states. In the $P = (-)^{j+1}$ case Γ is diagonal, and the triplet ($s = 1, \ell = j \pm 1$) states does not mix. Thus it is convenient to classify the first-order mass spectra by j and ℓ . These spectra can be obtained from Eq. (18) by the following substitution:

$$\gamma \rightarrow \gamma + \phi(\ell, j) \quad (52)$$

where the function $\phi(\ell, j)$ depends on both a spin state of the system and the tensor rank of mediating field. We have:

$$\phi_s = \begin{cases} \frac{1 + \delta^2 \pm \sqrt{(1 + \delta^2)^2 + 16\delta^2\ell(\ell + 1)}}{8\ell(\ell + 1)}, & \ell = j, \\ \frac{1 + \delta^2}{4\ell}, & \ell = j + 1, \\ -\frac{1 + \delta^2}{4(\ell + 1)}, & \ell = j - 1, \end{cases} \quad (53)$$

$$\phi_v = \begin{cases} -\frac{1 \pm \sqrt{1 + 4\delta^2\ell(\ell + 1)}}{4\ell(\ell + 1)}, & \ell = j, \\ -\frac{1}{2\ell} - \frac{1 - \delta^2}{2(2\ell - 1)}, & \ell = j + 1, \\ \frac{1}{2(\ell + 1)} + \frac{1 - \delta^2}{2(2\ell + 3)}, & \ell = j - 1, \end{cases} \quad (54)$$

$$\phi_g = \begin{cases} -\frac{3(1 \pm \sqrt{1 + 4\delta^2\ell(\ell + 1)})}{4\ell(\ell + 1)}, & \ell = j, \\ -\frac{3}{2\ell} - \frac{1 - \delta^2}{2(2\ell - 1)}, & \ell = j + 1, \\ \frac{3}{2(\ell + 1)} + \frac{1 - \delta^2}{2(2\ell + 3)}, & \ell = j - 1. \end{cases} \quad (55)$$

The Eqs. (18), (52) and (54) reproduce the muonium spectrum [15] and (if $m_1 = m_2 = m$) the positronium spectrum [4].

6. Solvable simulation of first-order mass spectra

Solving the Schrödinger equation (25) perturbatively is due to the fact that spin interaction term in the operator (28) depends on ρ as ρ^{-3} . Below we construct some exactly solvable model which reproduces the spectrum of perturbation theory.

Let us modify the operator (28) as follows

$$\Gamma \rightarrow \tilde{\Gamma} = \frac{Z(\mathbf{n})}{2\rho^2}, \quad (56)$$

where

$$Z(\mathbf{n}) = \gamma + 2\{\Sigma(\mathbf{n})/\mathbf{L}^2\}_{\text{ordered}}. \quad (57)$$

The operator Z acts on angle and spin variables. It is not defined on states which contain the S-wave, but we refuse these states from the very beginning. On other states Z is supposed to be Hermitian. Thus it must be somehow ordered if $\Sigma(\mathbf{n})$ and \mathbf{L}^2 do not commute.

It is easy to examine by means of Eqs. (50)–(51) that in the first order of perturbation theory the Hamiltonian $\tilde{H} = H^{(0)} + \alpha^2 \tilde{G}$ has the same spectrum as the original Hamiltonian H . This result does not depend on the ordering rule used in Z .

Below we show that the new Schrödinger equation is exactly solvable. Of course, the exact solution and corresponding spectrum depend on the ordering rule. One can consider, for example, the following rules:

$$\{\Sigma/\mathbf{L}^2\}_{\text{ordered}} = \frac{1}{2}(\Sigma|\mathbf{L}|^{-2} + |\mathbf{L}|^{-2}\Sigma), \quad (58)$$

$$\{\Sigma/\mathbf{L}^2\}_{\text{ordered}} = |\mathbf{L}|^{-1}\Sigma|\mathbf{L}|^{-1} \quad (59)$$

$$\{\Sigma/\mathbf{L}^2\}_{\text{ordered}} = \int_0^\infty dt e^{-\frac{t}{2}\mathbf{L}^2} \Sigma e^{-\frac{t}{2}\mathbf{L}^2}, \quad (60)$$

where $|\mathbf{L}| = \sqrt{\mathbf{L}^2}$. The last rule is inspired by the Feynman representation of an inverse operator: $A^{-1} = \int_0^\infty dt \exp(-tA)$.

