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Special Theory of Relativity without special assumptions and tachyonic motion

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The most general form of transformations of space-time coordinates in Special Theory of Relativity based solely on physical assumptions is described. Only the linearity of space-time transformations and the constancy of the speed of light are used as assumptions. The application to tachyonic motion is indicated.

Key words: *Minkowski space-time, Lorentz transformations, tachyons, Maxwell equations*

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1. Introduction

In almost all textbooks on Special Relativity [1] it is claimed that fundamental transformations of space-time coordinates $(\vec{x}, t)$ are of the form of Lorentz transformations

$$t \to t' = \gamma \left(t - \frac{\vec{V} \cdot \vec{x}}{c^2} \right), \tag{1.1}$$

$$\vec{x} \to \vec{x}' = \vec{x} + (\gamma - 1) \frac{(\vec{V} \cdot \vec{x})}{V^2} \vec{V} - \gamma \vec{V} t, \tag{1.2}$$

where

$$\gamma = \left(1 - \frac{V^2}{c^2} \right)^{-1/2} \tag{1.3}$$

is the famous Lorentz factor with $\vec{V}$ being the relative velocity of two observers and c being the velocity of light.

However, it turns out that **such statement is not quite true**. In fact, historically [2] H. A. Lorentz derived his transformations from the requirement of covariance of the vacuum Maxwell equations under linear transformations of space-time coordinates. Now it is known that in any medium the Maxwell equations are covariant not only under linear transformations but also under arbitrary transformations of space-time coordinates. Therefore, the Lorentz derivation of Lorentz transformations used additional particular assumption that the basic Maxwell equations are the vacuum Maxwell equations.

Also Einstein [3], in his derivation of the space-time transformations, used particular information on the Doppler effect and assumed the equality of two-way velocities of light. Again, we can see additional particular assumptions. It is easy to show that without these assumptions we can get more general forms of Lorentz transformations [4].

In modern textbooks [5] the Lorentz transformations are usually derived from the assumption that they leave the Minkowski interval invariant

$$\mathrm{d}s^2 = c^2 (\mathrm{d}t)^2 - (\mathrm{d}\vec{x})^2. \tag{1.4}$$

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It is quite understandable that all approaches of this kind use additional assumptions without sufficient physical justification!

In the present paper, which I dedicate to Professor N.N. Bogolyubov (Jr.), we shall present a new derivation of fundamental transformations of space-time coordinates based solely on physical assumptions. As a result, we get a slightly more general form of fundamental transformations of space-time coordinates. Based on this new form of Lorentz transformations we show that in the framework of Special Relativity we may equally well describe both sub- and super-luminal motions.

2. Physical approach to Special Relativity

To begin with, let us recall that Special Relativity is based only on two physically justified assumptions:

- The uniform motion is an invariant notion.
- There exists an invariant velocity.

From the first assumption we have the following transformations:

$$t \to t' = At + B_k x^k, \tag{2.1}$$
$$x^k \to x'^k = D^k_j x^j + E^k t \tag{2.2}$$

with some yet unknown coefficients A, B_k, D^k_j and E^k. Here the summation over the repeated indexes is meant.

From these transformations we get the following transformation rule for velocities of motions

$$v^k(t) = \frac{\mathrm{d}x^k(t)}{\mathrm{d}t} \to v'^k(t') = \frac{\mathrm{d}x'^k(t')}{\mathrm{d}t'} = \frac{D^k_j v^j(t) + E^k}{A + B_k v^k(t)}\,. \tag{2.3}$$

The requirement of the existence of an invariant velocity $\vec{C}$ means that the length of this velocity should be the same in each reference frame. Therefore, for the velocity $\vec{C}$ we have

$$\vec{C}'^2 = \vec{C}^2 \tag{2.4}$$

and in view of the transformation rule (1.3) for velocities, this leads to the relation

$$\vec{C}^2 \left[A + B_k C^k\right]^2 = \sum_{k=1}^{3} \left[D^k_j C^j + E^k\right] \left[D^k_l C^l + E^k\right]. \tag{2.5}$$

