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Critical line of the Φ^4 scalar field theory on a 4D cubic lattice in the local potential approximation

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We establish the critical line of the one-component Φ^4 (or Landau-Ginzburg) model on a simple four-dimensional cubic lattice. Our study is performed in the framework of the non-perturbative renormalization group in the local potential approximation with a soft infra-red regulator. The transition is found to be of the second order even in the Gaussian limit where the first order would be expected according to some recent theoretical predictions.

Key words: *non-perturbative renormalization group, local potential approximation, lattice Φ^4 theory, numerical experiments*

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1. Introduction

It is a real pleasure and a great honor for the author to contribute, with this paper, to the festschrift dedicated to Professor Myroslav Holovko on the occasion of his 70th birthday. Myroslav is an expert of the collective variables (CV) method introduced by the Ukrainian school in the framework of which Wilson's ideas on the renormalization group (RG) [1] can be implemented with great effect [2]. Here we expose recent post-Wilsonian advances on the RG in the framework of statistical field theory. Obviously, many of the ideas exposed here could be easily transposed to the CV "world" by the readers of references [3, 4] where the links between the CV method and standard statistical field theory are established.

These recent past years, Wilson's approach to the RG [1, 5] has been the subject of a revival in both statistical physics and quantum field theory. Since the seminal work of Wilson, two main formulations of the non-perturbative renormalization group (NPRG) have been developed in parallel. Very similar to the works of the Ukrainian school on the CV formalism, we have the approaches initiated independently and in parallel by Wetterich et al. [6–9] on the one hand, and Parola et al. on the other hand [11–13]. In this corpus of works one is interested to establish and solve the flow equations of the Gibb's free energy by means of non-perturbative methods. In an alternative formulation, Polchinski and his followers consider the flow of the Wilsonian action [14, 15] rather than that of the free energy, which makes the method more abstract and less predictive than that of Wetterich, although more in accord with Wilson's ideas. The link between these two formulations can, however, be established, see for instance references [16, 17]. Other non-perturbative methods based either on the CV or Monte Carlo methods are also the subject of active studies and are discussed, for instance, in reference [18] and in references cited herein.

The NPRG has proved its capability of describing *both universal and non universal* quantities for various models of statistical and condensed matter physics *near or even far* from criticality. It has been recently extended to the models defined on a lattice [19]. Successful applications to the three-dimensional (3D) Ising, XY, Heisenberg models [20] and Φ^4 model [21] are noteworthy. Here we extend the study of reference [21] on the Φ^4 model in three dimensions of space to the case $D = 4$; due to the recent publication by Loh of a novel numerical method, it was made possible to compute the lattice Green's functions [22, 23]. The $D = 4$ version of the Φ^4 model on a lattice describes the field of a Higgs boson on the lattice in interaction with itself [24]; thus, our conclusions concerning the type of transition that it undergoes, are of theoretical importance.

Like in our former study of the $D = 3$ version of the model, we work in the framework of the local potential approximation [6–8, 21] but here we consider only the case of the Litim-Machado-Dupuis infrared cut-off introduced in refs [20, 25] This regulator has been shown to give much better results than other sharp regulators in [21]. Like in references [11, 12, 21, 26, 27], the flow equations are numerically integrated out for the so-called threshold functions [7] rather than for the potential. The resulting flow equations belong to the class of quasi-linear parabolic partial differential equations (PDE) for which several efficient and unconditionally convergent numerical algorithms have been developed by mathematicians [28]. Like in references [11, 12, 21, 26, 27] we made use of an algorithm proposed by Douglas-Jones [28, 29] to solve our NPRG flow equations, both *above* and *below* the critical temperature; this yields an easy and precise determination of the critical point. The critical line of the model is obtained for a large range of parameters; unfortunately, and contrary to the case $D = 3$ [21, 30], we were unable to find available Monte Carlo simulations to compare our data with. We stress that, in the wide range of parameters considered in our study (see table 1), we exclude the occurrence of a first order transition. This conclusion seems to be in agreement with a general analysis of the criticality of the model made in references [31, 32].

Our paper is organized a follows: in section 2 we briefly review the basic definitions and results concerning the statistical mechanics of scalar fields on a lattice. Section 3 is devoted to theoretical and technical aspects of the NPRG on the lattice. We then present our numerical experiments and discuss the results in section 4. We conclude in section 5

2. Prolegomena

2.1. Model

Let us consider some arbitrary field theory defined on a 4D hyper-cubic lattice

$$\Lambda = a\mathbb{Z}^4 = \{\mathbf{r}|\mathbf{r}_\mu/a \in \mathbb{Z}; \mu = 1, \dots, 4\}, \tag{2.1}$$

where a is the lattice constant. The real, scalar field $\varphi_\mathbf{r}$ is defined on each point of the lattice. It is convenient to start with a finite hyper-cubic subset of points $\{\mathbf{r}\} \subset \Lambda$ and to assume periodic boundary conditions (PBC) for the $\varphi_\mathbf{r}$ before taking the infinite volume limit, although no difficulties are expected to arise from this operation.

In the case of short-range interactions between the fields, the action of the theory can quite generally be written as [24]

$$\mathscr{S}[\varphi] = \frac{1}{2Na^4} \sum_{\{\mathbf{q}\in\mathscr{B}\}} \widetilde{\varphi}_{-\mathbf{q}} \epsilon_0(\mathbf{q}) \widetilde{\varphi}_\mathbf{q} + a^4 \sum_{\{\mathbf{r}\}} U(\varphi_\mathbf{r}), \tag{2.2}$$

where $[\varphi]$ is a shortcut notation for $\{\varphi_\mathbf{r}\}$ and

$$\widetilde{\varphi}_\mathbf{q} = a^4 \sum_{\{\mathbf{r}\}} \mathrm{e}^{-\mathrm{i}\mathbf{r}\mathbf{q}} \varphi_\mathbf{r} \tag{2.3}$$

is the Fourier transform of the field and the N momenta $\{\mathbf{q}\}$ are restricted to the first Brillouin zone $\mathscr{B} = [-\pi/a, \pi/a]^{\otimes 4}$ of the reciprocal lattice. The inverse transformation reads:

$$\varphi_\mathbf{r} = \frac{1}{Na^4} \sum_{\{\mathbf{q}\in\mathscr{B}\}} \mathrm{e}^{\mathrm{i}\mathbf{r}\mathbf{q}} \widetilde{\varphi}_\mathbf{q}. \tag{2.4}$$

Note that, in the thermodynamic limit (a fixed, $N \to \infty$), $\sum_{\{\mathbf{q}\}} \to (Na^4) \int_\mathbf{q}$, where $\int_\mathbf{q} \equiv \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}q_1}{2\pi} \dots \frac{\mathrm{d}q_4}{2\pi}$. In equation (2.2), the spectrum $\epsilon_0(\mathbf{q})$ accounts for the next-neighbor interactions. For a simple cubic (SC) lattice it is equal to

$$\epsilon_0(\mathbf{q}) = (2/a^2) \sum_{\mu=1}^{4} [1 - \cos(q_\mu a)]. \tag{2.5}$$

Obviously one has $\epsilon_0(\mathbf{q}) \sim \mathbf{q}^2$ for $\mathbf{q} \to 0$ and $\max_{\mathbf{q}} \epsilon_0(\mathbf{q}) = \epsilon_0^{\max} = 16/a^2$. For convenience we will also define $k_{\max} \equiv 4/a$ by $\epsilon_0^{\max} = k_{\max}^2$.