The radial reduction of the Schrödinger equation can be performed by the following choice of the wave functions $\Psi(\rho)$ as the eigenfunctions of \mathbf{J}^2 , J_3 and P :

$$\Psi(\rho) = \frac{1}{\rho} \sum_i \psi_i(\rho) \varphi^i(\mathbf{n}). \quad (61)$$

Here the summa in r.h.s. of Eq. (61) runs over $i = A, 0$ if $P = (-)^j$, and over $i = -, +$ if $P = (-)^{j+1}$. Substituting this function into the new Schrödinger equation and collecting coefficients at bispinor harmonics φ^A and φ^0 (or at φ^- and φ^+) one obtains the pair of coupled Rarita-Schwinger equations. In the matrix form they are:

$$\mathbf{H}\Psi(\rho) = \varepsilon\Psi(\rho), \quad (62)$$

where

$$\Psi(\rho) = [\psi_i(\rho)] \quad (63)$$

is two-component column wave function,

$$\mathbf{H} = -\frac{1}{2} \left\{ \frac{d}{d\rho^2} - \frac{1}{\rho^2} \mathbf{K} \right\} - \frac{1}{\rho}, \quad (64)$$

and

$$\mathbf{K} = [K_{ik}] = [\langle i | \mathbf{L}^2 + \alpha^2 Z | k \rangle]. \quad (65)$$

The form of 2×2 symmetric matrix \mathbf{K} depends on both the parity and the tensor structure of interaction:

$$\mathbf{K} = \begin{bmatrix} j(j+1) + \alpha^2 \gamma & \frac{\alpha^2 \eta \delta}{2\sqrt{j(j+1)}} \\ \frac{\alpha^2 \eta \delta}{2\sqrt{j(j+1)}} & j(j+1) + \alpha^2 \left(\gamma - \frac{\xi - \delta^2 - \zeta(1-\delta^2)}{4j(j+1)} \right) \end{bmatrix} \quad (66)$$

for the parity $P = (-)^j$, and

$$\mathbf{K} = \begin{bmatrix} (j+1)(j+2) + \alpha^2 \left(\gamma - \frac{1}{4(j+1)} \left[\xi - \delta^2 + \zeta \frac{1-\delta^2}{2j+1} \right] \right), & \alpha^2 \zeta \frac{1-\delta^2}{4(2j+1)} \frac{3\sqrt{j(j+1)}}{j(j+1)+1} \\ \alpha^2 \zeta \frac{1-\delta^2}{4(2j+1)} \frac{3\sqrt{j(j+1)}}{j(j+1)+1}, & (j-1)j + \alpha^2 \left(\gamma + \frac{1}{4j} \left[\xi - \delta^2 - \zeta \frac{1-\delta^2}{2j+1} \right] \right) \end{bmatrix} \quad (67)$$

for the parity $P = (-)^{j+1}$. We note that in general case where $\zeta \neq 0$ (including the cases of vector and gravitational interaction; c.f. Eqs.(32),

(33)) the operators Σ and \mathbf{L}^2 does not commute. Thus in calculating of \mathbf{K} we chosen the ordering rule (60). The use of (58) or (59) leads to off-diagonal elements of \mathbf{K} which are singular at $j = 1$ (besides of $j = 0$). But there is no any physical reason for such a singularity.

Using now an appropriate unitary (even orthogonal, to be sharp) transformation, the matrix \mathbf{K} can be diagonalized, so that the coupled Rarita-Schwinger equations split into a pair of one-dimensional Schrödinger equations with effective Hamiltonians $H_{\tilde{\ell}(i,k)}$ of the form (42) but with non-integer $\tilde{\ell}(i,k)$:

$$\tilde{\ell}(i,k) = -\frac{1}{2} + \sqrt{\frac{1}{4} + K_{(i,k)}}, \quad (68)$$

with

$$K_{(i,k)} = \frac{1}{2} \left[K_{ii} + K_{kk} \pm \sqrt{(K_{ii} - K_{kk})^2 + 4K_{ik}^2} \right] \quad (i \neq k), \quad (69)$$

where $i = A, k = 0$ if $P = (-)^j$, and $i = -, k = +$ if $P = (-)^{j+1}$.