We may treat this relation as an equation for the square of the invariant velocity $\vec{C}^2$ which [4] may have no solution, one solution or two solutions either of the same modulus or quite different. Let us consider the case when the solution of (1.5) is the square of the usual velocity of light. In a more general case, equation (1.5) may have two solutions that correspond to the inequality of two-way velocities of light. Equation (1.5) leads to some conditions on the parameters A, B_k, D^k_j and E^k. Having solved these conditions, we arrive at the following general form of space-time transformations

$$t \to t' = At + \vec{B}\cdot\vec{x}, \tag{2.6}$$

$$\vec{x} \to \vec{x}' = \sqrt{A^2 - c^2\vec{B}^2}(R\vec{x}) + \frac{A - \sqrt{A^2 - c^2\vec{B}^2}}{\vec{B}^2}(R\vec{B})(\vec{B}\cdot x) + c^2(R\vec{B}). \tag{2.7}$$

Here A and $\vec{B}$ are arbitrary parameters and R denotes an arbitrary orthogonal matrix. The transformations make sense only if

$$A^2 - c^2\vec{B}^2 > 0. \tag{2.8}$$

We shall see below that this condition leads to a restriction on velocities of motion.

Standard Lorentz transformations (0.1) and (0.2) are obtained with the choice

$$A = \gamma, \qquad \vec{B} = -\frac{\gamma}{c^2}\vec{V}, \qquad R = I. \tag{2.9}$$

In two-dimensional space-time, the transformations (1.6) and (1.7) reduce to

$$t' = At + Bx, \tag{2.10}$$
$$x' = Ax + c^2 Bt. \tag{2.11}$$

For brevity, we shall refer to our form of Lorentz transformations as the A, B, C form of Lorentz transformations.

In N-dimensional space-time, the Jacobian of generalized Lorentz transformations is equal to

$$J = \left(A^2 - c^2\vec{B}^2\right)^{N/2}. \tag{2.12}$$

Correspondingly, in two-dimensional space-time we have:

$$J = A^2 - c^2 B^2, \tag{2.13}$$

in three-dimensional space-time we have:

$$J = \left(A^2 - c^2\vec{B}^2\right)^{3/2} \tag{2.14}$$

and in four-dimensional space-time we have:

$$J = \left(A^2 - c^2\vec{B}^2\right)^{2}. \tag{2.15}$$

The A, B, C transformations form a group with the following group composition law

$$A_{2,1} = A_2 A_1 + c^2(\vec{B}_2 \cdot \vec{B}_1), \tag{2.16}$$
$$\vec{B}_{2,1} = \sqrt{A_1^2 - \vec{B}_1^2 c^2}\, R_1^{-1}\vec{B}_2 + \frac{A_1 - \sqrt{A_1^2 - \vec{B}_1^2 c^2}}{\vec{B}_1^2}(\vec{B}_2 \cdot R_1\vec{B}_1)\vec{B}_1 + A_2\vec{B}_1\,. \tag{2.17}$$

The composition law for space rotations is much more complicated. In fact, having twice repeated the space-time transformations (1.6) and (1.7) we arrive at the composed rotation matrix R_{21} of the form

$$R_{21} = \Omega_{21}(A_2, \vec{B}_2, R_2; A_1, \vec{B}_1, R_1)R_2 R_1\,, \tag{2.18}$$

where Ω_{21} is an orthogonal matrix which satisfies the following equation

$$\sqrt{A_2^2 - c^2\vec{K}}\vec{K} + \frac{A_2 - \sqrt{A_2^2 - c^2\vec{L}}}{\vec{L}^2}(\vec{K}\cdot L)\vec{L} + A_1\vec{L} =$$
$$\Omega_{21}\left[\sqrt{A_1^2 - c^2\vec{K}^2}\vec{L} + \frac{A_1 - \sqrt{A_1^2 - c^2\vec{K}^2}}{\vec{K}^2}(\vec{K}\cdot\vec{L})\vec{K} + A_2\vec{K}\right]. \tag{2.19}$$

Here

$$\vec{K} = R_2 R_1 \vec{B}_1\,, \qquad \vec{L} = R_2\vec{B}_2\,. \tag{2.20}$$

Unfortunately, it is not easy to solve equation (1.19)and find an explicit form of Ω_{21}. It certainly is an orthogonal matrix because the lengths of vectors on both sides of (1.19)are the same and equal to

$$\left(A_2\vec{K} + A_1\vec{L}\right)^2 + c^2\left[(\vec{K}\cdot L)^2 - \vec{K}^2\vec{L}^2\right]. \tag{2.21}$$

From (1.19)it is clear that the vectors in both sides of (1.19)lie in the plane spanned by the vectors $\vec{K}$ and $\vec{L}$. The rotation Ω_{21}, therefore, is around the axis parallel to $\vec{K} \times \vec{L}$. The angle of this rotation may be calculated from the scalar product of the vectors on both sides of (1.19).