Note that in a system of units where the dimension of wave-vector q_μ is $[q_\mu] = +1$, the dimension of the fields are $[\varphi_{\mathbf{r}}] = 1$ and $[\widetilde{\varphi}_{\mathbf{q}}] = -3$ so that the kinematic part of the action $\mathscr{S}[\varphi]$ is dimensionless. Henceforth we shall only consider the Landau-Ginzburg polynomial form $U(\varphi) = (r/2)\,\varphi^2 + (g/4!)\,\varphi^4$. Since $[\varphi_{\mathbf{r}}] = 1$ and $[a_4 U(\varphi)] = 0$, it follows that $[r] = 2$ and $[g] = 0$. Therefore, in the thermodynamic limit, the physics of the model depends only upon the two dimensionless parameters $\overline{r} = ra^2$ and the dimensionless (only in $D = 4$) $\overline{g} = g$.

Another way of writing the action (2.2), which is useful for numerical investigations, is [24, 30]

$$\mathscr{S}[\psi] = \sum_{\{\mathbf{n}\}} \left[-2\kappa \sum_{\mu=1}^{4} \psi_{\mathbf{n}} \psi_{\mathbf{n}+\mathbf{e}_\mu} + \psi_{\mathbf{n}}^2 + \lambda \left(\psi_{\mathbf{n}}^2 - 1\right)^2 - \lambda \right], \tag{2.6}$$

where the 4 unit vectors $\mathbf{e}_\mu$ constitute an orthogonal basis set for $\mathbb{R}^4$. The field ψ and the parameters (κ, λ) are all dimensionless and they are related to the bare field φ and dimensionless parameters ($\overline{r}$, g) through the relations

$$\psi_{\mathbf{n}} = \sqrt{\frac{1}{2\kappa}}\, a\, \varphi_{\mathbf{r}} \text{ with } \mathbf{r} = a\mathbf{n}\,, \tag{2.7a}$$

$$\overline{r} = \frac{1-2\lambda}{\kappa} - 8\,, \tag{2.7b}$$

$$g = \frac{6\lambda}{\kappa^2}\,. \tag{2.7c}$$

2.2. Thermodynamic and correlation functions

The thermodynamic and structural properties of the model are coded in the partition function [33]

$$Z[h] = \int \mathscr{D}\varphi \exp\left\{-S[\varphi] + (h|\varphi)\right\}, \tag{2.8}$$

where the dimensionless functional measure is given by

$$\mathscr{D}\varphi = \prod_{\mathbf{n}} \mathrm{d}\psi_{\mathbf{n}}\,, \tag{2.9}$$

where $\mathbf{r} = a\mathbf{n}$, the dimensionless $\psi_{\mathbf{n}}$ is defined at equation (2.7a), h is an external lattice field, and the dimensionless scalar product in (2.8) is defined as

$$(h|\varphi) = a^4 \sum_{\mathbf{r}} h_{\mathbf{r}} \varphi_{\mathbf{r}}\,. \tag{2.10}$$

The order parameter is given by

$$\phi_{\mathbf{r}} = \langle \varphi_{\mathbf{r}} \rangle = \frac{1}{a^4} \frac{\partial W[h]}{\partial h_{\mathbf{r}}}\,, \tag{2.11}$$

where the brackets $\langle \cdots \rangle$ denote statistical ensemble averages and the Helmholtz free energy $W[h] = \ln Z[h]$. Note that in the continuous limit, i.e., $L = Na$ fixed, $a \to 0$, the partial derivatives tend to functional derivatives, i.e., $a^{-4}\partial \cdots / \partial h_{\mathbf{r}} \to \delta \cdots / \delta h(\mathbf{r})$.

It follows from first principles that $W[h]$ is a convex function of the N variables $\{h_{\mathbf{r}}\}$; it is also the generator of the connected correlation functions $G^{(n)}(\mathbf{r}_1 \ldots \mathbf{r}_n) = a^{-4n} \partial^n W[h] / \partial h_{\mathbf{r}_1} \cdots \partial h_{\mathbf{r}_n}$, where $\partial \cdots / \partial h_{\mathbf{r}}$ denotes a partial derivative with respect to one of the N variables $h_{\mathbf{r}}$.

The Legendre transform of $W[h]$, i.e., the Gibbs free energy, will be provisionally denoted as follows:

$$\hat{\Gamma}[\phi] = (h|\phi) - W[h]\,. \tag{2.12}$$

$\hat{\Gamma}[\phi]$ is also — as a Legendre transform — a convex function of the N conjugated field variables $\{\phi_{\mathbf{r}}\}$ It follows from equations (2.8) and (2.12) that the Gibbs potential is given implicitly by the functional relation

$$\exp\left(-\hat{\Gamma}[\phi]\right) = \int \mathscr{D}\varphi \, \exp\left\{-\mathscr{S}[\varphi] + \left(\varphi - \phi \,\middle|\, \frac{\delta \hat{\Gamma}}{\delta \phi}\right)\right\}, \tag{2.13}$$

where the abusive notation $\delta\cdots/\delta\phi(\mathbf{r}) \to a^{-4}\partial\cdots/\partial\phi_{\mathbf{r}}$ has been used for clarity.

The functional $\hat{\Gamma}[\phi]$ is the generator of the so-called vertex functions $\hat{\Gamma}^{(n)}(\mathbf{r}_1\ldots\mathbf{r}_n) = a^{-4n}(\partial/\partial\phi_{\mathbf{r}_1})\ldots(\partial/\partial\phi_{\mathbf{r}_n})\hat{\Gamma}[\phi]$. Finally, as is well known [33], the matrix $\hat{\Gamma}^{(2)}(\mathbf{r}_1,\mathbf{r}_2)$ is the inverse of matrix $G^{(2)}(\mathbf{r}_1,\mathbf{r}_2) = \langle\varphi_{\mathbf{r}_1}\varphi_{\mathbf{r}_2}\rangle - \langle\varphi_{\mathbf{r}_1}\rangle\langle\varphi_{\mathbf{r}_2}\rangle$; i.e., for 2 arbitrary points of the lattice $(\mathbf{x},\mathbf{y}) \in \Lambda$ one has

$$a^4 \sum_{\mathbf{z}\in\Lambda} G^{(2)}(\mathbf{x},\mathbf{z})\hat{\Gamma}^{(2)}(\mathbf{z},\mathbf{y}) = \frac{1}{a^4}\delta_{\mathbf{x},\mathbf{y}}. \tag{2.14}$$

3. State of the art on lattice NPRG

3.1. Lattice NPRG

An elegant procedure to implement the lattice NPRG was given by Dupuis et al. in references [19, 20]; it extends to the lattice the ideas of Wetterich [6, 7] for the continuum, i.e., the limit $a \to 0$ of the model; it is very similar to the Reatto and Parola hierarchical reference theory of liquids [11–13]. We add a quadratic term to the action (2.2)

$$\Delta\mathscr{S}_k[\varphi] = \frac{1}{2}\frac{1}{Na^4}\sum_{\{\mathbf{q}\}}\varphi_{-\mathbf{q}}\widetilde{R}_k(\mathbf{q})\varphi_{\mathbf{q}}, \tag{3.1}$$

where $\widetilde{R}_k(\mathbf{q})$ is positive-definite, has the dimension $[\widetilde{R}_k] = 2$ and acts as a $\mathbf{q}$ dependent mass term. The regulator $\widetilde{R}_k(\mathbf{q})$ is chosen in such a way that it acts as an infra-red (IR) cut-off which leaves the high-momentum modes unaffected and gives a mass to the low-energy ones. Roughly $\widetilde{R}_k(\mathbf{q}) \sim 0$ for $||\mathbf{q}|| > k$ and $\widetilde{R}_k(\mathbf{q}) \sim Z_k k^2$ for $||\mathbf{q}|| < k$. The scale k in momentum space varies from $\Lambda \sim a^{-1}$, some undefined microscopic scale of the model yet to be defined precisely, to $k = 0$ the macroscopic scale. To each scale "k" there corresponds a k-system defined by its microscopic action $\mathscr{S}_k[\varphi] = \mathscr{S}[\varphi] + \Delta\mathscr{S}_k[\varphi]$. We denote its partition function by $Z_k[h]$, its Gibbs free energy by $\hat{\Gamma}_k[\phi]$, etc. The generalization of equation (2.13) is then

$$\exp\left(-\Gamma_k[\phi]\right) = \int\mathscr{D}\phi\,\exp\left\{-\mathscr{S}[\varphi] + \left(\varphi-\phi\left|\frac{\delta\Gamma_k[\phi]}{\delta\phi}\right.\right) - \frac{1}{2}\left(\varphi-\phi\left|\widetilde{R}_k\right|\varphi-\phi\right)\right\}, \tag{3.2}$$

where the so-called average effective action $\Gamma_k[\phi]$, which was introduced by Wetterich in the first stages of the NPRG, is defined as a modified Legendre transform of $W_k[h]$ which includes the explicit subtraction of $\Delta\mathscr{S}_k[\phi]$ [6, 7], i.e.,

$$\Gamma_k[\phi] = \hat{\Gamma}_k[\phi] - \Delta\mathscr{S}_k[\phi]\ . \tag{3.3}$$

Note that the functional $\Gamma_k[\phi]$ is not necessarily a convex functional of the classical field ϕ by contrast with $\hat{\Gamma}[\phi]$ which is the true Gibbs free energy of the k-system.

