Condensed Matter Physics 2011, Vol. 14, No 1, 14001: 1–6
DOI:10.5488/CMP.14.14001
http://www.icmp.lviv.ua/journal

Rapid Communication

Analytical and numerical studies of creation probabilities of hierarchical trees

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Received February 18, 2011

We consider the creation conditions of diverse hierarchical trees both analytically and numerically. A connection between the probabilities to create hierarchical levels and the probability to associate these levels into a united structure is studied. We argue that a consistent probabilistic picture requires the use of deformed algebra. Our consideration is based on the study of the main types of hierarchical trees, among which both regular and degenerate ones are studied analytically, while the creation probabilities of Fibonacci, scale-free and arbitrary trees are determined numerically.

Key words: *probability, hierarchical tree, deformation*

PACS: *02.50.-r, 89.75.-k, 89.75.Fb*

1. Introduction

As it is shown for diverse systems, ranging from the World Wide Web [1] to biological [2] and social [3] networks, real networks are governed by strict organizing principles displaying the following properties: i) most networks have a high degree of clustering; ii) many networks have been found to be scale-free [4] which means that the probability distribution over node degrees, being the set of the numbers of links with neighbors, follows the power law.

A formal basis of the theory of hierarchical structures is represented by the fact that hierarchically constrained objects are related to an ultrametric space whose geometrical image is the Cayley tree with nodes and branches corresponding to elementary cells and their links [5]. One of the first theoretical pictures [6] has been devoted to the diffusion process on either uniformly or randomly multifurcating trees. The consequent study of hierarchical structures has shown [7] that their evolution is reduced to an anomalous diffusion process in ultrametric space that arrives at a steady-state distribution over hierarchical levels, which represents the Tsallis power law inherent to non-extensive systems [8]. The principal peculiarity of the Tsallis statistics is known to be governed by a deformed algebra [9].

This paper briefly represents the results of our study of creation conditions of a vast variety of hierarchical trees on the basis of methods initially developed within the quantum calculus [10]. An extended version of our analysis is published elsewhere [11]. The outline of the paper is as follows. In section 2, we discuss the statistical peculiarities of the picture of hierarchical structure creation to demonstrate that effective energies of hierarchical levels remain to be additive values, while the set of corresponding probabilities becomes both non-additive and non-multiplicative due to the coupling between different levels. Further consideration is based on an analytical and numerical study of the main types of hierarchical trees in section 3. Section 4 is devoted to the discussion of obtained results.

2. Statistical peculiarities of hierarchical ensembles

As pointed out above, the stationary creation probability of the l-th hierarchical level takes the Tsallis form

$$p_l = p_0 \exp_q\left(-\frac{\varepsilon_l}{\Delta}\right), \qquad \exp_q(x) := [1+(1-q)x]_+^{\frac{1}{1-q}}, \qquad [y]_+ \equiv \max(0,y). \tag{2.1}$$

Here, p_0 is the top-level probability fixed by the normalization condition, $q \geqslant 0$ is a deformation parameter, ε_l is an effective energy of the l number, Δ is an effective temperature. Although the energy is a key concept of the network optimization theory, it is not always possible to match its value to a given graph. However, basing on heuristic ideas, it is always possible to attach an effective value of energy to some phenomenological parameter. Also, for our purpose it is convenient to consider the nodes of the hierarchical tree as particles of a statistical ensemble, while its edges represent couplings between these particles.

In contrast to the statistical theory of complex networks [12], the hierarchical systems under our consideration cannot simultaneously display the properties of additivity of effective energies and multiplicativity of related probabilities. The cornerstone of our approach is that the creation of a hierarchical structure does not break the law of the energy conservation, so that the energies ε_l remain to be additive values:

$$\epsilon_n := \sum_{l=0}^{n} \varepsilon_l. \tag{2.2}$$

Within the statistical theory of random networks [12], effective energies ε_l are reduced to a constant for microcanonical ensemble and are fixed by the set of particular probabilities p_l according to the relation $\varepsilon_l = -\Delta \ln(p_l) + \text{const}$ for both canonical and grand canonical ensembles with an effective temperature Δ. On the other hand, due to the coupling between different levels, the hierarchy essentially deforms the corresponding probabilities p_l, which become non-multiplicative. Indeed, the probability P_n to create an n-level hierarchical structure is connected with total energy ϵ_n by means of the relation $\epsilon_n = -\Delta \ln_q(P_n)$ with the deformed logarithm $\ln_q(x) := \left(x^{1-q}-1\right)/(1-q)$. Then, the condition (2.2) leads to the additivity of these logarithms: $\ln_q(P_n) = \sum_{l=0}^{n} \ln_q(p_l)$, and one obtains the probability relation

$$P_n := p_0 \otimes_q p_1 \otimes_q p_2 \otimes_q \cdots \otimes_q p_n, \tag{2.3}$$

where the deformed product is defined as $x \otimes_q y := \left[x^{1-q} + y^{1-q} - 1\right]_+^{\frac{1}{1-q}}$. Thus, in contrast to ordinary statistical systems, the creation probability P_n of a hierarchical structure is equal to the *deformed* production of specific probabilities p_l related to the levels $l = 0, 1, \ldots, n$.

The above law of the deformed multiplicativity determines the probabilities p_l to create a set of hierarchical levels simultaneously. Another problem emerges when we consider the connection between the creation probability of a given hierarchical level l and the same for each node at this level. For simplicity let us consider a regular tree, whose nodes multifurcate to generate a set of the N_l nodes determined with inherent probabilities $\pi = p_0/N_l$ where p_0 is their top magnitude being a normalization constant. If one permits additivity of the node probabilities, we arrive at the total probability of the l level realization to be independent of their numbers: $p_l := N_l \pi = p_0$. Since the probability p_l to create a hierarchical level decays with the level number l, we are forced again to replace the trivial additive connection of the level probability p_l with the node value π by a deformed sum.

Finally, since the creation probabilities of the hierarchical levels go beyond probabilities related to non-hierarchical structures, the standard normalization condition based on the use of the usual sum is broken as well. With the growth of the difference $|q-1|$, the probability (2.1) increases at arbitrary values of the energy ε_l with respect to the non-deformed value related to the parameter $q = 1$. On the other hand, the deformed sum $x \oplus_q y := x + y + (1-q)xy$ decreases with the growth of the parameter $q > 1$. As a result, one can anticipate that a self-consistent probabilistic picture of hierarchical ensembles is reached if one proposes the normalization condition

$$p_0 \oplus_q p_1 \oplus_q \cdots \oplus_q p_n = 1, \qquad q > 1 \tag{2.4}$$

that is deformed to fix the top level probability p_0.

Taking into account the above statements, one obtains an explicit form of the creation probability of a hierarchical structure [11]

$$P_n = \exp_q \left[\frac{\sum_{l=0}^{n} p_l^{1-q} - (n+1)}{1-q} \right] = \left(\sum_{l=0}^{n} p_l^{1-q} - n \right)_+^{\frac{1}{1-q}} . \tag{2.5}$$

The relations (2.5) mean the decrease of the creation probability with the growth of the hierarchical tree in accordance with the difference equation

$$P_{n-1}^{1-q} - P_n^{1-q} = 1 - p_n^{1-q}. \tag{2.6}$$

In the non-deformed limit $q \to 1$, relations (2.3) and (2.5) are reduced to the ordinary rule $P_n = \prod_{l=0}^{n} p_l$ (respectively, equation (2.6) reads $P_n/P_{n-1} = p_n$), while at $q = 2$ the creation probability (2.5) takes a maximal value.

According to equation (2.5) the subsequent step in the definition of the creation probability P_n of a hierarchical structure is the determination of a set of probabilities $\{p_l\}_0^n$ related to different hierarchical levels.

3. Level probabilities for different hierarchical trees

First, we consider a regular tree whose nodes multifurcate at the level l with constant branching index $b > 1$ to generate a set of the $N_l = b^l$ nodes determined with inherent probabilities $\pi = p_0/N_l = p_0 b^{-l}$. Within naive proposition, one could permit additivity of the node probabilities to arrive at the total probability of the l level realization to be $p_l := N_l \pi = p_0$. Thus, within the condition of additivity of the node probabilities, the related values $p_l = p_0 = (n+1)^{-1}$ for all levels appear to be independent of their numbers $l = 0, 1, \ldots, n$.

