Condensed Matter Physics, 2012, Vol. 15, No 4, 43004: 1–9
DOI: 10.5488/CMP.15.43004
http://www.icmp.lviv.ua/journal

Entropy production in open quantum systems: exactly solvable qubit models

V.G. Morozov[1], G. Röpke[2]

[1] Moscow State Technical University of Radioengineering, Electronics, and Automation, Vernadsky Prospect 78, 119454 Moscow, Russia

[2] University of Rostock, FB Physik, Universitätsplatz 3, D–18051 Rostock, Germany

Received July 3, 2012, in final form August 23, 2012

We present analytical results for the time-dependent information entropy in exactly solvable two-state (qubit) models. The first model describes dephasing (decoherence) in a qubit coupled to a bath of harmonic oscillators. The entropy production for this model in the regimes of "complete" and "incomplete" decoherence is discussed. As another example, we consider the damped Jaynes-Cummings model describing a spontaneous decay of a two-level system into the field vacuum. It is shown that, for all strengths of coupling, the open system passes through the mixed state with the maximum information entropy.

Key words: *information entropy, open quantum systems, qubit models, decoherence, quantum entanglement*

PACS: *03.65.Ud, 03.65.Yz*

1. Introduction

The notion of information entropy has attracted a renewed interest over the last few decades in connection with fundamental problems in the theory of open quantum systems and quantum entanglement (see, e.g., [1–4] and references therein). Although the literature concerning diverse aspects of this topic is now quite voluminous, not much is known about the entropy behavior in concrete open systems exhibiting especially intriguing features of quantum dynamics: memory effects, dephasing (decoherence), entanglement, etc.

The simplest models describing many fundamental dynamic properties of open quantum systems are two-state systems. Such systems themselves deserve thorough studies as the elementary carriers of quantum information (qubits) [5, 6]. It is also important to note that some of two-state models admit *exact* solutions. The latter fact allows one to gain a valuable insight into general properties of open quantum systems. From this point of view, it is of interest to analyze the time behavior of information entropy in exactly solvable models.

In this paper we present exact analytic results for the time-dependent information entropy in two physically reasonable qubit models which are frequently used in discussing different problems in the theory of open quantum systems.

2. Entropy of a qubit

We consider a two-state quantum system (qubit) coupled to a reservoir. In what follows, the qubit and the reservoir will be referred to as subsystems A and B, respectively.

Suppose that at time t the state of the combined system (qubit plus reservoir) is described by some density matrix $\varrho_{AB}(t)$. Then, the reduced density matrix of the qubit is defined as

$$\varrho_A(t) = \mathrm{Tr}_B \varrho_{AB}(t), \tag{1}$$

where Tr_B denotes the trace over the reservoir degrees of freedom. The von Neumann (information) entropy of the qubit is given by

$$S_A(t) = -\mathrm{Tr}_A \left[\varrho_A(t) \ln \varrho_A(t)\right]. \tag{2}$$

It is convenient to use the “spin” representation for a qubit by writing its orthonormal basis $|0\rangle$ and $|1\rangle$ as

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3}$$

Then, all operators referring to a qubit can be expressed in terms of the Pauli matrices $\vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$. In particular, the density matrix (1) can be written in the form [7, 8]

$$\varrho_A(t) = \frac{1}{2}\left[1 + \vec{\sigma} \cdot \vec{v}(t)\right], \tag{4}$$

where

$$\vec{v}(t) = \mathrm{Tr}_A \left[\vec{\sigma}\, \varrho_A(t)\right] \tag{5}$$

is the so-called Bloch vector. In [9] we have shown that there exists another representation for the qubit density matrix, which is better suited to calculate $\ln \varrho_A(t)$ in (2):

$$\varrho_A(t) = \frac{1}{2}\sqrt{1 - v^2(t)}\, \exp\left[\vec{\sigma} \cdot \vec{u}(t)\right], \qquad u = \frac{1}{2} \ln\left(\frac{1+v}{1-v}\right). \tag{6}$$

Here, $v(t)$ is the modulus of the Bloch vector and $\vec{u} \upuparrows \vec{v}$. Strictly speaking, the representation (6) is valid only for a mixed state of a qubit with $v < 1$. Note, however, that the limit $v \to 1$ can be taken directly in the entropy (2) after calculating the trace. Using expressions (6), one easily derives from (2)

$$S_A(t) = \ln 2 - \frac{1}{2}(1+v)\ln(1+v) - \frac{1}{2}(1-v)\ln(1-v). \tag{7}$$

For a pure state ($v \to 1$), we have $S_A = 0$, as it should be. The entropy has its maximum $S_A = \ln 2$ in the mixed state with $v = 0$, when the density matrix (4) is diagonal and $\langle 0|\varrho_A|0\rangle = \langle 1|\varrho_A|1\rangle = 1/2$.