The choice of the regulator $\widetilde{R}_k(\mathbf{q})$ would not affect the exact results but it matters as soon as approximations are introduced. We have retained the Litim-Dupuis-Machado (LMD) regulator introduced by Dupuis and Machado [19, 20] for the lattice as an extension of Litim's regulator widely used for off-lattice field theories [25]. Sharp cut-off regulators often yield unphysical behaviors, notably in the local potential approximation, and should be avoided, see *e. g.* [21, 27]. The LMD regulator reads

$$\widetilde{R}_k(\mathbf{q}) = \left[\epsilon_k - \epsilon_0(\mathbf{q})\right]\Theta\left[\epsilon_k - \epsilon_0(\mathbf{q})\right], \tag{3.4}$$

where $\epsilon_k = k^2$ and Θ is the Heavyside's step function. At scale "k", the effective spectrum of the k-model of action $\mathscr{S}_k[\varphi]$ is clearly

$$\epsilon_k^{\mathrm{eff}}(\mathbf{q}) = \epsilon_0(\mathbf{q}) + \left[\epsilon_k - \epsilon_0(\mathbf{q})\right]\Theta\left[\epsilon_k - \epsilon_0(\mathbf{q})\right]\ . \tag{3.5}$$

We note that for $\epsilon_0(\mathbf{q}) > \epsilon_k$, the regulator $\widetilde{R}_k(\mathbf{q})$ vanishes in agreement with the fact that the high energy modes are of affected, i.e., one has $\epsilon_k^{\mathrm{eff}}(\mathbf{q}) = \epsilon_0(\mathbf{q})$. Conversely, for $\epsilon_0(\mathbf{q}) < \epsilon_k$, a constant massive contribution is associated with the low-energy modes, with a tendency to a freezing of their fluctuations, i.e., one has $\epsilon_k^{\mathrm{eff}}(\mathbf{q}) = \epsilon_k$.

It is easy to show that the average effective action satisfies the exact flow equation [6–8, 19, 20]

$$\partial_k\Gamma_k[\phi] = \frac{1}{2}\sum_{\mathbf{q}\in\mathscr{B}}\partial_k\widetilde{R}_k(\mathbf{q})\,\widetilde{G}_k^{(2)}(\mathbf{q},-\mathbf{q})\ , \tag{3.6}$$

where $\widetilde{G}_k^{(2)}$ is the Fourier transform of the connected pair correlation function of the k-system defined as

$$\widetilde{G}_k^{(2)}(\mathbf{p},\mathbf{q}) = a^8 \sum_{\mathbf{x},\mathbf{y}\in\Lambda} \exp\left(\mathrm{i}\mathbf{p}\cdot\mathbf{x} + \mathrm{i}\mathbf{q}\cdot\mathbf{y}\right) G_k^{(2)}(\mathbf{x},\mathbf{y}) \,. \tag{3.7}$$

For an homogeneous configuration of the field $\phi_{\mathbf{r}} = \phi$ we have, on the one hand, $\Gamma_k[\phi] = Na^4 U_k(\phi)$, where the potential $U_k(\phi)$ is a simple function of the field ϕ and, on the other hand, we have the conservation of momentum at each vertex which implies, with the usual abusive notation, $\widetilde{G}_k^{(2)}(\mathbf{q},-\mathbf{q}) = Na^4\widetilde{G}_k^{(2)}(\mathbf{q})$; from these remarks it follows that:

$$\partial_k U_k(\phi) = \frac{1}{2}\frac{1}{Na^4}\sum_{\mathbf{q}} \frac{\partial_k \widetilde{R}_k(\mathbf{q})}{\widetilde{\Gamma}_k^{(2)}(\mathbf{q}) + \widetilde{R}_k(\mathbf{q})} \,, \tag{3.8a}$$

$$= \frac{1}{2}\int_{\mathbf{q}\in\mathscr{B}} \frac{\partial_k \widetilde{R}_k(\mathbf{q})}{\widetilde{\Gamma}_k^{(2)}(\mathbf{q}) + \widetilde{R}_k(\mathbf{q})} \,, \tag{3.8b}$$

where the second line (3.8b) is valid in the thermodynamic limit (a fixed, $N\to\infty$). Note that in order to establish the equation (3.8) we also took into account the equation (2.14) in Fourier space for the $k-$system, i.e., $\widetilde{G}_k^{(2)}(\mathbf{q}) = 1/[\widetilde{\Gamma}_k^{(2)}(\mathbf{q}) + \widetilde{R}_k(\mathbf{q})]$, for an homogeneous system. The reader will agree that equation (3.8), which is *exact*, is an extremely complicated equation since the vertex function $\widetilde{\Gamma}_k^{(2)}(\mathbf{q},-\mathbf{q})$, which is the Fourier transform of the second-order functional derivative of $\widetilde{\Gamma}[\phi]$ with respect to the classical field ϕ, functionally depends upon ϕ.

The implicit solution (3.2) of (3.8) allows us to precisely establish the initial conditions. The initial value $k=\Lambda$ of the momentum scale k of the flow is chosen such that $\widetilde{R}_\Lambda(\mathbf{q}) \sim \infty$ for all values of $\mathbf{q}$; hence, since $\exp[-1/2\,(\chi|\widetilde{R}_\Lambda|\chi)] \propto \delta[\chi]$, where $\delta[\chi]$ is the Dirac functional, it follows from (3.2) that $\Gamma_\Lambda[\phi] = \mathscr{S}[\phi]$. Physically it means that all fluctuations are frozen and the mean-field theory becomes exact. When the running momentum goes from $k=\Lambda$ to $k=0$, all the modes $\widetilde{\varphi}_{\mathbf{q}}$ are progressively integrated out and the effective average action evolves from its microscopic limit $\Gamma_\Lambda[\phi] = \mathscr{S}[\phi]$ to its final macroscopic expression $\Gamma_{k=0}[\phi] = \Gamma[\phi]$.

3.2. Local models and the initial condition of the flow

Some members of our family of k-systems are nice fellows. It follows from (3.5) that, for $\Lambda > k > k_{\max}$, or equivalently $\epsilon_k > \epsilon_0(\mathbf{q})$ for all vectors $\mathbf{q}$ of the first Brillouin zone, we have $\epsilon_k^{\mathrm{eff}} \equiv \epsilon_k$ which means that the action $\mathscr{S}_k[\varphi]$ of the k-system is local and reads $\mathscr{S}_k[\varphi] = a^4\sum_{\{\mathbf{r}\}}\left[U(\varphi_{\mathbf{r}}) + (1/2)\,\epsilon_k\,\varphi_{\mathbf{r}}^2\right]$. Therefore, at scale "$k$", we have a theory of independent fields on a lattice, which is trivial.

