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# A new exactly solvable spatially one-dimensional quantum superradiance fermi-medium model and its quantum solitonic states*

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A new exactly solvable spatially one-dimensional quantum superradiance model describing a charged fermionic medium interacting with an external electromagnetic field is proposed. The infinite hierarchy of quantum conservation laws and many-particle Bethe eigenstates that model quantum solitonic impulse structures are constructed. The Hamilton operator renormalization procedure subject to a physically stable vacuum is described, the quantum excitations and quantum solitons, related to the thermodynamical equilibrity of the model, are discussed.

**Key words:** *charged fermionic medium, quantum superradiance model, Bethe eigenstates, quantum solitons, renormalization, conservation laws, quantum inverse spectral problem, Yang-Baxter identity*

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## 1. Fermionic medium and superradiance model description

We shall describe the quantum superradiance properties [1–9] of a model of a one-dimensional many particle charged fermionic medium interacting with an external electromagnetic field. The Dirac type $N$-particle Hamiltonian operator of the model is expressed as

$$H_N := \mathrm{i}\sum_{j=1}^{N}\sigma_3^{(j)}\frac{\partial}{\partial x_j}\otimes\mathbf{1}-\mathrm{i}\beta\mathbf{1}\otimes\int_{\mathbb{R}}\mathrm{d}x\varepsilon^{+}\varepsilon_x+\alpha\sum_{j=1}^{N}\sigma_1^{(j)}\otimes\mathscr{E}(x_j), \tag{1.1}$$

where $\sigma_3^{(j)},\sigma_1^{(j)}$, $j=1,\dots N$, are the usual Pauli matrices, $\alpha\in\mathbb{R}_+$ is an interaction constant, $0<\beta<1$ is the light speed in the linearly polarized fermionic medium,

$$\mathscr{E}(x) := \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \varepsilon^{+}(x) \end{pmatrix}$$

is the one-mode polarization matrix operator at particle location $x\in\mathbb{R}$ with quantized electric field bose-operators $\varepsilon(x),\varepsilon^{+}(x):\Phi_{\mathrm{B}}\to\Phi_{\mathrm{B}}$ acting in the corresponding Fock space $\Phi_{\mathrm{B}}$ and satisfying the commutation relationships:

$$[\varepsilon(x),\varepsilon^{+}(y)]=\delta(x-y), \qquad [\varepsilon(x),\varepsilon(y)]=0=[\varepsilon^{+}(x),\varepsilon^{+}(y)]$$

for all $x,y\in\mathbb{R}$. We note that throughout the sequel we employ units for which the standard constants $\hbar=1=c$.

*Authors with great pleasure devote their work to Professor Mykhaylo Kozlovskii in honor of his 60$^{\text{th}}$ Birthday Jubilee.

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By construction, the $N$-particle Hamiltonian operator (1.1) acts in the Hilbert space $L_2^{(\mathrm{as})}(\mathbb{R}^N;\mathbb{C}^2)\otimes\Phi_{\mathrm{B}}$, where $L_2^{(\mathrm{as})}(\mathbb{R}^N;\mathbb{C}^2)$ denotes the square-integrable antisymmetric vector functions on $\mathbb{R}^N, N\in\mathbb{Z}_+$. Correspondingly, the Fock space $\Phi_{\mathrm{B}}$ allows for the standard representation as the direct sum

$$\Phi_{\mathrm{B}}:=\bigoplus_{n\in\mathbb{Z}_+}L_2^{(\mathrm{s})}(\mathbb{R}^n;\mathbb{C}), \tag{1.2}$$

where $L_2^{(\mathrm{s})}(\mathbb{R}^n;\mathbb{C})$ denotes the space of symmetric square-integrable scalar functions on $\mathbb{R}^n, n\in\mathbb{Z}_+$. Similarly, the corresponding fermionic Fock space

$$\Phi_{\mathrm{F}}:=\bigoplus_{n\in\mathbb{Z}_+}L_2^{(\mathrm{as})}(\mathbb{R}^n;\mathbb{C}^2), \tag{1.3}$$

can be used to represent [10–13] the Hamiltonian operator (1.1) in the second quantized form

$$\mathbf{H}:=\mathrm{i}\int_{\mathbb{R}}\mathrm{d}x\left[\psi_1^+\psi_{1,x}-\psi_2^+\psi_{2,x}-\beta\varepsilon^+\varepsilon_x+\mathrm{i}\alpha\left(\varepsilon\psi_2^+\psi_1+\varepsilon^+\psi_1^+\psi_2\right)\right], \tag{1.4}$$