The square modulus of the Bloch vector (5) can in general be written as

$$v^2(t) = 4\, v_+(t)\, v_-(t) + v_3^2(t), \tag{8}$$

where

$$v_\pm(t) = \mathrm{Tr}_A[\sigma_\pm \varrho_A(t)], \tag{9}$$

and $\sigma_\pm = \left(\sigma_1 \pm i\sigma_2\right)/2$. The two terms in (8) have different physical interpretations. The quantities $v_+(t) = \langle 0|\varrho_A(t)|1\rangle$ and $v_-(t) = \langle 1|\varrho_A(t)|0\rangle$ are often referred to as the *coherences*. They describe the environmentally induced dephasing [1]. On the other hand, the time behavior of the component of the Bloch vector $v_3 = \langle 1|\varrho_A(t)|1\rangle - \langle 0|\varrho_A(t)|0\rangle$ is determined by energy exchange between an open system and its environment, which is responsible for complete statistical equilibrium in the combined system. Thus, formulas (7) and (8) are convenient for studying the role of different relaxation mechanisms in the entropy production.

3. Entropy production in a dephasing model

We start with a simple spin-boson model describing a qubit coupled to a reservoir of harmonic oscillators [1, 10–12]. The Hamiltonian of the model is (in our units $\hbar = 1$)

$$H = H_A + H_B + H_I = \frac{\omega_0}{2}\sigma_3 + \sum_k \omega_k b_k^\dagger b_k + \sigma_3 \sum_k \left(g_k b_k^\dagger + g_k^* b_k\right), \tag{10}$$

where ω_0 is the energy difference between the excited state $|1\rangle$ and the ground state $|0\rangle$ of the qubit. Bosonic operators $b_k^\dagger$ and b_k correspond to the kth reservoir mode with frequency ω_k.

Note that σ_3 commutes with the Hamiltonian (10). As a consequence, the populations $\langle 0|\varrho_A(t)|0\rangle$ and $\langle 1|\varrho_A(t)|1\rangle$ do not depend on time. In other words, there is no relaxation to a complete equilibrium between the qubit and the environment; that is, the model is *nonergodic*. However, we shall see below that this model exhibits a dephasing relaxation and entropy production without energy exchange between the qubit and the environment.

Let us first specify the initial density matrix $\varrho_{AB}(0)$ of the combined system. Usually (see, e.g., [1, 10–12]) it is assumed that the subsystems A and B are uncorrelated, and the reservoir is in thermal equilibrium at some temperature T. In this paper we will concentrate on the entropy production in time-dependent *entangled* quantum states of the combined system. We assume that at time $t = 0$, the combined system is prepared in a *pure* quantum state which is a direct product

$$|\psi_{AB}(0)\rangle = \big(a_0|0\rangle + a_1|1\rangle\big) \otimes |0_B\rangle, \tag{11}$$

where $|a_0|^2 + |a_1|^2 = 1$, and $|0_B\rangle$ denotes the ground state of the reservoir. The initial state $|0_B\rangle$ is chosen only for simplicity's sake. The subsequent discussion may easily be extended to the case of an arbitrary initial state $|\psi_B(0)\rangle$ of the reservoir. The density matrix corresponding to (11) is

$$\varrho_{AB}(0) = |\psi_{AB}(0)\rangle\langle\psi_{AB}(0)|. \tag{12}$$

Since the evolution of the combined system is unitary, the initial state (11) evolves after time t into the *pure* state

$$|\psi_{AB}(t)\rangle = \exp(-\mathrm{i}Ht)\,|\psi_{AB}(0)\rangle, \tag{13}$$

so that the density matrix of the combined system is given by

$$\varrho_{AB}(t) = |\psi_{AB}(t)\rangle\langle\psi_{AB}(t)|. \tag{14}$$

In principle, the qubit density matrix $\varrho_A(t)$, and then the entropy $S_A(t)$ can be calculated by using (14). It is more convenient, however, to calculate the modulus of the Bloch vector, $v(t)$, and then apply formula (7). Since v_3 is constant and is determined by the amplitudes a_0 and a_1 in (11), we need only to consider the coherences $v_\pm(t)$. Note that expression (9) can be rewritten as