The partition function $Z_k[h] = \prod_{\mathbf{r}} z_k(\overline{h}_{\mathbf{r}})$ is a product of one-site partition functions with

$$z_k(\overline{h}) = \int_{-\infty}^{+\infty} \mathrm{d}\overline{\varphi}\, \exp\left(-\overline{U}(\overline{\varphi}) - \frac{1}{2}\overline{\epsilon}_k\overline{\varphi}^2 + \overline{h}\overline{\varphi}\right) \,, \tag{3.9}$$

where we have introduced the dimensionless variables $\overline{\varphi} = a\varphi$, $\overline{h} = a^3 h$, and $\overline{\epsilon}_k = a^2\epsilon_k$. Note that $U(\varphi) = a^4\,\overline{U}(\overline{\varphi})$. The Helmholtz free energy and Wetterich effective action can be written as lattice sums

$$W_k[h] = \sum_{\mathbf{r}} \ln z_k(\overline{h}_{\mathbf{r}}), \tag{3.10a}$$

$$\Gamma_k[\phi] = \sum_{\mathbf{r}} \gamma_k(\overline{\phi}_{\mathbf{r}}) \,, \tag{3.10b}$$

where the convex functions $\ln z_k(\overline{h})$ and $\gamma_k(\overline{\phi})$ are related by a Legendre transform $\gamma_k(\overline{\phi}) + \ln z_k(\overline{h}) = \overline{\phi}\,\overline{h}$, with, for instance $\overline{\phi} = \mathrm{d}\ln z_k(\overline{h})/\mathrm{d}\,\overline{h}$. In general, the quantities $\ln z_k(\overline{h})$ and $\gamma_k(\overline{\phi})$ cannot be computed analytically but can be easily evaluated numerically for any value of $\Lambda > k > k_{\max}$.

It is interesting to note that the implicit equation (3.2) now reads

$$\exp\left[-\gamma_k\left(\overline{\phi}\right)\right] = \int_{-\infty}^{+\infty} \mathrm{d}\overline{\varphi}\, \exp\left\{-\overline{U}(\overline{\phi}) + \left[\mathrm{d}\gamma_k(\overline{\phi})/\mathrm{d}\overline{\phi}\right]\,(\overline{\varphi}-\overline{\phi}) - \frac{1}{2}\,\overline{\epsilon}_k\,\overline{\varphi}^2\right\} \,, \tag{3.11}$$

which leads us to two remarks. First, the choice $\Lambda = \infty$ implies $\gamma_\Lambda = U$ since we can replace the Gaussian $\exp[-(1/2)\,\overline{\epsilon}_\Lambda\,\overline{\varphi}^2]$ by a delta function $\delta(\varphi)$ in equation (3.11). Our initial condition for the flow of $\Gamma_k[\phi]$ is now perfectly defined.

Our second remark is that one can derive from the equation (3.11), i.e., from its solution!, the flow equation within the range $\Lambda > k > k_{\max}$. A short calculation reveals that

$$k\partial_k\gamma_k\left(\overline{\phi}\right) = \frac{\overline{\epsilon}_k}{\overline{\epsilon}_k + \gamma_k''\left(\overline{\phi}\right)} \,. \tag{3.12}$$

Noting that, for a homogeneous system, $\gamma_k(\overline{\phi}) = a^4 U_k(\phi)$, where $U_k(\phi)$ is the local potential defined in previous section 3.1, the flow equation for the local potential reads

$$\partial_t U_k = -\frac{a^4\epsilon_k}{\epsilon_k + U_k''} \,, \tag{3.13}$$

with $\partial_t = -k\partial_k$. Clearly, equation (3.13) can also be obtained directly from (3.8) in the range $\Lambda > k > k_{\max}$.

We are now in position to exemplify the initial conditions which can be used to solve the flow equation (3.8) for the local potential

- either $\Lambda = \infty$ and $U_\Lambda = U$ (mean field theory as initial conditions). In this case, the flow equation (3.13) must be solved numerically for $\Lambda > k > k_{\max}$. Note that $U_\Lambda(\phi)$ can be non-convex;

- or $\Lambda = k_{\max} = 4/a$ and $U_\Lambda(\phi) \equiv a^{-4}\gamma_\Lambda(\overline{\phi})$. In this case, $\gamma_\Lambda(\overline{\phi})$ should be evaluated numerically (local field theory as initial conditions). Note that $U_\Lambda(\phi)$ is necessarily convex.

In our numerical experiments we retained the second term of the alternative.

3.3. The local potential approximation

3.3.1. The general case

A non-perturbative, but intuitive approximation to solve the flow equation (3.8) is to make an ansatz on the functional form of $\Gamma_k[\phi]$. In the local potential approximation (LPA), one neglects the renormalization of the spectrum and assumes that [19, 20]

$$\text{(LPA ansatz)}\;\; \Gamma_k\left[\phi\right] = \frac{1}{2Na^4}\sum_{\{\mathbf{q}\}}\phi_{-\mathbf{q}}\epsilon_0\left(\mathbf{q}\right)\phi_{\mathbf{q}} + a^4\sum_{\{\mathbf{r}\}}U_k(\phi_{\mathbf{r}}) \,. \tag{3.14}$$

For a uniform configuration of the classical field $\phi_{\mathbf{r}} = \phi$ and, in the thermodynamic limit, the flow equation (3.8b) becomes:

$$\partial_k U_k(\phi) = \frac{1}{2}\int\limits_{\mathbf{q}\in\mathscr{B}} \frac{\partial_k\widetilde{R}_k\left(\mathbf{q}\right)}{\epsilon_0(\mathbf{q}) + \widetilde{R}_k\left(\mathbf{q}\right) + U_k''(\phi)} \,, \tag{3.15}$$

where $U_k''(\phi)$ denotes the second-order derivation of $U_k(\phi)$ with respect to the order parameter ϕ. Equation (3.15) is a non-linear parabolic PDE. These are good pieces of news since mathematicians have worked hard to provide us with numerical methods for solving such equations. The equation should be supplemented by initial and boundary conditions which will be exemplified in section 4.1

3.3.2. The LMD regulator

With the LMD regulator (3.4), the loop-integral in the r.h.s. of equation (3.15) can be worked out analytically which leaves us with a much simplified flow equation for the potential

$$\partial_t U_k = -\mathscr{N}(\epsilon_k)\mathscr{L}(\omega_k) \,, \tag{3.16}$$

where the RG time "t" is defined by $k = \Lambda e^{-t}$, so that $\partial_t = -k\partial_k$, $\omega_k(\phi) \equiv U_k''(\phi)/\epsilon_k$ is a dimensionless renormalized inverse susceptibility,

$$\mathscr{L}(x) = \frac{1}{1+x} \tag{3.17}$$

is the threshold function [7] which takes a very simple expression with the LMD regulator and finally

$$\mathscr{N}(\epsilon) = \int_{\mathbf{q}\in\mathscr{B}} \Theta\left[\epsilon - \epsilon_0(\mathbf{q})\right] \tag{3.18}$$

denotes the (normalized) number of states (note that we set $a = 1$ to simplify the algebra). It proves convenient to introduce also the density of states

$$\mathscr{D}(\epsilon) = \int_{\mathbf{q}\in\mathscr{B}} \delta\left[\epsilon - \epsilon_0(\mathbf{q})\right] , \tag{3.19}$$

so that

$$\mathscr{N}(\epsilon) = \int_0^{\epsilon} \mathrm{d}\epsilon' \, \mathscr{D}(\epsilon') . \tag{3.20}$$

The two functions $\mathscr{D}(\epsilon)$ and $\mathscr{N}(\epsilon)$ are obviously related to the lattice Green function which, for a SC lattice, reads [22, 23, 34, 35]

$$G(\tau) = \frac{1}{\pi^4}\int_0^{\pi} \mathrm{d}q_1 \ldots \int_0^{\pi} \mathrm{d}q_4 \frac{1}{\tau - \sum_{\mu=1}^{4}\cos(q_\mu)} . \tag{3.21}$$