which acts on the tensored Fock space $\Phi:=\Phi_{\mathrm{F}}\otimes\Phi_{\mathrm{B}}$, where the spaces $\Phi_{\mathrm{F}}$ and $\Phi_{\mathrm{B}}$, defined respectively, by (1.2) and (1.3), can also be represented as

$$\begin{aligned}
\Phi_{\mathrm{B}} &:= \bigoplus_{n\in\mathbb{Z}_+}\mathrm{span}\Big\{\int_{\mathbb{R}^n}\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\chi_n(x_1,x_2,\ldots,x_n)\\
&\qquad\times\prod_{j=1}^{n}\varepsilon^+(x_j)\,|0\rangle_{\mathrm{B}};\chi_n\in L_2^{(\mathrm{s})}(\mathbb{R}^n;\mathbb{C})\Big\},\\
\Phi_{\mathrm{F}} &:= \bigoplus_{n\in\mathbb{Z}_+}\mathrm{span}\Big\{\int_{\mathbb{R}^n}\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\varphi_n^{(m)}(x_1,x_2,\ldots,x_n)\\
&\qquad\times\prod_{j=m+1}^{n}\psi_1^+(x_j)\prod_{k=1}^{m}\psi_2^+(x_k)|0\rangle:0\leqslant m\leqslant n;\varphi_n^{(m)}\in L_2^{(\mathrm{as})}(\mathbb{R}^n;\mathbb{C}^2)\Big\},
\end{aligned} \tag{1.5}$$

and $|0\rangle_{\mathrm{B}}\in\Phi_{\mathrm{B}}$, $|0\rangle_{\mathrm{F}}\in\Phi_{\mathrm{F}}$ are the corresponding vacuum bose- and fermi-states, satisfying the determining conditions

$$\psi_1(x)\,|0\rangle_{\mathrm{F}}=0=\psi_2(x)\,|0\rangle_{\mathrm{F}},\qquad \varepsilon(x)\,|0\rangle_{\mathrm{B}}=0 \tag{1.6}$$

for all $x\in\mathbb{R}$. The creation and annihilation operators $\psi_j(x),\psi_k^+(y):\Phi_{\mathrm{F}}\to\Phi_{\mathrm{F}}$, $j,k=1,2$, satisfy the anti-commuting

$$\begin{aligned}
\{\psi_j(x),\psi_k^+(y)\} &= \delta_{j,k}\delta(x-y),\\
\{\psi_j(x),\psi_k(y)\} &= 0=\{\psi_j^+(x),\psi_k^+(y)\}
\end{aligned} \tag{1.7}$$

and commuting

$$\begin{aligned}
[\varepsilon(x),\psi_j(y)] &= 0=[\varepsilon(x),\psi_j^+(y)],\\
[\varepsilon^+(x),\psi_j(y)] &= 0=[\varepsilon^+(x),\psi_j^+(y)]
\end{aligned} \tag{1.8}$$

relationships for all $x,y\in\mathbb{R}$.

As we are interested in describing the so-called super-resonance processes in our fermionic medium induced by an external electromagnetic field, in particular, a possibility of generating strong localized photonic impulses, it is first necessary to investigate the bound photonic medium states and their spectral energy characteristics. Toward this aim, we make an important note that it has been observed that the spectral properties of the Hamiltonian operator (1.4) can be analyzed in great detail owing to the fact that the related Heisenberg nonlinear dynamical system

$$\begin{aligned}
\psi_{1,t} &= \mathrm{i}[\mathbf{H},\psi_1]=\psi_{1,x}+\mathrm{i}\alpha\varepsilon^+\psi_2,\\
\psi_{2,t} &= \mathrm{i}[\mathbf{H},\psi_2]=-\psi_{2,x}+\mathrm{i}\alpha\varepsilon\psi_1,\\
\varepsilon_t &= \mathrm{i}[\mathbf{H},\varepsilon]=-\beta\varepsilon_x+\mathrm{i}\alpha\psi_1^+\psi_2
\end{aligned} \tag{1.9}$$

is a quantum exactly solvable Hamiltonian flow on the quantum operator manifold $\mathbf{M} := \{(\psi_1, \psi_2, \varepsilon; \varepsilon^+, \psi_2^+, \psi_1^+) \in \mathrm{End}\Phi^6\}$, where the notation $\mathrm{End}\Phi^6$ denotes the space of all endomorphisms from the linear operator space $\Phi^6$ to itself. The system (1.9) can be linearized by means of the quantum Lax type spectral problem