$$v_\pm(t) = \mathrm{Tr}_{AB}\left[\sigma_\pm(t)\varrho_{AB}(0)\right], \tag{15}$$

where $\sigma_\pm(t)$ are the Heisenberg picture operators and the trace is taken over all degrees of freedom of the combined system. In the model (10), equations of motion for $\sigma_\pm(t)$ can be solved exactly. The result reads [9]

$$\sigma_\pm(t) = \exp\left[\pm\mathrm{i}\omega_0 t \mp R(t)\right]\sigma_\pm, \tag{16}$$

where the operator $R(t)$ acts only on the reservoir states and is given by

$$R(t) = \sum_k \left[\alpha_k(t) b_k^\dagger - \alpha_k^*(t) b_k\right], \qquad \alpha_k(t) = 2g_k \frac{1-\mathrm{e}^{\mathrm{i}\omega_k t}}{\omega_k}. \tag{17}$$

Substituting the expression (16) into (15) and recalling formula (12), we find

$$v_\pm(t) = v_\pm(0)\exp\left[\pm\mathrm{i}\omega_0 t - \gamma_{\mathrm{vac}}(t)\right] \tag{18}$$

with the so-called *vacuum decoherence function* [1]

$$\gamma_{\mathrm{vac}}(t) = -\ln\langle 0_B|\exp[R(t)]|0_B\rangle = -\sum_k \ln\langle 0_B|\exp\left[\alpha_k(t) b_k^\dagger - \alpha_k^*(t) b_k\right]|0_B\rangle. \tag{19}$$

After simple algebra which we omit, we obtain

$$\gamma_{\mathrm{vac}}(t) = \int_0^\infty \mathrm{d}\omega\, J(\omega)\,\frac{1-\cos\omega t}{\omega^2}, \tag{20}$$

where the continuum limit of the reservoir modes is performed, and the spectral density $J(\omega)$ is introduced by the rule

$$\sum_k 4|g_k|^2 f(\omega_k) = \int_0^\infty \mathrm{d}\omega\, J(\omega) f(\omega). \tag{21}$$

Now using the solution (18) and taking into account that $v(0) = 1$, we find from equation (8)

$$v^2(t) = v_3^2 + \left(1 - v_3^2\right) \exp\left[-2\gamma_{\text{vac}}(t)\right]. \tag{22}$$

Formulas (7) and (22) determine the time evolution of the qubit entropy. To go beyond these formal relations, one needs some information on the spectral density $J(\omega)$. In many cases of physical interest (see, e.g., [12, 13]), $J(\omega)$ may be considered to be a reasonably smooth function which has a power-law behavior $J(\omega) \propto \omega^s$ ($s > 0$) at frequencies much less than some "cutoff" frequency Ω, characteristic of the reservoir modes. In the limit $\omega \to \infty$, $J(\omega)$ is assumed to fall off at least as some negative power of ω. For the spectral density, we shall take the expression which is most commonly used in the theory of spin-boson systems [10–13]:

$$J(\omega) = \lambda_s \Omega^{1-s} \omega^s \mathrm{e}^{-\omega/\Omega}, \tag{23}$$

where λ_s is a dimensionless coupling constant. The case $s = 1$ is usually called the "Ohmic" case, the case $s > 1$ is "super-Ohmic", and the case $0 < s < 1$ is "sub-Ohmic".

Substituting the spectral density (23) into (20) and doing standard integrals, one gets

$$\begin{aligned} \gamma_{\text{vac}}(t) &= \lambda_s \Gamma(s-1) \left\{ 1 - \frac{\cos\left[(s-1)\arctan(\Omega t)\right]}{\left(1+\Omega^2 t^2\right)^{(s-1)/2}} \right\}, \qquad (s \neq 1), \\ \gamma_{\text{vac}}(t) &= \frac{\lambda_1}{2} \ln\left(1 + \Omega^2 t^2\right), \qquad (s = 1), \end{aligned} \tag{24}$$

where $\Gamma(s)$ is the Euler gamma function. Some important properties of $\gamma_{\text{vac}}(t)$ can easily be seen directly from the above expressions. First, $\gamma_{\text{vac}}(t)$ is a monotonously increasing function of time for $s \leqslant 1$. Second, in the super-Ohmic case $\gamma_{\text{vac}}(t)$ has a long-time limit:

$$\gamma_{\text{vac}}(\infty) \equiv \lim_{t\to\infty} \gamma_{\text{vac}}(t) = \lambda_s \Gamma(s-1), \qquad (s > 1). \tag{25}$$

Finally, the $\gamma_{\text{vac}}(t)$ monotonously saturates to $\gamma_{\text{vac}}(\infty)$ for $1 < s \leqslant 2$ and is a nonmonotonous function of time for $s > 2$.