Note that we have, in the sense of distributions, for $\eta \to 0+$, $1/(\tau + \mathrm{i}\eta) = \mathscr{P}(1/\tau) + \mathrm{i}\pi\delta(\tau)$, where $\mathscr{P}$ is Cauchy principal part. With this remark, the comparison of equations (3.19) and (3.21) reveals at first glance that

$$\mathscr{D}(\epsilon) = \frac{1}{2a^2}\frac{1}{\pi}\,\mathrm{Im}\,G(\tau) , \tag{3.22}$$

with $\tau = 4 - a^2/2\epsilon$. Note that the interval of the spectrum $0 \leqslant \epsilon_k \leqslant \epsilon_0^{\max}$ corresponds to the interval $-4 \leqslant \tau \leqslant 4$ for the auxiliary variable τ. Recently, in reference [22, 23], Loh has obtained a novel integral representation of the Green's function of simple hyper cubic lattices. The resulting one-dimensional integral obtained for $G(\tau)$ involves non-oscillating, well behaved functions and it can thus be computed precisely by means of a Gauss quadrature. From the results of reference [22, 23], we obtained:

- For $0 \leqslant \epsilon \leqslant 2$

$$\mathscr{N}(\epsilon) = \frac{1}{2} + \int_0^{\infty} p_{02}(\epsilon, x)\mathrm{d}x, \tag{3.23a}$$

$$p_{02}(\epsilon, x) = \frac{1}{4\pi} I_\mathrm{e}(x) K_\mathrm{e}(x)^3 \frac{\left\{3\exp(-2x) - \exp[(\epsilon-2)x] - 2\exp[-(\epsilon+2)x]\right\}}{\epsilon} , \tag{3.23b}$$

  where $I_\mathrm{e}(x) = I_0(x)\exp(-x)$, $K_\mathrm{e}(x) = K_0(x)\exp(x)$, $I_0(x)$ and $K_0(x)$ being the modified Bessel Functions of the first and the second class, respectively.

- For $2 \leqslant \epsilon \leqslant 4$

$$\mathscr{N}(\epsilon) = 1 - \int_0^{\infty} p_{24}(\epsilon, x)\mathrm{d}x, \tag{3.24a}$$

$$p_{24}(\epsilon, x) = \frac{I_\mathrm{e}(x)K_\mathrm{e}(x)}{4\pi}\left\{ I_\mathrm{e}(x)^2 \frac{\exp((2-\epsilon)x) - \exp(-2x)}{\epsilon} - K_\mathrm{e}(x)^2 \frac{\exp\left[-(2+\epsilon)x\right] - \exp(-6x)}{\epsilon} \right\} , \tag{3.24b}$$

  while, for negative values of ϵ, one uses $\mathscr{N}(-|\epsilon|) = 1 - \mathscr{N}(|\epsilon|)$ and one of the equations (3.23) or (3.24).

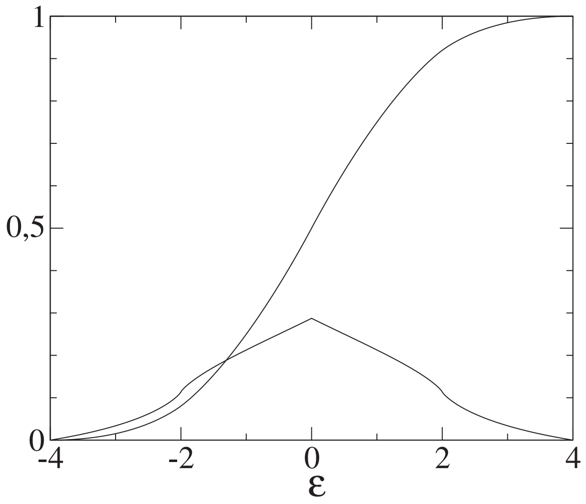

Figure 1. Density and number of states, respectively $\mathscr{D}(\epsilon)$ (bottom) and $\mathscr{N}(\epsilon)$ (top), for the simple $D=4$ cubic lattice.

The functions $\mathscr{N}(\epsilon)$ and $\mathscr{D}(\epsilon)$ were computed from the expressions (3.23) and (3.24) and are displayed in figure 1. The Bessel functions involved in equations (3.23) and (3.24) were evaluated with the double-precision FORTRAN codes i0 and k0 of the specfun library of the Netlib distribution [36] while we made use of the code DQAGIE of the quadpack library, of the same distribution, for the numerical integrations.

3.4. Various limits

We first note that, for $\infty > k > k_{\text{max}}$, one has the trivial identity $\mathscr{N}(\epsilon_k) = a^{-4}$. Therefore, the LMD flow equation (3.16) is identical to the exact NPRG equation (3.13) for the local potential. LMD approximation is thus exact for local theories [21].

Secondly, we consider the scaling limit $k \to 0$. We have

$$\mathscr{N}(\epsilon) = \int\limits_{\mathbf{q}\in\mathscr{B}} \Theta\left[\epsilon - \epsilon_0(\mathbf{q})\right] \sim \int\limits_{\mathbf{q}\in\mathscr{B}} \Theta(k^2 - \mathbf{q}^2) \sim v_4 k^4 \,, \tag{3.25}$$

where $v_4 = 1/(32\pi^2)$ is a geometrical factor, then, the flow equation (3.16) reduces to

$$\partial_t U_k = -v_4 k^4 \mathscr{L}(\omega_k) \,, \qquad (k \to 0) \,, \tag{3.26}$$

which is, of course, the LPA flow equation for the continuous (off-lattice) theory with Litim regulator [27, 37–39]. In the scaling limit, the lattice and off-lattice versions of the Φ^4 model share the same fixed-points and critical exponents, if any.

Let us briefly discuss the Gaussian fixed points solutions of equation (3.26). A general discussion, i.e., for arbitrary dimension D and regulator $\mathscr{L}$, can be found in reference [27] while the case of a sharp cut-off was discussed for the first time in the inspiring paper of Hasenfratz-Hazenfratz [40].

Fixed point solutions make sense only for an equation involving exclusively dimensionless functions and variables and emerge in general in the limit $k \to 0$. We introduce the dimensionless field $x = k^{-1}\phi$ and potential $u_k(x) = k^{-4} U_k(\phi)$. The adimensioned flow equation can thus be written

$$\partial_t u_k = 4u_k - x u_k' - \frac{v_4}{1 + u_k''} \,, \tag{3.27}$$

with $u_k' \equiv \mathrm{d}u_k/\mathrm{d}x$. A fixed point $u^\star(x)$ satisfies $\partial_t u^\star(x) = 0$ for all x. $u''^\star(x) = 0$ is obviously a special solution. By integration it gives $u'^\star(x) = 0$ ($\mathbb{Z}2$ symmetry) and $u^\star(x) = v_4/4$, this is the Gaussian fixed point. In order to study the stability of the fixed point, we linearize (3.27). Let us define

$$u_k(x) = u^\star(x) + h_k(x) \tag{3.28}$$

and expand equation (3.27) in powers of h, it yields

$$\partial_t h = Dh - v_4 h''^2, \tag{3.29a}$$
$$Dh = 4h - xh' + v_4 h'' . \tag{3.29b}$$