In standard treatments of Special Relativity, the problem of combining two rotations is not well-understood [7] and there is a widely accepted erroneous suggestion that a correction of the composition law of two Lorentz transformations by an additional rotation is needed. As a matter of fact, our treatment of Lorentz transformations shows that no correction of the group theoretical composition law is needed. The correct statement is that Lorentz transformations without rotations do not form a subgroup of space-time transformations because the composition of two Lorentz transformations with $R_1 = R_2 = I$ is a Lorentz transformation with $R_{21} \neq I$. It is a specific property of Lorentz transformation that there always takes place an additional relativistic rotation.

The neutral element for the composition laws (1.16)and (1.17)is given by $A = 1,\ \vec{B} = 0,$ R=I. The parameters of the reverse transformation are given by

$$A^{-} = \frac{A}{A^2 - \vec{B}^2 c^2}, \qquad \vec{B}^{-} = -\frac{R\vec{B}}{A^2 - \vec{B}^2 c^2}, \qquad R^{-} = R^{-1}. \tag{2.22}$$

3. Velocities of motion

From the transformation rules (1.6) and (1.7) for space-time coordinates we get the following transformation rule for velocities of motion

$$\vec{V}' = \frac{\sqrt{A^2 - c^2\vec{B}^2}(R\vec{V}) + \frac{A - \sqrt{A^2 - c^2\vec{B}^2}}{\vec{B}^2}(R\vec{B})(\vec{B}\cdot\vec{V}) + c^2(R\vec{B})}{A + \vec{B}\cdot\vec{V}}. \tag{3.1}$$

Assuming that there exists a reference frame (for which we choose unprimed coordinates) in which a physical object is at rest we find that in a transformed reference system this object moves uniformly with the velocity (we omit the unnecessary prime on the left hand side)

$$\vec{V} = \frac{c^2}{A} R\vec{B}. \tag{3.2}$$

Using this relation in the formula for Lorentz transformations (1.6) and (1.7), we may replace $\vec{B}$ by $\frac{A}{c^2}R^{-1}\vec{V}$ and arrive at the following form of the Lorentz transformations

$$t' = A\left[t + \frac{(\vec{V}\cdot R\vec{x})}{c^2}\right], \tag{3.3}$$

$$\vec{x}' = A\left[\sqrt{1 - \frac{\vec{V}^2}{c^2}}(R\vec{x}) + \left(1 - \sqrt{1 - \frac{\vec{V}^2}{c^2}}\right)\frac{\left(\vec{V}\cdot R\vec{x}\right)}{\vec{V}^2}\vec{V} + \vec{V}t\right]. \tag{3.4}$$

This is almost the same as the standard Lorentz transformation with the exception of the arbitrary factor A. Assuming that this factor is a function of the velocity $\vec{V}$ the composition rule

$$A(\vec{V}_{21}) = A(\vec{V}_2)A(\vec{V}_1)\left(1 + \frac{\vec{V}_1\cdot\vec{V}_2}{c^2}\right) \tag{3.5}$$

becomes a functional equation for the function $A(\vec{V})$ with the only solution

$$A = \gamma. \tag{3.6}$$

Here, of course, $\vec{V}_{21}$ denotes the relativistic composition of the velocities $\vec{V}_1$ and $\vec{V}_2$. However, we stress the fact that the choice (2.6) is not dictated by Special Theory of Relativity.