In discussing the properties of the qubit entropy in the model (10), it is necessary to distinguish two cases: the regime of "complete decoherence", and the regime of "incomplete decoherence". In the former case ($s \leqslant 1$) we have $\gamma_{\text{vac}}(t) \to \infty$ as $t \to \infty$, and hence $v_\pm(t) \to 0$. Then, it follows directly from (7) and (22) that the limiting value of the qubit entropy is given by

$$S_A(\infty) = \ln 2 - \frac{1}{2}\left(1 + |v_3|\right)\ln\left(1 + |v_3|\right) - \frac{1}{2}\left(1 - |v_3|\right)\ln\left(1 - |v_3|\right), \qquad (s \leqslant 1). \tag{26}$$

The maximum qubit entropy $S_{\max}(\infty) = \ln 2$ corresponds to the initial state with equal populations ($v_3 = 0$). In the case of "incomplete decoherence", we have

$$S_A(\infty) = \ln 2 - \frac{1}{2}\left(1 + v_\infty\right)\ln\left(1 + v_\infty\right) - \frac{1}{2}\left(1 - v_\infty\right)\ln\left(1 - v_\infty\right), \qquad (s > 1), \tag{27}$$

where

$$v_\infty = \left\{ v_3^2 + \left(1 - v_3^2\right)\exp\left[-2\gamma_{\text{vac}}(\infty)\right] \right\}^{1/2}. \tag{28}$$

Figure 1 illustrates the time behavior of the qubit entropy for different values of the parameter s and for different coupling strengths. It should be emphasized at once that, for the case under consideration here, the entropy production has no "thermodynamic" meaning. Indeed, at any time t, the combined system is in the *pure* quantum state (13). Note, however, that this state is *entangled* [1], i.e., it cannot be written as a direct product of states of the subsystems. Thus, the information entropy $S_A(t)$ may be regarded as a measure of entanglement.

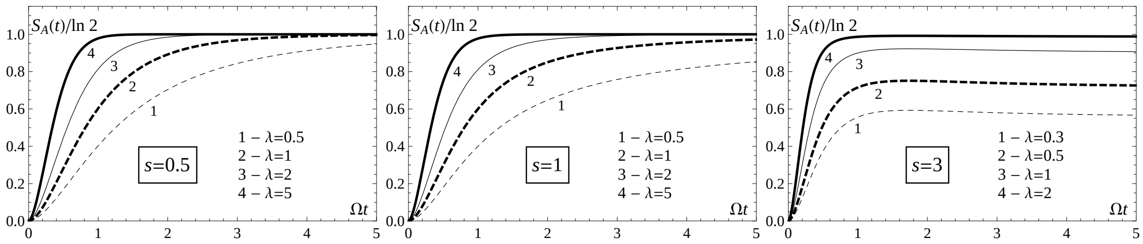

Figure 1. Time evolution of the qubit entropy in the dephasing model (10). In all cases $\lambda \equiv \lambda_s$. The qubit is initially in the pure state with equal populations ($v_3 = 0$).

Within the framework of the model (10) with the initial condition (11), the structure of the entangled state (13) can easily be established. Using the fact that the Hamiltonian (10) contains only the qubit operator σ_3, we obtain

$$|\psi_{AB}(t)\rangle = a_0 \mathrm{e}^{-\mathrm{i}\omega_0 t/2} |\psi_B^{(-)}(t)\rangle \otimes |0\rangle + a_1 \mathrm{e}^{\mathrm{i}\omega_0 t/2} |\psi_B^{(+)}(t)\rangle \otimes |1\rangle \tag{29}$$

with

$$|\psi_B^{(\pm)}(t)\rangle = \exp\big(-\mathrm{i}H_B^{(\pm)}t\big)|0_B\rangle, \qquad H_B^{(\pm)} = \sum_k \omega_k b_k^\dagger b_k \pm \sum_k \Big(g_k b_k^\dagger + g_k^* b_k\Big). \tag{30}$$