These equations are exactly solvable and lead to the mass spectrum (12)–(17) but with another effective “principal quantum number”:

$$\nu \longrightarrow \tilde{\nu} = \tilde{\nu}(n_r, j, P) = n_r + \tilde{\ell}(i,k) + 1. \quad (70)$$

The calculation of effective “principal quantum number” $\tilde{\nu}$ is straightforward by the use of Eqs. (66)–(70). Here we do not write down these rather cumbersome formulae.

7. The confinement problem

The Todorov recipe was observed on the systems with Coulomb-like interaction. Here we demonstrate that this rule appears useful for the construction of relativistic potential model of mesons.

It is well known that the spectra of heavy quarkoniums are described satisfactory (and modulo spin effects) by means of the nonrelativistic potential model with the short-range Coulomb potential (7) and the long-range linear potential [17]:

$$V_v(r) = ar, \quad (71)$$

where $a > 0$ is a constant.

The description of light mesons needs the application of relativistic models. They frequently are built as single-particle wave equations with

the vector short-range potential and the scalar long-range one [22]. Other models treat mesons as an extended objects or a composite two-quark relativistic systems. One of them which is concise and elegant, and which reflects principal features of the light meson spectroscopy, is the covariant oscillator model. Few versions of this model are given in Refs. [23].

The nonrelativistic, single-particle and oscillator models appears related to one another by the Todorov recipe. Given the single-particle Klein-Gordon equation (1) with the scalar potential (71) and the vector one (7), this rule fixes unambiguously the form of the two-particle wave equation (3)–(5) which is the relativization of the nonrelativistic potential model.

If $m_a = 0$ and $\alpha = 0$ the wave equation reduces to the oscillator problem and yields the exact solution for the mass spectrum:

$$M^2 = 8a \left[\ell + 2n_r + \frac{3}{2} \right]. \quad (72)$$

The spectrum falls on the family of straight lines in the (M^2, ℓ) -plane known in the hadron spectroscopy as the leading (for $n_r = 0$) and daughter's ($n_r > 0$) Regge trajectories. This structure and $(\ell + 2n_r)$ -degeneracy of the spectrum (72) are characteristic of actual light meson spectra, if to neglect the rest mass contribution and fine spin effects.

In the general case the equation (3)–(5), (7), (71) is not exactly solvable. Here we use the oscillator approximation to estimate the spectrum for ℓ large.

The substitution $\Psi(\mathbf{r}) = \frac{1}{r}\psi(r)Y_\ell^\mu(\mathbf{n})$ reduces the equation (3)–(5), (71) to the form

$$\psi''(r) + [Q_M - U_{M\ell}(r)]\psi(r) = 0, \quad (73)$$

where

$$U_{M\ell}(r) = U_M(r) + \ell(\ell + 1)/r^2. \quad (74)$$

The function $U_{M\ell}(r)$ has a local minimum at some point r_0 depending on M , ℓ and satisfying the condition

$$U'_{M\ell}(r_0) = 0. \quad (75)$$

Thus one can expand the potential (74) at the minimum,

$$U_{M\ell}(r) \approx U_{M\ell}(r_0) + \frac{1}{2}U''_{M\ell}(r_0)(r - r_0)^2, \quad (76)$$

and search a solution of this oscillator problem. A quantization condition then reads:

$$Q_M - U_{M\ell}(r_0) = \sqrt{\frac{1}{2}U''_{M\ell}(r_0)}(2n_r + 1). \quad (77)$$

If $n_r \sim 1$, the approximate solution differs exponentially little from the exact solutions of the problem.