It is clear that, using (2.2), from the positivity condition $A^2 - c^2\vec{B}^2 > 0$ we get

$$\vec{V}^2 < c^2. \tag{3.7}$$

Let us now, contrary to the previous case, assume that in the unprimed reference frame some object moves with an infinite velocity. Then, in the primed reference frame its velocity $\vec{W}$ is equal to

$$\vec{W} = \frac{A}{\vec{B}^2} R\vec{B}. \tag{3.8}$$

Writing the vector parameter $\vec{B}$ in the space-time transformation (1.6) and (1.7) in terms of this velocity we get

$$t' = A\left(t + \frac{\vec{W}\cdot R\vec{x}}{\vec{W}^2}\right), \tag{3.9}$$

$$\vec{x}' = A\left[\sqrt{1-\frac{c^2}{\vec{W}^2}}\,(R\vec{x}) + \left(1-\sqrt{1-\frac{c^2}{\vec{W}^2}}\right)\frac{\left(\vec{W}\cdot R\vec{x}\right)}{\vec{W}^2}\vec{W} + \frac{c^2}{\vec{W}^2}\vec{W}t\right] \tag{3.10}$$

with an arbitrary factor A. Again, assuming that this factor is a function of the velocity $\vec{W}$ from the functional equation which follows from the composition law (1.16)we get

$$A(\vec{W}) = \sqrt{1-\frac{c^2}{\vec{W}^2}} \tag{3.11}$$

and from the positivity condition (1.8) we get

$$\vec{W}^2 > c^2. \tag{3.12}$$

Therefore, in our version of Relativity Theory we may equally well describe both sub-luminal and super-luminal motions. However, we must remember that for the super-luminal objects (tachyons), there does not exist the rest reference frame.

It should be also stressed that the velocity of the relative motion of two reference frames is equal to

$$c^2\frac{\vec{B}}{A} \tag{3.13}$$

and, therefore, from the positivity condition (1.8) it is always less than the velocity of light.

4. Velocity dependent tensors

Physical quantities have definite tensorial properties. In Special Relativity a particular role is played by tensors or pseudo-tensors which are form-invariant functions of the velocity $\vec{v}$ of moving objects. Pseudo-tensors of (K, L) type and of D rank on the right hand side of the transformation rule are homogeneous functions of A and $\vec{B}$ of $K - L + ND$ degree, where N is the dimension of the space-time and D is the degree of the Jacobian on the right hand side of the transformation rule. The velocity transformation rule (2.1) on the right hand side is a homogeneous function of A and $\vec{B}$ of 0 degree. Therefore, all velocity dependent (pseudo) tensors should satisfy the condition

$$K - L + ND = 0. \tag{4.1}$$

For pure tensors we have $D = 0$ and consequently $K = L$. Therefore, only mixed tensors with the same degree of contra- and covariance may be functions of the velocity. The example of such a tensor is the velocity tensor $V^{\mu}_{\nu}(\vec{v})$ discussed in [6].

It is interesting to note that transformation rules of tensors or pseudo-tensors which are form-invariant functions of velocities convert into sets of functional equations from which the components of these tensors can be found.

5. Four-momentum of objects

As an application of the presented formalism we shall construct the four-momenta for moving objects. For the contra-variant components of the four-momenta we have $K = 1, \quad L = 0$. Since a momentum is certainly velocity dependent on (3.1) it follows that in the four-dimensional space-time we should have $D = -1/4$. Therefore, the contra-variant components P^μ of the four-momenta transform as follows:

$$P'^0 = J^{-1/4}\left[AP^0 + \vec{B}\cdot\vec{P}\right], \tag{5.1}$$

$$\vec{P}' = J^{-1/4}\left[c^2(R\vec{B})P^0 + \sqrt{A^2 - c^2\vec{B}^2}(R\vec{P}) + \frac{A - \sqrt{A^2 - c^2\vec{B}^2}}{\vec{B}^2}(\vec{B}\cdot\vec{P})(R\vec{B})\right], \tag{5.2}$$

where J is the Jacobian (1.15). Assuming that momentum is a form-invariant function of the velocity, we get the following set of functional equations for the component of momentum

$$\begin{aligned} P^0(\vec{V}') &= J^{-1/4}\left[AP^0(\vec{V}) + \vec{B}\cdot\vec{P}(\vec{V})\right], & (5.3)\\ \vec{P}(\vec{V}') &= J^{-1/4}\left[c^2(R\vec{B})P^0(\vec{V}) + \sqrt{A^2 - c^2\vec{B}^2}(R\vec{P}(\vec{V}))\right. \\ &\quad \left. + \frac{A - \sqrt{A^2 - c^2\vec{B}^2}}{\vec{B}^2}(\vec{B}\cdot\vec{P}(\vec{V}))(R\vec{B})\right], & (5.4) \end{aligned}$$

where $\vec{V}'$ is linked with $\vec{V}$ by the transformation rule (2.1).