$$\frac{\mathrm{d}f}{\mathrm{d}x} = l(x;\lambda)f, \tag{1.10}$$

where the generalized eigenfunction $f \in \Phi^3$, and the operator matrix $l(x;\lambda) \in \mathrm{End}\Phi^3$ equals

$$l(x;\lambda) := \begin{pmatrix} -\frac{\mathrm{i}\lambda}{3-\beta} & \mathrm{i}\xi_1\psi_1 & \mathrm{i}\xi_2\psi_2 \\ \mathrm{i}\xi_1\psi_1^+ & -\frac{\mathrm{i}\lambda}{2\beta} & \mathrm{i}\xi_3\varepsilon \\ \mathrm{i}\xi_2\psi_2^+ & \mathrm{i}\xi_3\varepsilon^+ & \frac{\mathrm{i}\lambda(3+\beta)}{2\beta(3-\beta)} \end{pmatrix} \tag{1.11}$$

for all $x \in \mathbb{R}$, with $\lambda \in \mathbb{C}$ being an arbitrary time-independent spectral parameter, and

$$\begin{aligned} \xi_1 &:= \xi_1(\alpha,\beta) = -18\alpha\left[\frac{(9-3\beta)(\beta+1)}{\beta+3}\right]^{1/2}\left(\frac{12\beta}{\beta+3}\right)^{1/2}\frac{\beta+3}{2\beta^2+3\beta+3}, \\ \xi_2 &:= \xi_2(\alpha,\beta) = 6\alpha(3-3\beta)^{1/2}\left(\frac{12\beta}{\beta+3}\right)^{1/2}\frac{(9-3\beta)(\beta+1)}{(\beta-1)(2\beta^2+3\beta+3)}, \\ \xi_3 &:= \xi_3(\alpha,\beta) = 72\alpha\beta\frac{(3(1-\beta))^{1/2}}{(\beta-1)(2\beta^2+3\beta+3)}\left[\frac{(9-3\beta)(\beta+1)}{\beta+3}\right]^{1/2} \end{aligned} \tag{1.12}$$

being constants, depending on the interaction parameter $\alpha \in \mathbb{R}_+$ and the light speed $\beta$ in the polarized fermionic medium $0 < \beta < 1$.

**Remark 1.** *Concerning the quantum Lax type spectral problem (1.10) and its determination, one can consult [10, 14–17], where the corresponding analytical tools are developed and described in detail.*

The quantum dynamical system (1.4) may be also regarded as an exactly solvable approximation of the three-level quantum model studied in [18] subject to its superradiance properties. Concerning the studies of such superradiance Dicke type one-dimensional models, it is necessary to mention the work [19] in which it was shown that the well-known quantum Bloch-Maxwell dynamical system

$$\begin{aligned} \psi_{1,t} &= \mathrm{i}[\hat{\mathbf{H}},\psi_1] = \mathrm{i}\alpha\varepsilon^+\psi_2, \\ \psi_{2,t} &= \mathrm{i}[\hat{\mathbf{H}},\psi_2] = \mathrm{i}\alpha\varepsilon\psi_1, \\ \varepsilon_t &= \mathrm{i}[\hat{\mathbf{H}},\varepsilon] = -\beta\varepsilon_x + \mathrm{i}\alpha\psi_1^+\psi_2 \end{aligned} \tag{1.13}$$

generated by the the reduced quantum Hamiltonian operator

$$\hat{\mathbf{H}} := -\mathrm{i}\int_{\mathbb{R}} \mathrm{d}x\left[\beta\varepsilon^+\varepsilon_x - \mathrm{i}\alpha\left(\varepsilon\psi_2^+\psi_1 + \varepsilon^+\psi_1^+\psi_2\right)\right] \tag{1.14}$$

in the strongly degenerate Fock space $\Phi$ is also exactly solvable. Moreover, it has the corresponding Lax type operator spectral problem [10, 15, 20] in the space $\Phi^3$. However, the important problem of constructing the stable physical vacuum for the Hamiltonian (1.14) was on the whole not discussed in [19], and neither was the problem of studying the related thermodynamics of quantum excitations over it. More interesting quantum one-dimensional models with the Hamiltonian similar to (1.4) describing the quantum interaction of just fermionic particles and only bosonic particles with an external electromagnetic field were studied, respectively, in [21] and [22]. In these investigations, the quantum localized Bethe states were constructed and analyzed in detail. The corresponding classical version of the quantum dynamical system (1.9), called the *three-wave model*, was studied in [20, 23, 24] and elsewhere.