To avoid this trivial situation, we propose to replace the common additive connection of the level probability with the node value π by the deformed one. Such deformation leads to the required level distribution in the binomial form [11]

$$p_l = \frac{[1 + (1-q)b^{-l}]_+^{b^l} - 1}{1-q} p_0 . \tag{3.1}$$

This probability increases with the growing number l of hierarchical level at $q < 1$ and decays at $q > 1$. From the physical point of view, the creation probability of a lower hierarchical level should be less than for higher levels, so that one ought to conclude that only the case $q > 1$ is meaningful. Characteristically, the form of this distribution very weakly depends on both deformation parameter q and branching index b excluding the domain $2 - q \ll 1$, where the probability does not decay that fast at small values of the branching index b. With large growth of the parameter $b \gg 1$ or level number l, the dependence p_l decreases faster to exponentially reach the minimum value

$$p_\infty = \frac{\mathrm{e}^{1-q} - 1}{1-q} p_0 = p_0 \ln_q \mathrm{e}. \tag{3.2}$$

There is a distinctive feature in the behavior of the regular hierarchical tree near the limit value $q = 2$ where the dependence (3.1) has no singularity. This feature is corroborated with the dependence of the top level probability p_0 on the deformation parameter. This probability increases monotonously with the q-growth to reach sharply the limit value $p_0 = 1$ in the point $q = 2$. Obviously, this means an anomalous increase of probabilities p_l for the whole set of hierarchical levels. Though, within the domain $2 - q \ll 1$, the ordinary normalization condition $\sum_{l=0}^{n} p_l = 1$ is violated appreciably, the definition of the deformed sum shows that the deformed normalization condition (2.4) can be recovered with the q parameter growing. However, beyond the border $q = 2$, this condition is not satisfied at all. As a result, we arrive at the conclusion that physically meaningful values of the deformation parameter are concentrated within the domain $q \in [1, 2]$.

The difference between regular and degenerate trees is that *all* nodes multifurcate at each level in the former case, while the *only one* node branches in the latter. In this sense, the degenerate tree can be considered as an antipode of the regular one to be studied analytically. Taking into account this peculiarity, the creation probability of the lth hierarchical level takes the form [11]

$$p_l = \frac{\left[1+(1-q)b^{-l}\right]\prod_{m=1}^{l}\left[1+(1-q)b^{-m}\right]^{b-1} - 1}{1-q} p_0 \,. \tag{3.3}$$

Similarly to the case of the regular tree, this distribution decays exponentially fast to the limit probability p_∞ determined by equation (3.2).

Above, we have considered two conceptual examples of hierarchical trees with self-similar structure, i.e., regular and degenerate trees. By contrast, a scale-free tree has rather random structure, but the probability distribution over hierarchical levels tends to the power-law form inherent in self-similar statistical systems. In this case, the probability distribution over tree levels is determined by the discrete difference equation [7]

$$p_{l+1} - p_l = -p_l^q/\Delta, \qquad l = 0, 1, \dots, n \tag{3.4}$$

accompanied with the deformed normalization condition (2.4) (Δ being a distribution dispersion).

In figure 1 we compare the probability distributions over hierarchical levels of scale-free, regular,