The reservoir states $|\psi_B^{(\pm)}(t)\rangle$ are normalized but $\langle\psi_B^{(+)}(t)|\psi_B^{(-)}(t)\rangle \neq 0$. It is possible, however, to write the states appearing in (29) as linear combinations (the argument t is omitted for brevity)

$$|\psi_B^{(\pm)}\rangle = \beta_0^{(\pm)}|\Phi_0\rangle + \beta_1^{(\pm)}|\Phi_1\rangle, \qquad |0\rangle = \alpha_0|\phi_0\rangle + \alpha_1|\phi_1\rangle, \qquad |1\rangle = \alpha_0'|\phi_0\rangle + \alpha_1'|\phi_1\rangle, \tag{31}$$

where the amplitudes may be determined from the following conditions: a) the pairs of states ($|\Phi_0\rangle$,$|\Phi_1\rangle$) and ($|\phi_0\rangle$,$|\phi_1\rangle$) are orthonormal; b) the terms with $|\Phi_0\rangle\otimes|\phi_1\rangle$ and $|\Phi_1\rangle\otimes|\phi_0\rangle$ cancel when expressions (31) are inserted into (29). Then, the state of the combined system takes the form

$$|\psi_{AB}(t)\rangle = A_0(t)|\Phi_0(t)\rangle \otimes |\phi_0(t)\rangle + A_1(t)|\Phi_1(t)\rangle \otimes |\phi_1(t)\rangle \tag{32}$$

with amplitudes satisfying $|A_0(t)|^2 + |A_1(t)|^2 = 1$. Explicit expressions for $A_0(t)$ and $A_1(t)$ are rather cumbersome and are not given here.

Formula (32) is an example of the so-called *Schmidt decomposition* of quantum states of a combined system [1]. Amplitudes $A_0(t)$ and $A_1(t)$ play a role of the corresponding *Schmidt coefficients*. From (32) it follows that the reduced density matrices of the subsystems may be written as

$$\varrho_A(t) = |A_0|^2|\phi_0\rangle\langle\phi_0| + |A_1|^2|\phi_1\rangle\langle\phi_1|, \qquad \varrho_B(t) = |A_0|^2|\Phi_0\rangle\langle\Phi_0| + |A_1|^2|\Phi_1\rangle\langle\Phi_1|. \tag{33}$$

Using these formulas, it is easy to show that

$$S_A(t) = S_B(t) = -|A_0(t)|^2 \ln|A_0(t)|^2 - |A_1(t)|^2 \ln|A_1(t)|^2, \tag{34}$$

which is the well-known consequence of the Schmidt decomposition theorem for entangled quantum states [1].

4. Spontaneous decay of a qubit

Now we consider the entropy production in an exactly solvable model which describes a spontaneous decay of a two-level system into a field vacuum [1, 14, 15]. The total Hamiltonian of the model is

$$H = H_A + H_B + H_I = \frac{\omega_0}{2}\sigma_3 + \sum_k \omega_k b_k^\dagger b_k + \sum_k \Big(g_k b_k^\dagger \sigma_- + g_k^* b_k \sigma_+\Big), \tag{35}$$

where the index k labels the photon modes with frequencies ω_k.

The initial state of the combined system is again assumed to be given by (11), i.e., $v(0) = 1$ and, consequently, $S_A(0) = 0$. To obtain the entropy of the qubit at times $t > 0$ we need to calculate $v(t)$. To do this, we will closely follow the approach taken in the works [14, 15].

It is convenient to work in the interaction picture with the unperturbed Hamiltonian $H_0 = H_A + H_B$. Introducing the interaction picture state vector $|\widetilde{\psi}_{AB}(t)\rangle = \exp\left(\mathrm{i}H_0 t\right)|\psi_{AB}(t)\rangle$, and applying the standard procedure, one obtains

$$|\widetilde{\psi}_{AB}(t)\rangle = \exp_+\left[-\mathrm{i}\int_0^t \mathrm{d}t'\, H_I(t')\right]|\psi_{AB}(0)\rangle, \tag{36}$$

where $\exp_+[\cdots]$ is the chronologically ordered exponent, and

$$H_I(t) = \sum_k \left(g_k \mathrm{e}^{-\mathrm{i}(\omega_0-\omega_k)t} b_k^\dagger \sigma_- + g_k^* \mathrm{e}^{\mathrm{i}(\omega_0-\omega_k)t} b_k \sigma_+\right) \tag{37}$$

is the interaction picture Hamiltonian. As noted in the work [14] (see also [1, 15]), there exists a simple representation for the state (36), which can be derived using the following properties of the Hamiltonian (37):

$$\begin{gathered} H_I(t)\,|0\rangle\otimes|0_B\rangle = 0, \qquad H_I(t)\,|1\rangle\otimes|0_B\rangle = \sum_k g_k \mathrm{e}^{-\mathrm{i}(\omega_0-\omega_k)t}\,|0\rangle\otimes b_k^\dagger|0_B\rangle, \\ H_I(t)\,|0\rangle\otimes b_k^\dagger|0_B\rangle = g_k^* \mathrm{e}^{\mathrm{i}(\omega_0-\omega_k)t}\,|1\rangle\otimes|0_B\rangle. \end{gathered} \tag{38}$$