Let us start the analysis with the linearized RG equation

$$\partial_t h = Dh . \tag{3.30}$$

We search a solution under the form $h(x,t) = \exp(\lambda t) H(y = \beta x)$ which yields the eigenvalue problem $(D-\lambda)H = 0$ which can be rewritten as Hermite equation:

$$H''(y) - 2y\,H'(y) + 2n\,H(y) = 0\ , \tag{3.31}$$

with $4 - \lambda = n$. Hermite's equation (3.31) admits in general solutions without definite parity (Weber's functions). Only if n is a positive integer, do the solutions $H_n(y)$ have the same parity as n. Such solutions are polynomials, namely the Hermite's polynomials [41]. Imposing $\mathbb{Z}2$ symmetry, therefore, leads to a discretization of the spectrum $4 - \lambda_p = 2p$, where p is positive integer. The general linearized solution of (3.30) is then

$$h(x,t) = \sum_{p=0}^{\infty} c_p \exp(\lambda_p t)\ H_{2p}(x/\sqrt{2v_4})\ , \tag{3.32a}$$
$$= \sum_{p=0}^{\infty} \hat{c}_p \exp(\lambda_p t)\ \chi_p(x)\ , \tag{3.32b}$$

where $\chi_p(x)$ is a convenient redefinition of Hermite's polynomial H_{2p} such that its coefficient of degree $2p$ is one. We have $\chi_0(x) = 1$, $\chi_1(x) = x^2 - v_4/2$, $\chi_2(x) = x^4 - 6v_4x^2 + 3v_4^2$, etc

Clearly for $p = 0$ we have a trivial constant solution. $p = 1$ corresponds to $\lambda_1 = 2$, thus $\chi_1(x)$ is a relevant field. The case $p = 2$ corresponds to $\lambda_2 = 0$ and $\chi_2(x)$ is a marginal field. For all $p \geqslant 3$, the eigenvalue $\lambda_p < 0$ (for instance $\lambda_3 = -2$) corresponds to irrelevant solutions $\chi_p(x)$. The stability of the marginal field $\chi_2(x)$ can be obtained by finding a solution of equation (3.29a) equal to χ_2 at the dominant order. An analysis similar to that of reference [40] reveals that χ_2 is in fact irrelevant beyond the linear approximation. The picture of the scaling fields $\chi_p(x)$ in $D = 4$ is thus consistent with the critical point [33]. The usual analysis [33] then yields for the critical exponent ν the classical value $\nu = 1/\lambda_1 = 0.5$. Since Fisher's exponent $\eta = 0$ in the LPA, all other (classical) exponents are deduced from scaling relations.

It is generally admitted, and was confirmed by the recent numerical studies of Codello [42], that there is no other fixed point than the Gaussian fixed point in $D = 4$. We have just shown that the LPA/LMD theory, albeit approximate, supports the existence of this fixed point.

4. Numerical experiments

4.1. A change of variables

We pointed out in section 3.4 that in the asymptotic limit $k \to 0$, the lattice and off-lattice LPA flow equations bear the same form. In the ordered phase, their behaviors are both singular due to the simple pole $\omega = -1$ in the threshold function $\mathscr{L}(\omega)$ [see equation (3.17)]. This point has been studied at length in references [26, 27]. Specializing this discussion to the case $D = 4$ we note that in the limit $k \to 0$, $\omega_k(\phi) = U''_k(\phi)/\epsilon_k \to -1$ for $-\phi_0(k) < \phi < \phi_0(k)$ where $\phi_0(k)$ is a precursor of the spontaneous magnetization $\phi_0 = \lim_{k\to 0}\phi_0(k)$. It follows that the threshold function $\mathscr{L}$ diverges in this interval as k^{-2}. This yields a universal behavior $\mathscr{L}(\phi)/\mathscr{L}(\phi = 0) = 1 - \phi^2/\phi_0^2$. Moreover, as a consequence, $U_k(\phi)$ becomes convex as $k \to 0$, in particular, it becomes constant for $-\phi_0 < \phi < +\phi_0$.

The divergence of the threshold function makes it impossible to obtain numerical solution of the non-linear PDE (3.16) in the ordered phase, and we really deal with *stiff* equations. In order to remove stiffness, one is led to make the change of variables $U_k(M) \Longrightarrow L_k(M) = \mathscr{L}[\omega_k(M) \equiv U''_k(M)/\epsilon_k]$. We then obtain the equations

$$L''_k(\phi) = \frac{2\epsilon_k}{\mathscr{N}(\epsilon_k)} \left[\frac{1}{L_k(\phi)} - 1\right] + \frac{\epsilon_k}{\mathscr{N}(\epsilon_k)} \frac{1}{L_k(\phi)^2} \partial_t L_k(\phi), \tag{4.1}$$

where $k = \Lambda \mathrm{e}^{-t}$.

In contradistinction with equations (3.16), the quasi-linear parabolic PDE (4.1) can easily be integrated out. As in references [11, 21, 26, 27] we made use of the fully implicit predictor-corrector algorithm of Douglas-Jones [29]. This algorithm is unconditionally stable and convergent and introduces an error of $\mathscr{O}[(\Delta t)^2] + \mathscr{O}\left[(\Delta\phi)^2\right]$ (Δt and $\Delta\phi$ discrete RG time and field steps, respectively) and can be used below and above the critical point as well. In the ordered phase we note that [27] $L_k(\phi) \propto k^{-2}\left[\phi_0(k)^2 - \phi^2\right]$ for $-\phi_0 < \phi < +\phi_0$ which obviously does not preclude us from obtaining a numerical solution of equation (4.1).

The initial conditions on the local potential U_k at $k = \Lambda$ are easily transposed to the field L_k. It follows from the discussion in the end of section 3.2 that the simplest choice is $\Lambda = k_{\mathrm{max}} = 4/a$ and $L_\Lambda = \mathscr{L}[a^{-4}\gamma''_{k_{\mathrm{max}}}(\overline{\phi})]$, where $\gamma_{k_{\mathrm{max}}}$ is the local Wetterich function and $\overline{\phi} = a\phi$ for all values of the order parameter ϕ.

Of course, in practice, a cut-off must be imposed on ϕ, and boundary conditions must then be introduced such that the PDE is solved only on the interval $-\phi_{\mathrm{max}} < \phi < \phi_{\mathrm{max}}$ for all k with some specifications on the boundaries. We made a consistent choice [21, 27] $L_k(\pm\phi_{\mathrm{max}}) = a^{-4}\mathscr{L}[a^{-4}\gamma''{}_k(\overline{\phi}_{\mathrm{max}})]$. Here, Wetterich effective function $\gamma_k(\overline{\phi}_{\mathrm{max}})$ is evaluated in the first approximation of the hopping parameter expansion (see, e.g., reference [24]) by assuming the validity of the local approximation.

4.2. Solving the flow equations

We solved equation (4.1) using the Douglas-Jones algorithm [29]. For most of our numerical experiments we used $\Delta t = 10^{-4}$, a maximum of $N_t = 3\ 10^5$ time steps, $\Delta\phi = 10^{-4}$ and $N_\phi = 30000$ field steps (i.e., $\phi_{\mathrm{max}} = 3.$). Note that the functions $\mathscr{N}(\epsilon)$ and $\mathscr{D}(\epsilon)$ can be computed once for all with the desired precision.

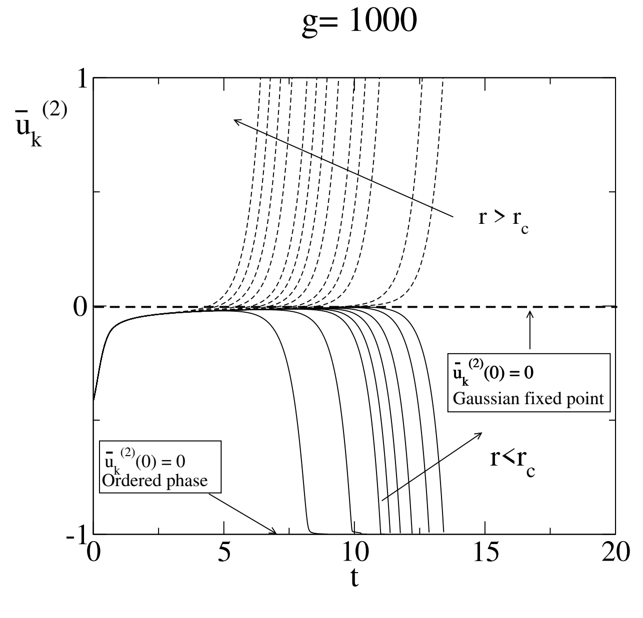

Figure 2. The coupling constant $\overline{u}^{(2)}_k \equiv [\mathrm{d}^2U_k(\phi = 0)/\mathrm{d}\phi^2]/\epsilon_k$ as a function of the RG time $t = \ln\Lambda/k$ at $g = 1000$. For $r > r_{\mathrm{c}}$, the flow escapes to infinity (dotted lines) while , for $r < r_{\mathrm{c}}$, the flow reaches the low temperature fixed point $\overline{u}^{(2)}_k = -1$ (solid lines). For $t \to \infty$, the dashed line $\overline{u}^{(2)}_k = 0$ (Gaussian fixed point value) separates the two regimes.