It is also worthy to mention here that the spectral operator problem (1.11) makes sense [5, 6, 8] only if the light speed inside the polarized fermionic medium is less than the light speed in a vacuum. This is ensured by the dynamical stability of the quantum Hamiltonian system (1.9) following from the existence of an additional infinite hierarchy of conservation laws, suitably determined on the quantum operator phase space **M**. Consequently, one can expect that the quantum dynamical system (1.9) also possesses many-particle localized photonic states in the Fock space $\Phi$, which are called [9, 19, 25–29] *quantum solitons*, whose spatial range is inverse to the number of interior particles, and which can be interpreted as special Dicke type superradiance laser impulses. In particular, the quantum stability, solitonic formation aspects and construction of the physical ground state [10, 14, 17] related to the unbounded a priori from below Hamiltonian operator (1.4) will be the main focus of the succeeding sections.

## 2. Bethe eigenstates and the energy localization

In this section we construct some of the finite-particle Bethe eigenstates [10, 14, 16, 17, 30] for the quantum Hamiltonian operator (1.4) and discuss their energy localization property. The localization manifests itself in the fact that the energy of a many-particle cluster appears to be less [10, 14, 30] than that of the corresponding system of free particles, giving rise to the formation of the so-called quantum solitonic states localized in the space.

The following number operators

$$\mathbf{N}_{\mathrm{F}} := \int_{\mathbb{R}} \mathrm{d}x \left(\psi_1^+\psi_1 + \psi_2^+\psi_2\right), \mathbf{N}_{\mathrm{B}} := \int_{\mathbb{R}} \mathrm{d}x \left(\varepsilon^+\varepsilon + \psi_2^+\psi_2\right) \tag{2.1}$$

commute with each other and with the Hamiltonian operator (1.4):

$$[\mathbf{N}_{\mathrm{F}}, \mathbf{N}_{\mathrm{B}}] = 0, \qquad [\mathbf{H}, \mathbf{N}_{\mathrm{F}}] = 0 = [\mathbf{H}, \mathbf{N}_{\mathrm{B}}]. \tag{2.2}$$

Hence, the Bethe eigenstates of the Hamiltonian operator (1.4) can be indexed by two integers $N, M \in \mathbb{Z}_+$: the state $|(N,M)\rangle \in \Phi$ satisfies the determining equation

$$\mathbf{H}\,|(N,M)\rangle = E\,|(N,M)\rangle\,, \tag{2.3}$$

where $E \in \mathbb{R}$ is the energy and

$$\begin{aligned} \mathbf{N}_{\mathrm{F}}\,|(N,M)\rangle &= N\,|(N,M)\rangle\,, \\ \mathbf{N}_{\mathrm{B}}\,|(N,M)\rangle &= M\,|(N,M)\rangle\,. \end{aligned} \tag{2.4}$$

Owing to (2.4), the state

$$\begin{aligned} |(N,M)\rangle \;=\; & \bigoplus_{n=0}^{N} \int_{\mathbb{R}^{N+M}} \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\mathrm{d}y_1\mathrm{d}y_2\ldots\mathrm{d}y_{N-n}\mathrm{d}z_1\mathrm{d}z_2\ldots\mathrm{d}z_{M-N+n} \\ & \times\varphi^{(n)}_{(M,N)}(x_1,x_2,\ldots,x_n;y_1,y_2,y_3,\ldots,y_{N-n};z_1,z_2,\ldots,z_{M-N+n}) \\ & \times\prod_{j=1}^{n}\psi_1^+(x_j)\prod_{k=1}^{N-n}\psi_2^+(y_k)\prod_{l=1}^{M-N+n}\varepsilon^+(z_l)\,|0\rangle \end{aligned} \tag{2.5}$$

for any $M, N \in \mathbb{Z}_+$, where for bounded states $\varphi^{(n)}_{(M,N)} \in L_2^{(\mathrm{as})}(\mathbb{R}^N;\mathbb{C}) \times L_2^{(\mathrm{s})}(\mathbb{R}^{M-N+n};\mathbb{C}), 0 \leqslant n \leqslant N$, one can easily construct [16, 21, 30] the Bethe state

$$|(1,1)\rangle = \int_{\mathbb{R}} \mathrm{d}y_1\varphi(y_1)\psi_2^+(y_1)\,|0\rangle + \int_{\mathbb{R}} \mathrm{d}x_1 \int_{\mathbb{R}} \mathrm{d}z_1\chi(x_1;z_1)\psi_1^+(x_1)\varepsilon^+(z_1)\,|0\rangle\,. \tag{2.6}$$