Recalling the expression (11), one can easily show that at any time t, the state (36) is a superposition of $|0\rangle\otimes|0_B\rangle$, $|1\rangle\otimes|0_B\rangle$, and $|0\rangle\otimes b_k^\dagger|0_B\rangle$:

$$|\widetilde{\psi}_{AB}(t)\rangle = \left[a_0|0\rangle + c_1(t)|1\rangle\right]\otimes|0_B\rangle + \sum_k c_k(t)|0\rangle\otimes b_k^\dagger|0_B\rangle. \tag{39}$$

The amplitudes $c_1(t)$ and $c_k(t)$ satisfy the initial conditions $c_1(0) = a_1$, $c_k(0) = 0$, and the normalization condition

$$|a_0|^2 + |c_1(t)|^2 + \sum_k |c_k(t)|^2 = 1. \tag{40}$$

The qubit density matrix in the interaction picture can now be calculated using (39):

$$\widetilde{\varrho}_A(t) = \mathrm{Tr}_B\left\{|\widetilde{\psi}_{AB}(t)\rangle\langle\widetilde{\psi}_{AB}(t)|\right\} = \frac{1}{2} + \frac{1}{2}\left(2|c_1(t)|^2 - 1\right)\sigma_3 + a_0^* c_1(t)\sigma_+ + a_0 c_1^*(t)\sigma_-, \tag{41}$$

where the amplitudes $c_k(t)$ have been eliminated with the help of (40). Expressions for $v_\pm(t)$ and $v_3(t)$ follow immediately from (5) and the relation $\varrho_A(t) = \exp(-\mathrm{i}H_A t)\widetilde{\varrho}_A(t)\exp(\mathrm{i}H_A t)$:

$$v_+(t) = \mathrm{e}^{\mathrm{i}\omega_0 t} a_0 c_1^*(t), \qquad v_-(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t} a_0^* c_1(t), \qquad v_3(t) = 2|c_1(t)|^2 - 1. \tag{42}$$

Substituting these expressions into (8) and using $|a_0|^2 = 1 - |c_1(0)|^2$, we finally obtain

$$v^2(t) = 1 - 4|c_1(t)|^2\left[|c_1(0)|^2 - |c_1(t)|^2\right]. \tag{43}$$

Thus, the modulus of the Bloch vector and, consequently, the qubit entropy $S_A(t)$ are completely determined by $|c_1(t)|^2$ which is the time-dependent population of the excited state $|1\rangle$. It is important to note that the probability amplitude $c_1(t)$ obeys a closed equation [1, 15]

$$\dot{c}_1(t) = -\int_0^t \mathrm{d}t'\, f(t-t')c_1(t') \tag{44}$$

with the kernel

$$f(t) = \sum_k |g_k|^2\,\mathrm{e}^{\mathrm{i}(\omega_0-\omega_k)t} \equiv \int \mathrm{d}\omega\, J(\omega)\,\mathrm{e}^{\mathrm{i}(\omega_0-\omega)t}, \tag{45}$$

where $J(\omega)$ is the spectral density of the field. In some important special cases of $J(\omega)$, equation (44) can be solved to give exact analytic solutions for $c_1(t)$. We shall restrict ourselves to the so-called damped Jaynes-Cummings model (see, e.g., [1, 15]) which describes the resonant coupling of a two-level atom to a single cavity mode of the field. In this model, the effective spectral density has the form

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{\left(\omega_0 - \omega\right)^2 + \lambda^2}, \tag{46}$$

where λ is a spectral width of the coupling, and the parameter γ_0 defines the characteristic time scale $\tau_A = 1/\gamma_0$ on which the state of the qubit changes. For the details of solving the equation (44) we refer to [1, 15]. Here, we quote the resulting expression for the population $|c_1(t)|^2$:

$$|c_1(t)|^2 = |c_1(0)|^2 \mathrm{e}^{-\lambda t} \left[\cosh(\Lambda t/2) + \frac{\lambda}{\Lambda} \sinh(\Lambda t/2)\right]^2, \tag{47}$$

where $\Lambda = \left(\lambda^2 - 2\gamma_0\lambda\right)^{1/2}$. Substituting the above expression into (43) and introducing the dimensionless coupling parameter

$$K = 2\gamma_0/\lambda, \tag{48}$$

we obtain

$$v^2(t) = 1 - 4|c_1(0)|^4 \, \mathrm{e}^{-\lambda t} F_K(t) \left[1 - \mathrm{e}^{-\lambda t} F_K(t)\right]. \tag{49}$$