In order to determine the critical point $r_c(g)$, one proceeds by dichotomy, g is fixed and one varies r. An illustration of the method is given in figure 2 in the case $g = 1000$. The renormalized coupling constant $\overline{u}_k^{(2)} \equiv U_k''(M=0)/\epsilon_k$, with $\epsilon_k = a^2 k^2$, discriminates the state of the system by its behavior in the limit $k \to 0$.

Of course, the Gaussian fixed point, characterized by $\overline{u}_k^{(2)} = 0$, is never reached but is approached only asymptotically for $r = r_c(g)$. As soon as $r \neq r_c(g)$, the flow deviates from the fixed point due to the relevant fields. For $r < r_c(g)$, the coupling constant $\overline{u}_k^{(2)} \to -1$ as t increases; this is the expected behavior in the ordered phase. For $r > r_c(g)$, $\overline{u}_k^{(2)} \to +\infty$ when $k \to 0$ (and thus $\epsilon_k \to 0$) since the compressibility $U_k''(\phi)$ remains finite for all values of the order parameter ϕ; the curves escape to $+\infty$ as can be seen in figures 2 and 3.

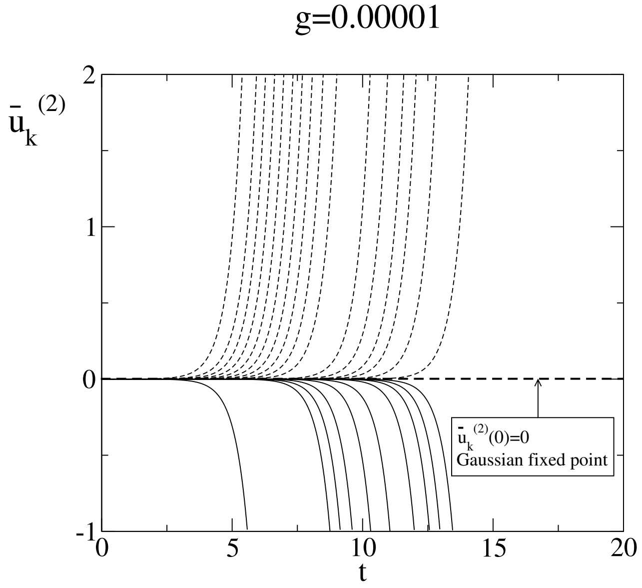

Figure 3. Same as figure 2 for $g = 0.00001$.

A few dichotomies of r thus yield a very precise estimate of $\overline{r}_c(g)$. We checked that our values for the parameters Δt, $\Delta\phi$, etc., give at least 8 stable figures for $\overline{r}_c(g)$. We report only 7 figures in the table 1 with the last figure rounded-up. Precision could be enhanced with codes in quadruple precision, but unfortunately no such public domain FORTRAN code exists for the calculation of Bessel functions. We explored a wide range of values of parameters with g varying in the range $g = 10^{-5}$ (the Gaussian limit) up to $g = 100000$ (Ising model limit), see respectively figures 3 and 2.

Recent Monte Carlo simulations suggest, according to the authors of reference [43], the existence of a weak first order transition, at low values of g, i.e., in the Gaussian limit. Since there are no other fixed points (FP) than the Gaussian FP in $D = 4$, it would mean that the flow stops at some finite value of k and does not reach the FP. Consequently, hysteresis phenomena should be observed in conjunction with the abortion of critical fluctuations. This scenario is in contradiction with our findings in the LPA/LMPD theory. Figure 4 displays the inverse compressibility $U_k''(\phi = 0)$ in the limit $k \to 0$ for $g = 0.00001$. The fixed point is attained and the expected linear classical behavior of $U_k''(\phi = 0) \Leftarrow (\delta\overline{r})$ is eventually obtained. A linear regression of the right hand part of the curve gives an exponent of $\gamma^{-1} = 0.99985$ in agreement with the classical value of the compressibility exponent $\gamma = 1$. A weak first order transition would yield a discontinuity at some value of r which is never observed for $g \geqslant 10^{-5}$. Numerically, it proved very difficult to consider smaller values of g smaller than 10^{-5}, and a code written in quadruple precision should be necessary to investigate further this question.

Table 1. Critical parameters of the Φ^4 scalar field theory on a 4D simple cubic lattice in the LPA approximation using the LMD regulator (3.4). From left to right: g, $\overline{r}_c(g)$. The data were obtained by fixing g and determining $\overline{r}_c(g)$ by dichotomy. An uncertainty of at most ± 1 affects the last digit.