Here, the functions $\varphi : \mathbb{R} \to \mathbb{C}$ and $\chi : \mathbb{R}^2 \to \mathbb{C}$ satisfy the generalized differential equations

$$\begin{aligned} &\frac{1}{\mathrm{i}}\frac{\partial\varphi(x_1)}{\partial x_1} - \alpha\chi(x_1;x_1) = E\varphi(x_1), \\ -&\frac{1}{\mathrm{i}}\frac{\partial\chi(x_1;x_2)}{\partial x_1} + \frac{\beta}{\mathrm{i}}\frac{\partial\chi(x_1;x_2)}{\partial x_2} - \alpha\varphi(x_1)\delta(x_1-x_2) = E\chi(x_1;x_2), \end{aligned} \tag{2.7}$$

having solutions

$$
\begin{aligned}
\varphi(x_1) &= S_+(\lambda,\mu)\exp\left[\mathrm{i}\left(p_1+q_1\right)x\right],\\
\chi(x_1,x_2) &= \left[\vartheta(x_1-x_2)+S_1(\lambda,\mu)\vartheta(x_2-x_1)\right]\exp\left[\mathrm{i}\left(p_1x_1+q_1x_2\right)\right],
\end{aligned}
\tag{2.8}
$$

where the momenta $\lambda,\mu\in\mathbb{R}$ and

$$
\begin{gathered}
p_1=(\beta-1)\lambda, \qquad q_1=2\mu, \qquad E=p_1-\beta q_1,\\
S_1(\lambda,\mu)=\frac{\lambda-\mu-\mathrm{i}\alpha^2/[4(1-\beta^2)]}{\lambda-\mu+\mathrm{i}\alpha^2/[4(1-\beta^2)]}, \qquad S_+(\lambda,\mu)=\frac{\alpha}{2(1-\beta)\lambda-\mu+\mathrm{i}\alpha^2/[4(1-\beta^2)]}.
\end{gathered}
\tag{2.9}
$$

In the same way, one can represent the other quantum Bethe state as

$$
|(2,1)\rangle=\int\limits_{\mathbb{R}^2}\mathrm{d}x_1\mathrm{d}x_2\varphi(x_1,x_2)\psi_1^+(x_1)\psi_2^+(x_2)\,|0\rangle+\int\limits_{\mathbb{R}^3}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\chi(x_1,x_2;x_3)\psi_1^+(x_1)\varepsilon^+(x_2)\,|0\rangle, \tag{2.10}
$$

where the functions $\varphi:\mathbb{R}^2\to\mathbb{C}$ and $\chi:\mathbb{R}^3\to\mathbb{C}$ satisfy the generalized differential equations:

$$
\begin{aligned}
&-\frac{1}{\mathrm{i}}\left[\frac{\partial\varphi(x_1,x_2)}{\partial x_1}-\frac{\partial\varphi(x_1,x_2)}{\partial x_2}\right]+\alpha\chi(x_2,x_1;x_2)+\alpha\chi(x_1,x_2;x_1)=E\varphi(x_1,x_2),\\
&-\frac{1}{\mathrm{i}}\left[\frac{\partial\chi(x_1,x_2;x_3)}{\partial x_1}+\frac{\partial\chi(x_1,x_2;x_3)}{\partial x_2}\right]+\frac{\beta}{\mathrm{i}}\frac{\partial\chi(x_1,x_2;x_3)}{\partial x_3}\\
&+\frac{\alpha}{2}\varphi(x_1,x_2)\delta(x_2-x_3)+\frac{\alpha}{2}\varphi(x_2,x_1)\delta(x_1-x_3)=E\chi(x_1,x_2;x_1),
\end{aligned}
\tag{2.11}
$$

with solutions similar to those of (2.8) and (2.9).