For the subluminal motions, assuming the unprimed reference frame $\vec{V} = 0$, the velocity in the primed reference frame is given by (2.2). Therefore, for the 0-th component of the momentum we have

$$P^0(\vec{V}) = \frac{1}{\sqrt{1 - \frac{\vec{V}^2}{c^2}}}\left[P^0(0) + \frac{\vec{V}\cdot R\vec{P}(0)}{c^2}\right] \tag{5.5}$$

and a more complicated expression for space components of the momentum. Here $P^0(0)$ and $\vec{P}(0)$ are arbitrary constants. Observing that energy should be an even function of velocity we should take $\vec{P}(0) = 0$ and then we end up with the four-momentum equal to

$$\begin{aligned} P^0(\vec{V}) &= \frac{P^0(0)}{\sqrt{1 - \frac{\vec{V}^2}{c^2}}}, & (5.6)\\ \vec{P}(\vec{V}) &= \frac{P^0(0)}{\sqrt{1 - \frac{\vec{V}^2}{c^2}}}\vec{V}. & (5.7) \end{aligned}$$

To have a standard expression for space components of a momentum we put $P^0(0) = M$, where M is the mass of the particle.

In the same way, for the covariant components of the four-momenta (for which $K = 0, \quad L = 1, \quad D = +1/4$) we get

$$P_0(\vec{V}) = \frac{P_0(0)}{\sqrt{1 - \frac{\vec{V}^2}{c^2}}}, \tag{5.8}$$

and

$$\vec{P}(\vec{V}) = -\frac{P_0(0)}{c^2}\frac{1}{\sqrt{1 - \frac{\vec{V}^2}{c^2}}}\vec{V}. \tag{5.9}$$

We get a standard expression for the momentum provided

$$P_0(0) = -Mc^2. \tag{5.10}$$

For a superluminal motion, we have to use the basic transformations of space-time coordinates in the form of (2.9) and (2.10). Proceeding similarly as above we end up with the following expression for momentum

$$P_0(\vec{W}) = \frac{P_0(0)}{\sqrt{1 - \frac{c^2}{\vec{W}^2}}} \tag{5.11}$$

and

$$\vec{P}(\vec{W}) = -\frac{P_0(0)\vec{W}}{\vec{W}^2\sqrt{1 - \frac{c^2}{\vec{W}^2}}}\,. \tag{5.12}$$

However, in this case it is not worthy to write $P_0(0)$ in terms of any mass parameter M as we did in (4.10) because for a tachyon there does not exist a rest frame in which we could measure its mass. From (4.11) and (4.12) we see that for $\vec{W} \to \infty$ the component P_0 remains finite while $\vec{P}$ tends to zero. It is, therefore, suggestive to assume that $P_0(0)$ is equal to some unknown kind of energy, for example, the dark energy present in the Universe.

Conclusion

The present paper provides a firm theoretical basis for tachyonic physics. We have shown that superluminal motions are permitted by Special Relativity in the same way as the subluminal motions are. There is no doubt that Special Relativity does not forbid the existence of tachyons for which there do not exist frames of rest.

From the physical point of view, there arises the most important problem of experimental discovery of tachyons. In the forthcoming paper we shall argue that in the famous experiment by G. Nimtz [8] the tachyons were indeed created in the first part of the paraffin block as a result of interaction of the incoming photons with phonons. After traveling through the gap between paraffin blocks the tachyons are subsequently annihilated by phonons in the second block with the creation of the final photons.

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Спеціальна теорія відносності без спеціальних припущень і тахіонний рух

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Описано найзагальнішу форму перетворень координат простір-час у спеціальній теорії відносності, що ґрунтується виключно на фізичних припущеннях. Як припущення використано тільки лінійність перетворень простір-час і постійність швидкості світла. Вказується на застосування до тахіонного руху.

Ключові слова: *простір-час Мінковського, перетворення Лоренца, тахіони, рівняння Максвелла*