g	$\overline{r}_c(g)$	g	$\overline{r}_c(g)$
$0.10\ 10^{-4}$	$-0.7746694\ 10^{-6}$	$0.70\ 10^{2}$	$-0.4200564\ 10^{1}$
$0.10\ 10^{-3}$	$-0.7746662\ 10^{-5}$	$0.75\ 10^{2}$	$-0.4456839\ 10^{1}$
$0.50\ 10^{-3}$	$-0.3873318\ 10^{-4}$	$0.80\ 10^{2}$	$-0.4709800\ 10^{1}$
$0.10\ 10^{-2}$	$-0.7746600\ 10^{-4}$	$0.85\ 10^{2}$	$-0.4959654\ 10^{1}$
$0.10\ 10^{-1}$	$-0.7745977\ 10^{-3}$	$0.90\ 10^{2}$	$-0.5206587\ 10^{1}$
$0.20\ 10^{-1}$	$-0.1549056\ 10^{-2}$	$0.95\ 10^{2}$	$-0.5450764\ 10^{1}$
$0.30\ 10^{-1}$	$-0.2323377\ 10^{-2}$	$0.100\ 10^{3}$	$-0.5692335\ 10^{1}$
$0.40\ 10^{-1}$	$-0.3097560\ 10^{-2}$	$0.110\ 10^{3}$	$-0.6168189\ 10^{1}$
$0.50\ 10^{-1}$	$-0.3871605\ 10^{-2}$	$0.120\ 10^{3}$	$-0.6635096\ 10^{1}$
$0.60\ 10^{-1}$	$-0.4645512\ 10^{-2}$	$0.130\ 10^{3}$	$-0.7093852\ 10^{1}$
$0.70\ 10^{-1}$	$-0.5419282\ 10^{-2}$	$0.140\ 10^{3}$	$-0.7545135\ 10^{1}$
$0.80\ 10^{-1}$	$-0.6192914\ 10^{-2}$	$0.150\ 10^{3}$	$-0.7989528\ 10^{1}$
$0.90\ 10^{-1}$	$-0.6966409\ 10^{-2}$	$0.160\ 10^{3}$	$-0.8427538\ 10^{1}$
0.10	$-0.7739766\ 10^{-2}$	$0.170\ 10^{3}$	$-0.8859610\ 10^{1}$
0.20	$-0.1546584\ 10^{-1}$	$0.180\ 10^{3}$	$-0.9286136\ 10^{1}$
0.30	$-0.2317839\ 10^{-1}$	$0.190\ 10^{3}$	$-0.9707466\ 10^{1}$
0.40	$-0.3087757\ 10^{-1}$	$0.200\ 10^{3}$	$-0.1012391\ 10^{2}$
0.50	$-0.3856355\ 10^{-1}$	$0.225\ 10^{3}$	$-0.1114543\ 10^{2}$
0.60	$-0.4623647\ 10^{-1}$	$0.250\ 10^{3}$	$-0.1214183\ 10^{2}$
0.70	$-0.5389649\ 10^{-1}$	$0.275\ 10^{3}$	$-0.1311605\ 10^{2}$
0.80	$-0.6154375\ 10^{-1}$	$0.300\ 10^{3}$	$-0.1407051\ 10^{2}$
0.90	$-0.6917840\ 10^{-1}$	$0.350\ 10^{3}$	$-0.1592776\ 10^{2}$
$0.10\ 10^{1}$	$-0.7680056\ 10^{-1}$	$0.400\ 10^{3}$	$-0.1772604\ 10^{2}$
$0.15\ 10^{1}$	-0.1147289	$0.450\ 10^{3}$	$-0.1947454\ 10^{2}$
$0.20\ 10^{1}$	-0.1523643	$0.500\ 10^{3}$	$-0.2118027\ 10^{2}$
$0.25\ 10^{1}$	-0.1897212	$0.550\ 10^{3}$	$-0.2284875\ 10^{2}$
$0.30\ 10^{1}$	-0.2268122	$0.600\ 10^{3}$	$-0.2448441\ 10^{2}$
$0.40\ 10^{1}$	-0.3002422	$0.650\ 10^{3}$	$-0.2609089\ 10^{2}$
$0.50\ 10^{1}$	-0.3727360	$0.700\ 10^{3}$	$-0.2767124\ 10^{2}$
$0.60\ 10^{1}$	-0.4443624	$0.750\ 10^{3}$	$-0.2922800\ 10^{2}$
$0.70\ 10^{1}$	-0.5151810	$0.800\ 10^{3}$	$-0.3076338\ 10^{2}$
$0.80\ 10^{1}$	-0.5852432	$0.850\ 10^{3}$	$-0.3227925\ 10^{2}$
$0.90\ 10^{1}$	-0.6545945	$0.900\ 10^{3}$	$-0.3377728\ 10^{2}$
$1.00\ 10^{1}$	-0.7232751	$0.950\ 10^{3}$	$-0.3525890\ 10^{2}$
$1.25\ 10^{1}$	-0.8922688	$0.10\ 10^{4}$	$-0.3672538\ 10^{2}$
$1.50\ 10^{1}$	$-0.1057756\ 10^{1}$	$0.12\ 10^{4}$	$-0.4246102\ 10^{2}$
$1.75\ 10^{1}$	$-0.1220112\ 10^{1}$	$0.14\ 10^{4}$	$-0.4802738\ 10^{2}$
$0.20\ 10^{2}$	$-0.1379637\ 10^{1}$	$0.16\ 10^{4}$	$-0.5346330\ 10^{2}$
$0.25\ 10^{2}$	$-0.1691160\ 10^{1}$	$0.18\ 10^{4}$	$-0.5879670\ 10^{2}$
$0.30\ 10^{2}$	$-0.1993908\ 10^{1}$	$0.20\ 10^{4}$	$-0.6404841\ 10^{2}$
$0.35\ 10^{2}$	$-0.2289055\ 10^{1}$	$0.25\ 10^{4}$	$-0.7691726\ 10^{2}$
$0.40\ 10^{2}$	$-0.2577512\ 10^{1}$	$0.30\ 10^{4}$	$-0.8953949\ 10^{2}$
$0.45\ 10^{2}$	$-0.2860003\ 10^{1}$	$0.40\ 10^{4}$	$-0.1144133\ 10^{3}$
$0.50\ 10^{2}$	$-0.3137118\ 10^{1}$	$0.50\ 10^{4}$	$-0.1391031\ 10^{3}$
$0.55\ 10^{2}$	$-0.3409347\ 10^{1}$	$0.60\ 10^{4}$	$-0.1637724\ 10^{3}$
$0.60\ 10^{2}$	$-0.3677103\ 10^{1}$	$0.70\ 10^{4}$	$-0.1884735\ 10^{3}$
$0.65\ 10^{2}$	$-0.3940740\ 10^{1}$	$0.10\ 10^{5}$	$-0.2627898\ 10^{3}$

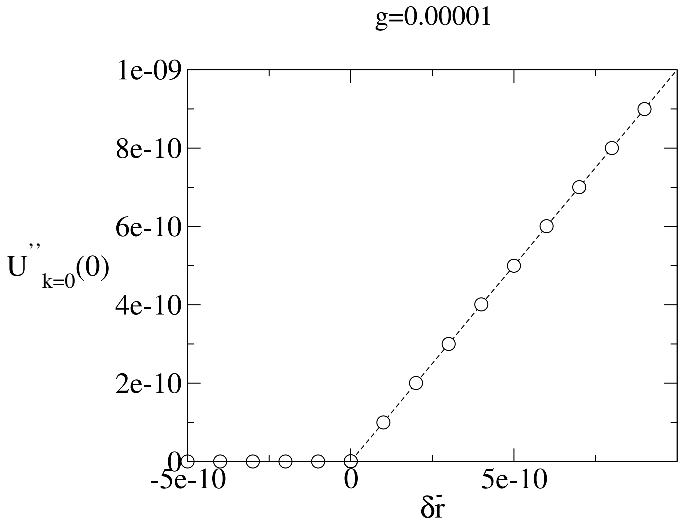

Figure 4. Inverse compressibility $U''_k(\phi = 0)$ in the limit $k \to 0$ for $g = 0.00001$ as a function of $\delta\overline{r} = \overline{r} - \overline{r}_{\mathrm{c}}$.

5. Conclusion

In this paper we have computed the critical line of the Φ^4 one-component model on the simple cubic lattice in four dimensions of space in the framework of the NPRG within the LPA approximation. We made use of only the smooth LMD regulator which is expected to give the better results. The flow equations have been solved for the threshold functions rather than for the potential. This trick allows one to obtain numerical solutions in the ordered phase where the PDE for the potential are stiff and fail to converge. A dichotomy process based on the generically different asymptotic behaviors of the dimensioned inverse susceptibility $U''_k(\phi = 0)/k^2$ in zero field, below and above the critical point, provides a very precise determination of the critical line $\overline{r}_{\mathrm{c}}(g)$. The model is trivial in the sense that all the solutions belong to the basin of attraction of the Gaussian fixed point for all the considered values of g. We did not observe a weak first order transition in the Gaussian limit $g \to 0$, at least, numerically, for $g > 10^{-5}$. A numerical exploration of still lower values of parameter g would require a quadruple precision code which is out of reach for the moment.

In reference [21], we obtained an excellent agreement between our estimates of the critical line of the $3D$ Φ^4 model on a simple three dimensional lattice and that of Monte Carlo simulations of Hasenbush [30]. In $D = 3$, the LPA approximation does not yield exact critical exponents contrary to the case $D = 4$ where the classical exponents are found. One can thus *a fortiori* expect an excellent agreement for the critical line between the theory and the simulations in 4D. Unfortunately, we were unable to find estimates of the critical line of the 4D version of the model by means of Monte Carlo simulations in the literature.

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Критична лінія скалярної теорії поля Φ^4 на чотиривимірній кубічній гратці в наближенні локального потенціалу

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Ми визначаємо критичну лінію однокомпонентної (чи Ландау-Гінзбурга) моделі Φ^4 на простій чотиривимірній кубічній ґратці. Наше дослідження здійснено в рамках непертурбативної ренормалізаційної групи в наближенні локального потенціалу з м’яким інфрачервоним регулятором. Показано, що перехід є другого роду навіть у гаусовій границі, де можна було б очікувати перший рід відповідно до деяких нещодавніх теоретичних передбачень.

Ключові слова: *непертурбатина ренормалізаційна група, наближення локального потенціалу, ґраткова теорія Φ^4, числові експерименти*