It is important to mention here that the eigenstates (2.6) and (2.10) become degenerate as $\beta\to1$, meaning that the corresponding bound quantum soliton states cannot be formed. The same statement is also true for an arbitrary eigenstate (2.5). To demonstrate this, we shall in the next section make use of the quantum spectral problem (1.11) to prove that the quantum dynamical system (1.9) possesses an infinite hierarchy of commuting conservation laws, thereby ensuring its complete quantum integrability and the formation of quantum solitons.

## 3. The quantum solitons

We now consider the following quantum operator Cauchy problem for the spectral equation (1.10) subject to the periodic conditions $l(x+2\pi;\lambda)=l(x;\lambda)\in\operatorname{End}\Phi^3$ for all $x\in\mathbb{R}$ and $\lambda\in\mathbb{C}$:

$$
\frac{\mathrm{d}F(x,y;\lambda)}{\mathrm{d}x}=\vdots\, l(x;\lambda)F(x,y;\lambda)\vdots\,, \tag{3.1}
$$

where $F(x,y;\lambda)\in\operatorname{End}\Phi^3$ is the corresponding fundamental transition operator matrix satisfying

$$
F(x,y;\lambda)\big|_{y=x}=1, \tag{3.2}
$$

and the operation $\vdots\cdot\vdots$ arranges operators $\psi_j,\psi_j^+$, $j=1,2$, $\varepsilon$ and $\varepsilon^+$, via the standard normal ordering [11, 31] that does not change the position of any other operators; for instance, $\vdots A\psi_1^+\psi_2\varepsilon^+B\vdots=\psi_1^+\varepsilon^+AB\psi_2$ for any $A,B\in\operatorname{End}\Phi$.

Construct now the operator products

$$
\begin{aligned}
\hat{\mathscr{F}}(x,y|\lambda,\mu) &:= \tilde{F}(x,y;\lambda)\tilde{\tilde{F}}(x,y;\mu),\\
\check{\mathscr{F}}(x,y|\lambda,\mu) &:= \tilde{\tilde{F}}(x,y;\mu)\tilde{F}(x,y;\lambda),
\end{aligned}
\tag{3.3}
$$

where

$$\tilde{F}(x,y;\lambda) := F(x,y;\lambda)\otimes 1,$$
$$\approx{F}(x,y;\mu) := 1\otimes F(x,y;\mu) \tag{3.4}$$

are for all $x,y\in\mathbb{R}$ , $\lambda,\mu\in\mathbb{C}$, the corresponding tensor products of operators acting in the space $\Phi^3\otimes\Phi^3$. The following proposition is crucial [10, 17, 31] for further analysis of integrability of the quantum dynamical system (1.9) and is proved by a direct computation.

**Proposition 1.** *The operator expressions (3.3) satisfy the following differential relationships:*

$$\frac{\partial}{\partial x}\hat{\mathscr{F}}(x,y|\lambda,\mu) = \vdots\hat{\mathscr{L}}(x;\lambda,\mu)\hat{\mathscr{F}}(x,y|\lambda,\mu)\vdots,$$
$$\frac{\partial}{\partial x}\check{\mathscr{F}}(x,y|\lambda,\mu) = \vdots\breve{\mathscr{L}}(x;\lambda,\mu)\check{\mathscr{F}}(x,y|\lambda,\mu)\vdots, \tag{3.5}$$

where the matrices

$$\hat{\mathscr{L}}(x;\lambda,\mu) = \tilde{l}(x;\lambda) + \approx{l}(x;\mu) - \alpha\hat{\triangle}(x;\lambda,\mu),$$
$$\breve{\mathscr{L}}(x;\lambda,\mu) = \tilde{l}(x;\lambda) + \approx{l}(x;\mu) - \alpha\check{\triangle}(x;\lambda,\mu), \tag{3.6}$$

and $\hat{\triangle}(x;\lambda,\mu), \check{\triangle}(x;\lambda,\mu)$ satisfy the algebraic relationship $P\hat{\triangle}(x;\lambda,\mu)P = \check{\triangle}(x;\lambda,\mu)$ for all $x\in R$, $\lambda,\mu\in C$, where $P\in \mathrm{End}\Phi^3\otimes\Phi^3$ is the standard transmutation operator in the space $\Phi^3\otimes\Phi^3$, that is $P(a\otimes b) := b\otimes a$ for any vectors $a,b\in\Phi^3$.