Eqs. (75) and (77) form the set of algebraic equations with r_0 and M to be found. Solving this set by a power series in ℓ leads to the asymptotic formulae:

$$r_0^2 = \frac{\ell + \frac{1}{2}}{a} - \frac{1}{\sqrt{2}a} \left(\frac{m_1 m_2}{4a} + \alpha \right) + O(\ell^{-1}) \quad (78)$$

and

$$M^2 = 8a \left[\ell + 2n_r + \frac{3}{2} - \sqrt{2}\alpha \right] + 2 \left(m_1^2 + m_2^2 + \sqrt{2}m_1 m_2 \right) + O(\ell^{-1}). \quad (79)$$

The latter represents the spectrum of the system at $\ell \gg n_r$ (but provides a good approximation even if $\ell \simeq 2 \div 3$). As to compare this formula to the Eq. (72), the influence of the short-range interaction and non-zero rest masses result in the parallel shift of the family of Regge trajectories as a whole. Considering the constants m_1 , m_2 (and, possibly, α) as adjustable parameters one can obtain trajectories for different families of light mesons.

Up to now we neglected a mass splitting due to a spin interaction. The majority of attempts to describe spin effects in hadron spectroscopy concerns with the heavy quark systems which can be treated as weakly relativistic systems. As usual, one takes the generalized Pauli-Fermi Hamiltonian with long-range scalar potential and short-range vector one (the potentials (71), (7) in our case) and corresponding spin corrections (19), (20) treated perturbatively [16,17,24]. But this scheme can fail when considering light mesons as corresponding to strongly relativistic domain $M \gg m_{1(2)}$. First of all we note that, as it follows from Eqs. (78), (79), in this domain $r_0^2 \sim \ell$ and $M^2 \sim \ell$. Thus $r_0 \sim M$, i.e., the radius of meson is proportional to its mass; here we took into account that the wave function at ℓ large is localized around r_0 . Then we have rough estimates:

$$\begin{aligned} V_s = ar \sim M, & & W_s \sim \ell/r \sim M & \implies & V_s \sim W_s, \\ V_v = -\alpha/r \sim M^{-1}, & & W_v \sim \ell/r^3 \sim M^{-1} & \implies & V_v \sim W_v. \end{aligned} \quad (80)$$

The spin corrections appears to be of the same order as the nonrelativistic potentials. But actual spin effects in light meson spectra are small. Moreover, the operators (19) and (20) was deduced within the perturbation theory [16,17], so they should satisfy inequality $W \ll V$ by construction. Second, the operators (19) and (20) are divergent if m_1 or/and $m_2 \rightarrow 0$. Consequently, divergent terms appear in the mass

spectrum (in contrast to the case of Coulomb-like interaction where the spectrum is not singular if m_1 or/and m_2 vanish).

The possible way to avoid these two problems is the following modification of operators (19) and (20): the rest masses of quarks involved in these operators should be replaced by the “constituent” masses:

$$m_a \rightarrow M_a = \sqrt{m_a^2 + Q_M} = \frac{M^2 + m_a^2 - m_{\bar{a}}^2}{2M}, \quad a = 1, 2, \quad (81)$$

$$\bar{a} = 3 - a$$

possessing the properties 1) $M_1 + M_2 = M$, 2) $M_a \approx M/2$ if $M \gg m_{1(2)}$, and 3) $M_a \rightarrow m_a$ in the nonrelativistic limit $M \rightarrow m_+$ (we suppose that $M^2 > |m_1^2 - m_2^2|$). The modified operators \overline{W} are regular at m_1 or/and $m_2 \rightarrow 0$, they became small as $\overline{W} \sim V(m_+/M)^2$ in strongly relativistic domain $M \gg m_{1(2)}$, and reduce to W in weakly relativistic domain $M \approx m_+$.