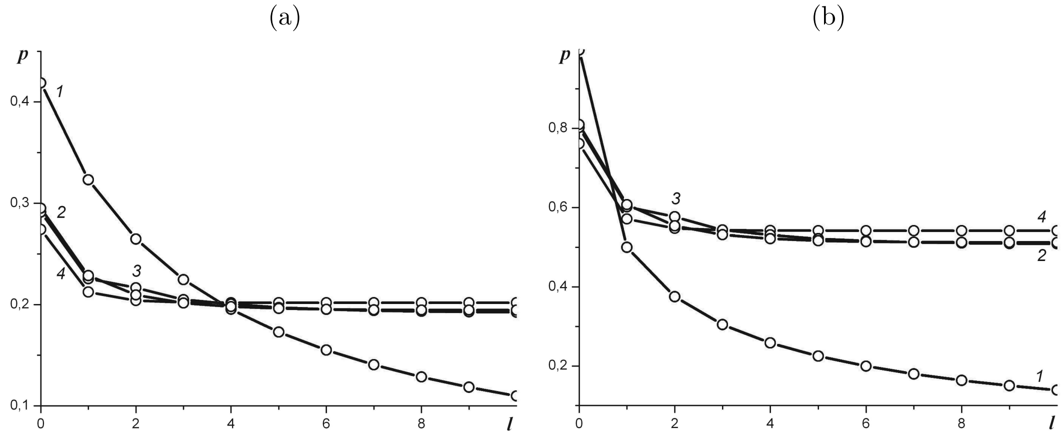

Figure 1. Probability distributions over hierarchical levels for scale-free, regular, Fibonacci and degenerate trees (curves 1-4, respectively) at $\Delta = 2$, $b = 2$, $n = 10$ and $q = 1.9$ (a) and $q = 1.9999$ (b).

degenerate and Fibonacci trees at different values of the deformation parameter. As it is seen, at all q-values, that the forms of these distributions are actually similar for all the above trees excluding the scale-free one. In the latter case, the level probability decays to zero as a power law, whereas there is a limit non-zero value (3.2) for the regular trees. In accordance with such a behavior, the creation probabilities depicted in figure 2 decay faster for the scale-free tree than in the case of the regular and degenerate ones. Characteristically, this difference appears only within the domain $2 - q \ll 1$ of the deformation parameter variation.

Finally, let us consider two examples of arbitrary trees, among which the former concerns the Fibonacci tree (number of nodes at its each level is equal to Fibonacci number), while the latter relates to the schematic evolution tree shown in figure 3a (in the latter case, the nodes identify substantial stages in the evolution of life, e.g., human is situated on the 24th level). Using the approach developed for the node and level probabilities, obeying the normalization condition, we show that the probability distributions of the Fibonacci tree depicted in figures 1, 2 do not actually differ from the related dependencies for both regular and degenerate trees. As concerns the evolution tree, its probability distributions (figure 3b) show that the presence of the stopped

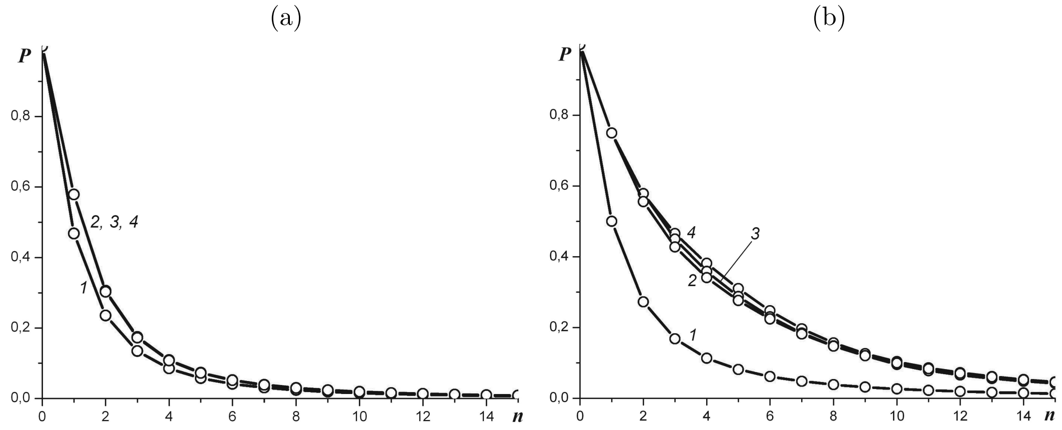

Figure 2. Creation probabilities of scale-free, regular, Fibonacci and degenerate hierarchical trees (curves 1-4, respectively) as function of the whole level number at $\Delta = 2$, $b = 2$ and $q = 1.9$ (a) and $q = 1.9999$ (b).