Using proposition 1, one can easily verify that there exists a scalar $\mathscr{R}$-matrix $\mathscr{R}(\lambda,\mu)$, $\mathscr{R}\in\mathrm{End}\mathbb{C}^9$, such that

$$\mathscr{R}(\lambda,\mu)\hat{\mathscr{L}}(x;\lambda,\mu) = \breve{\mathscr{L}}(x;\lambda,\mu)\mathscr{R}(\lambda,\mu) \tag{3.7}$$

holds for all $\lambda,\mu\in\mathbb{C}$ and $x\in\mathbb{R}$. This, owing to the equations (3.5), implies the main functional Yang-Baxter type [14, 17, 31] operator relationship

$$\mathscr{R}(\lambda,\mu)\hat{\mathscr{F}}(x,y|\lambda,\mu) = \check{\mathscr{F}}(x,y|\lambda,\mu)\mathscr{R}(\lambda,\mu) \tag{3.8}$$

is satisfied for any $x,y\in\mathbb{R}$ and $\lambda,\mu\in\mathbb{C}$, where, as a result of [22],

$$\mathscr{R}(\lambda,\mu) = (\lambda-\mu)P - \mathrm{i}\alpha\mathbf{1}. \tag{3.9}$$

Recalling now the periodicity condition, from (3.8), one easily deduces by means of the trace-operation that the monodromy operator matrix $T(x;\lambda) := F(x+2\pi,x;\lambda)$ satisfies the following commutation relationship for all $x\in\mathbb{R}$ and $\lambda,\mu\in\mathbb{C}$ :

$$\left[\mathrm{tr}T(x;\lambda), \mathrm{tr}T(x;\mu)\right] = 0. \tag{3.10}$$

Actually, it follows from (3.8) that

$$\mathrm{tr}\left\{T(x;\lambda)\otimes T(x;\mu)\right\} = \mathrm{tr}\left\{\mathscr{R}^{-1}T(x;\mu)\otimes T(x;\lambda)\right\} = \mathrm{tr}\left\{T(x;\mu)\otimes T(x;\lambda)\right\}. \tag{3.11}$$

Taking into account that $\mathrm{tr}(A\otimes B) = \mathrm{tr}A\cdot\mathrm{tr}B$ for any operators $A,B\in\mathrm{End}\Phi^3$, one easily obtains (3.10) from (3.11). Consequently, the $\lambda$-dependent operator functional

$$\gamma(\lambda) := \mathrm{tr}T(x,\lambda) \cong \Sigma_{j\in\mathbb{Z}_+}\gamma_j\lambda^{-j}, \tag{3.12}$$

as $|\lambda|\to\infty$ generates an infinite hierarchy of commuting conservation laws $\gamma_j:\Phi\to\Phi$, $j\in\mathbb{Z}_+$ :

$$\left[\gamma_j,\gamma_k\right] = 0 \tag{3.13}$$

for all $j,k \in \mathbb{Z}_+$, where, in particular,

$$\gamma_1 = \mathbf{N}_\mathrm{F} = \int_{\mathbb{R}} \mathrm{d}x(\psi_1^+\psi_1 + \psi_2^+\psi_2), \qquad \gamma_2 = \mathbf{N}_\mathrm{B} = \int_{\mathbb{R}} \mathrm{d}x(\varepsilon^+\varepsilon + \psi_2^+\psi_2),$$
$$\gamma_3 = \mathbf{P} = \mathrm{i}\int_{\mathbb{R}} \mathrm{d}x(\psi_1^+\psi_{1,x} + \psi_2^+\psi_{2,x} + \varepsilon^+\varepsilon_x),$$
$$\gamma_4 = \mathbf{H} = \mathrm{i}\int_{\mathbb{R}} \mathrm{d}x\left[\psi_1^+\psi_{1,x} - \psi_2^+\psi_{2,x} - \varepsilon^+\varepsilon_x + \mathrm{i}\alpha(\varepsilon\psi_2^+\psi_1 + \psi_1^+\psi_2\varepsilon^+)\right]. \tag{3.14}$$

Since the operator functional $\gamma_4 = \mathbf{H}$ is the Hamiltonian operator for the dynamical system (1.9), from (3.13) one obtains

$$\left[\mathbf{H}, \gamma_j\right] = 0 \tag{3.15}$$

for all $j \in \mathbb{Z}_+$; that is, all of functionals $\gamma_j : \Phi \to \Phi$, $j \in \mathbb{Z}_+$, are conservation laws.