The form of the function $F_K(t)$ depends on the value of K:

$$\begin{aligned}
F_K(t) &= \left[\cosh\left(\sqrt{1-K}\,\lambda t/2\right) + \frac{1}{\sqrt{1-K}} \sinh\left(\sqrt{1-K}\,\lambda t/2\right)\right]^2, && K < 1,\\
F_K(t) &= \left[\cos\left(\sqrt{K-1}\,\lambda t/2\right) + \frac{1}{\sqrt{K-1}} \sin\left(\sqrt{K-1}\,\lambda t/2\right)\right]^2, && K > 1,\\
F_K(t) &= (1 + \lambda t/2)^2, && K = 1.
\end{aligned} \tag{50}$$

Formulas (7) and (49) completely determine the qubit entropy in the Jaynes-Cummings model. Figure 2

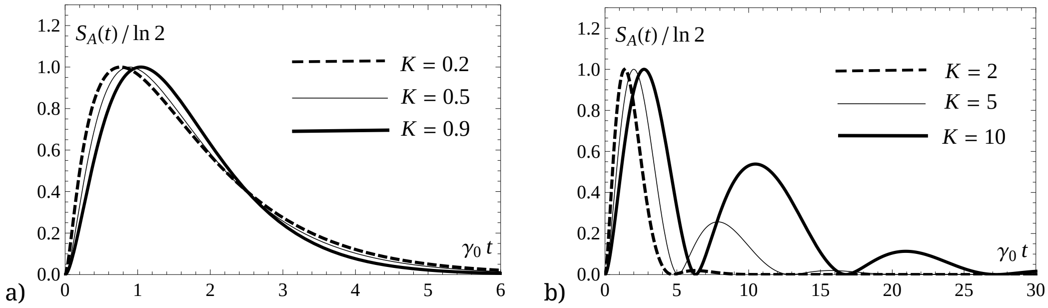

Figure 2. Time evolution of the qubit entropy in the damped Jaynes-Cummings model: a) Weak and moderate coupling ($K < 1$); b) Strong coupling ($K > 1$). In all cases the qubit is initially in the excited state $|1\rangle$, i.e., $|c_1(0)|^2 = 1$.

illustrates the time behavior of $S_A(t)$. We take the situation where the initial pure state $|1\rangle$ of the qubit evolves into the final pure state $|0\rangle$. Note that for all strengths of coupling, at some time t_m, the qubit passes through the mixed state with the *maximum* entropy $S_A(t_\mathrm{m}) = \ln 2$, i.e., with $v(t_\mathrm{m}) = 0$. It is interesting that this "maximally entangled" intermediate state of the combined system appears only in the case of the maximum initial population of the excited qubit state $|1\rangle$ ($|c_1(0)|^2 = 1$), as may be seen directly from (43). The time behavior of the qubit entropy for smaller values of $|c_1(0)|^2$ is shown in figure 3.

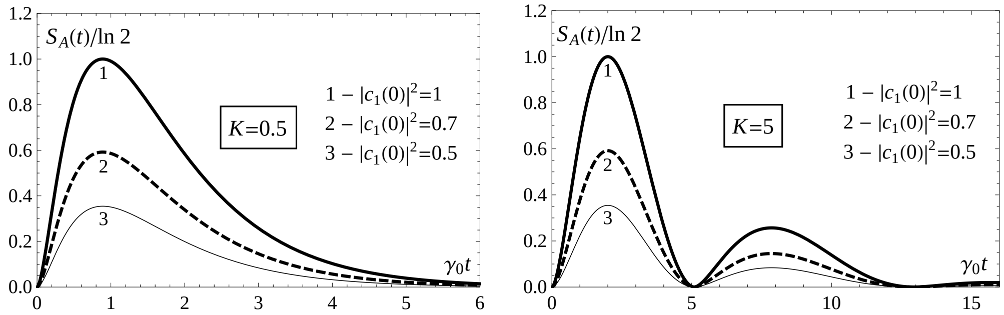

Figure 3. Time evolution of the qubit entropy in the damped Jaynes-Cummings model for different initial populations of the excited qubit state $|1\rangle$ in the cases of weak and strong coupling.

5. Conclusion

In this paper we have considered the time evolution of information entropy in two exactly solvable models of two-state open quantum systems (qubits). Calculations were based on the simple but quit general representation (7) for the qubit entropy.