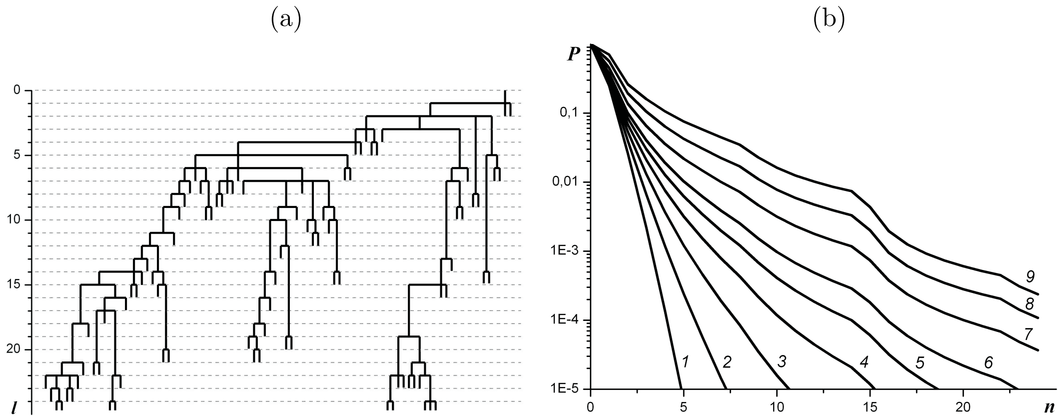

Figure 3. (a) Schematic representation of evolution tree (from Ref. [13]). (b) Creation probability of the evolution tree vs. the level number at: $q = 1.0001, 1.1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.9, 1.9999$ (curves 1-9, respectively).

branches (type of two rightmost ones in figure 3a) considerably decreases the creation probability of new hierarchical levels. Particularly, the probability of human appearance takes the values greater than 10^{-4} only at the deformation parameter $q = 1.9999$.

4. Concluding remarks

To avoid ambiguities it is worthwhile to stress that our consideration concerns rather the probabilistic picture of creation of the hierarchical trees themselves than hierarchical phenomena and processes evolving on these trees (for example, hierarchically constrained statistical ensembles [14], diffusion processes on multifurcating trees [6], et cetera).

The principal peculiarity of the probabilistic picture elaborated is a distinction between the deformed and non-deformed quantities. Thus, effective energies of hierarchical levels in equation (2.2) are non-deformed quantities because the creation of hierarchical structures does not break the conservation law of the energy being an additive value. Moreover, the node probabilities are determined using the non-deformed relations because these probabilities relate to the configuration of the hierarchical tree itself (in other words, they are determined for geometrical, rather than for probabilistic reasons). At the same time, the hierarchy appearance essentially deforms the probability relations (2.3)–(2.5) due to the coupling level probabilities p_l . Similarly, the definition of these probabilities through corresponding node values is based on the use of a deformed summation.

Making use of the deformed algebra leads to an increase of probabilities p_l for the whole set of hierarchical levels assuming an anomalous character near the point $q = 2$. The deformed normalization condition (2.4) is fulfilled only at $q \leqslant 2$, while it is broken beyond the limit $q = 2$. As a result, taking into account the fact that the creation probability of a lower hierarchical level should be less than the one for higher levels, the physically meaningful values of the deformation parameter belong to the domain $q \in [1, 2]$.

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Аналітичне і чисельне дослідження ймовірності утворення ієрархічних дерев

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Розглянуто аналітично і чисельно умови утворення різних ієрархічних дерев. Досліджено зв'язок між ймовірностями утворення ієрархічних рівнів та ймовірності об'єднання цих рівнів у єдину структуру. Показано, що побудова послідовної ймовірнісної картини вимагає використання деформованої алгебри. Даний розгляд заснований на дослідженні основних типів ієрархічних дерев, серед яких регулярне і вироджене досліджені аналітично, тоді як ймовірності утворення дерева Фібоначчі, безмасштабного та довільного дерева визначені чисельно.

Ключові слова: *ймовірність, ієрархічне дерево, деформація*