Moreover, making use of the exact operator relationships (3.8) one can construct the physically stable quantum states $|(N,M)\rangle \in \Phi$ for all $N,M \in \mathbb{Z}_+$ upon redefining the Fock vacuum $|0\rangle \in \Phi$, which is non-physical for the dynamical system (1.9), governed by the unbounded from below Hamiltonian operator (1.4). Following a renormalization scheme similar to those developed in [14, 17, 32], one can construct a new physically stable vacuum

$$|0\rangle_{\mathrm{phys}} := \prod_{q\leqslant\mu_j\leqslant Q} B^+(\mu_j)\ |0\rangle \tag{3.16}$$

by means of the new, commuting to each other, "creation" operators $B^+(\mu) : \Phi \to \Phi, \mu \in \mathbb{C}$, generated by suitable components of the monodromy operator matrix $T(x;\mu) : \Phi^3 \to \Phi^3$, $x \in \mathbb{R}$, whose commutation relationships with the Hamiltonian operator (1.4)

$$[\mathbf{H}, B^+(\mu)] = S(\mu;\alpha,\beta)B^+(\mu) \tag{3.17}$$

are parameterized by the two-particle scalar scattering factor $S(\mu;\alpha,\beta), \mu \in \mathbb{C}$, and where the values $q < Q \in \mathbb{R}$ are to be determined from the condition that quantum excitations over the physical vacuum (3.16) have positive energy. Since the physical vacuum (3.16) is an eigenstate of the Hamiltonian operator (1.4), the corresponding quantum eigenstates of the excitations can be represented as

$$\left|(\mu)\right\rangle := B^+(\mu)\ |0\rangle_{\mathrm{phys}} \tag{3.18}$$

for some $\mu \in \mathbb{R}$ and the new energy level can be taken into account in the renormalized Hamiltonian operator (1.4) by means of the chemical potentials $a_\mathrm{F}, a_\mathrm{B} \in \mathbb{R}$:

$$\mathbf{H}_a := \mathbf{H} - a_\mathrm{F}\mathbf{N}_\mathrm{F} - a_\mathrm{B}\mathbf{N}_\mathrm{B}\,, \tag{3.19}$$

which should be determined from the conditions

$$\mathbf{H}_a\ |0\rangle_{\mathrm{phys}} = 0, \qquad \left\langle(\mu)\ |\mathbf{H}_a\ |(\mu)\right\rangle > 0 \tag{3.20}$$

for any $\mu \in \mathbb{R}$. The physical vacuum state and quantum Hamiltonian renormalization construction described above make it possible to study the properties of superradiance quantum photonic impulse structures generated by interaction of the charged fermionic medium with an external electromagnetic field. Owing to the existence of quantum periodic eigenstates over the physically stable vacuum, one can also investigate the related thermodynamic properties of the model and analyze the generated superradiance photonic structures, which are important for explaining many [1] existing experiments.

## 4. Conclusion

The exact solvability of our model describing a one-dimensional many-particle charged fermionic medium interacting with an external electromagnetic field allows one to calculate diverse superradiance effects, which are closely related to the formation of the bound quantum solitonic states and their stability. The existence of the bound states is established by suitably applying the physical vacuum renormalization subject to which all quantum excitations are of positive energy. This procedure, based on the determining operator relationships (3.8), (3.16) and (3.17) enables one to describe the thermodynamic properties of the quantum dynamical system over a stable physical vacuum. In addition, it facilitates the analysis of the corresponding thermodynamic states of the resulting quantum photonic system and its superradiance properties. Our results indicate that a more detailed investigation of these and related topics is in order, which we plan to undertake elsewhere.

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# Нова точно розв'язувана просторово-одновимірна квантова модель супервипромінювального фермі-середовища та квантові солітонні стани

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Запропоновано нову точно розв'язувану просторово-одновимірну квантову супервипромінювальну модель, що описує заряджене ферміонне середовище, взаємодіюче із зовнішнім електромагнітним полем. Сконструйовано скінченну ієрархію законів збереження та багаточастинкові власні стани Бете, що моделюють квантові солітонні імпульсні структури. Описано процедуру ренормалізації оператора Гамільтона щодо стійкого фізичного вакуума, обговорюються квантові збудження та квантові солітони, асоційовані із термодинамічною рівновагою моделі.

**Ключові слова:** *заряджене ферміонне середовище, квантова обернена спектральна проблема, власні стани Бете, ренормалізація вакуума, квантова супервипрмінювальна модель, закони збереження, тотожність Янга-Бакстера*