Now let us estimate the effect of spin corrections \overline{W}_s to the scalar potential (71) (the effect of vector interaction is minor, as it follows from the estimates in (80) and the paragraph above). For this purpose we modify the scalar potential $\overline{V}_s = V_s + \overline{W}_s$ and substitute it (instead of V_s) into the effective potential U_M . Then we diagonalize the modified effective potential $\overline{U}_M(r)$ and apply the oscillator approximation to the resulting pair of split quasipotential equations. Finally, we come to the asymptotically linear Regge trajectories (79) with the same slope parameter $8a$, but each trajectory splits into three ones by parallel shift

$$\Delta M^2 = \begin{cases} 0, & \ell = j, \\ \pm 2a, & \ell = j \pm 1. \end{cases} \quad (82)$$

Note that the mass splitting does not depend on ℓ . Qualitatively this result correlates with actual light meson spectra as well as with theoretical results following from the string models [25,26].

As in the spinless case, the model is exactly solvable if $m_a = 0$ and $\alpha = 0$. The mass spectrum is determined by $M^2 = 4ax$, where x is the positive solution of the transcendental equation:

$$x^2 - 2(2n_r + 1)x - \kappa = \sqrt{(2\ell + 1)^2 x^2 + \kappa^2}. \quad (83)$$

which, in turns, reduces to a cubic algebraic equation, and

$$\kappa = \begin{cases} 0, & \ell = j, s = 0, \\ 1, & \ell = j, s = 1, \\ \ell + 1, & \ell = j + 1, s = 1, \\ -\ell, & \ell = j - 1, s = 1. \end{cases} \quad (84)$$

Here we do not write down an explicit form of the mass spectrum.

Of course, the minimal set of adjustable parameters makes the present potential model too poor to provide a sharp fit to an experimental data. In particular, the mass splitting (82) is unambiguously fixed by the typical hadron scale $8a$. This value (0% and 25% of $8a$) contradicts to some estimates of actual experimental data (about 5÷6%; see [26]).

In another version of the model we include operators \overline{W} similarly to the case of Coulomb-like interaction (see Eq. (22)), i.e., by the following modification of the effective potential:

$$U_M(r) \rightarrow \overline{U}_M(r) = U_M(r) + 2m_M \overline{W}_s + 2E_M \overline{W}_v. \quad (85)$$

In the strongly relativistic domain the present potential model leads to the same asymptotic spectrum (79) as the spinless model does. The mass splitting is small. It can be calculated perturbatively and fitted to experimental data by adjusting the parameters m_a and α .

8. Summary

In the present paper we have constructed the family of simple quantum-mechanical models which describe two-fermion relativistic systems with different Coulomb-like and confining interactions. We have embodied in these models some theoretical experience of studies in the relativistic two-body problem. We started with the quasipotential equations describing the scalar and vector interaction of two spinless particles. As it was observed by Todorov [5,7], these equations are solvable, and they have simple single-particle structure. Similar features are characteristic of the family of RDIT models [12,13] which describe an arbitrary relativistic Coulomb-like interaction including the gravitation. We generalize these equations to the case of two-fermionic systems. For this purpose we use the operators known from QFT which describe spin-dependent corrections to the scalar, vector and gravitational interacting [4,16–18]. We first treat these operators perturbatively and obtain the spectrum of muonium and its scalar and gravitational counterparts with accuracy up to α^4 . Then we modify the equations in such a way that they become exactly solvable and yield correct (within the same accuracy) mass spectra. These equations can be useful for accounting higher than α^4 (say, radiative) corrections to a particle interaction by means of the first order perturbation theory.

Also we have demonstrated that the Todorov recipe of constructing two-body equations permits a straightforward application to the case of confining interaction. The generalization to spinning particles has been

proposed too. In the weakly relativistic domain this equation reduces to well-known potential model [17] which is appropriate to the description of heavy mesons. In the strongly relativistic limit it yields a mass spectrum which reproduces qualitatively light-meson experimental data.

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ЕВРИСТИЧНІ МОДЕЛІ ДВО-ФЕРМІЙОННИХ РЕЛЯТИВІСТИЧНИХ СИСТЕМ ІЗ ВЗАЄМОДІЄЮ ПОЛЬОВОГО ТИПУ

Роботу отримано 18 липня 2002 р.

Затверджено до друку Вченою радою ІФКС НАН України

Рекомендовано до друку семінаром відділу теорії металів та сплавів

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