Our discussion was restricted to the special case of the qubit and the reservoir being initially uncorrelated and the reservoir being in the ground state, i.e., at zero temperature. This simplifying assumption allows one to study in detail the entropy production in entangled time-dependent quantum states of a combined system. It should be noted, however, that in many situations of physical interest, a factorized (uncorrelated) initial state at $T = 0$ cannot always be realized, so that the dynamics of thermal and correlated initial states, including the entropy behavior, is of great significance. For instance, it was shown in [9] that the dephasing model (10) admits exact solutions for a large class of physically reasonable correlated initial states at finite temperatures. It was found that, for a sufficiently strong coupling, initial qubit-environment correlations have a profound effect on the dephasing process and on the time behavior of entropy. It would be of interest to study the entropy behavior in further examples of exactly solvable models of open quantum systems in the presence of initial system-environment correlations.

Acknowledgement

This work was supported by DFG (Deutsche Forschungsgemeinschaft), SFB 652 (Sonderforschungsbereich — Collective Research Center 652).

References

1. Breuer H.-P., Petruccione F., The Theory of Open Quantum Systems, Oxford University Press, Oxford, 2002.
2. Zurek W.H., Rev. Mod. Phys., 2003, **75**, 715; doi:10.1103/RevModPhys.75.715.
3. Amico L., Fazio R., Osterloh A., Vedral V., Rev. Mod. Phys., 2008, **80**, 517; doi:10.1103/RevModPhys.80.517.
4. Horodecki R., Horodecki P., Horodecki M., Horodecki K., Rev. Mod. Phys., 2009, **81**, 865; doi:10.1103/RevModPhys.81.865.
5. The Physics of Quantum Information. Eds. D. Bouwmeester, A. Ekert, A. Zeilinger, Springer-Verlag, Berlin, 2000.
6. Valiev K.A., Phys.-Usp., 2005, **48**, 1; doi:10.1070/PU2005v048n01ABEH002024.
7. Allen L., Eberly J.H., Optical Resonance and Two-Level Atoms, Dover, New York, 1975.
8. Cohen-Tannoudji C., Diu B., Laloë F., Quantum Mechanics, Wiley, New York, 1977.
9. Morozov V.G., Mathey S., Röpke G., Phys. Rev. A, 2012, **85**, 022101; doi:10.1103/PhysRevA.85.022101.
10. Łuczka J., Physica A, 1990, **167**, 919; doi:10.1016/0378-4371(90)90299-8.
11. Unruh W.G., Phys. Rev. A, 1995, **51**, 992; doi:10.1103/PhysRevA.51.992.
12. Palma G. M., Suominen K.-A., Ekert A.K., Proc. R. Soc. London, Ser. A, 1996, **452**, 567; doi:10.1098/rspa.1996.0029.
13. Leggett A.J., Chakravarty S., Dorsey A.T., Fisher M.P.A., Garg A., Zwerger W., Rev. Mod. Phys., 1987, **59**, 1; doi:10.1103/RevModPhys.59.1.

14. Garraway B.M., Phys. Rev. A, 1997, **55**, 2290; doi:10.1103/PhysRevA.55.2290.
15. Breuer H.P., Kappler B., Petruccione F., Phys. Rev. A, 1999, **59**, 1633; doi:10.1103/PhysRevA.59.1633.

Продукування ентропії у відкритих квантових системах: точно розв'язні кубіт моделі

В.Г. Морозов[1], Г. Репке[2]

[1] Московський державний технічний університет радіотехніки, електроніки та автоматики, проспект Вернадського, 78, 119454 Москва, Росія

[2] Університет м. Росток, Університетська площа, 3, D–18051 м. Росток, Німеччина

Ми представляємо аналітичні результати для часовозалежної інформаційної ентропії в двостанових точно розв'язних (кубіт) моделях. Перша модель описує дефазування (декогеренцію) в кубіті, який є зв'язаний з резервуаром гармонічних осциляторів. Обговорюється продукування ентропії для цієї моделі у режимах "повної" та "неповної" декогеренції. Як інший приклад ми розглядаємо задемпфовану модель Джейнса-Каммінгса, яка описує спонтанне затухання дворівневої системи в польовому вакуумі. Показано, що для всіх сил зв'язку, відкрита система переходить через змішаний стан з максимумом інформаційної ентропії.

Ключові слова: *інформаційна ентропія, відкриті квантові системи, кубіт моделі, декогеренція, квантове заплутування*