Condensed Matter Physics 2010, Vol. 13, No 1, 13101: 1–12

http://www.icmp.lviv.ua/journal

CONDENSED MATTER PHYSICS

Two-loop RG functions of the massive ϕ^4 field theory in general dimensions

M.A. Shpot

Institute for Condensed Matter Physics of the National Academy of Sciences of Ukraine,
1 Svientsitskii Str., 79011 Lviv, Ukraine

Received September 28, 2009, in final form December 23, 2009

Two-loop Feynman integrals of the massive ϕ_d^4 field theory are explicitly obtained for generic space dimensions d. Corresponding renormalization-group functions are expressed in a compact form in terms of Gauss hypergeometric functions. A number of interesting and useful relations are given for these integrals as well as for several special mathematical functions and constants.

Key words: *Feynman integrals, special functions, renormalization group, field theory*

PACS: *11.10.Kk, 12.38.Bx, 02.30.Gp, 05.70.Jk*

1. Introduction

The dependence of critical exponents of statistical mechanical systems on the physical space dimension is one of the most fundamental features of critical phenomena [1]. Continuous change of the space dimension d has been the basic principle for constructing the famous Wilson-Fisher epsilon expansion [2] with $\varepsilon = 4-d$. However, the Renormalization Group (RG) [3], which provides the fundamental theoretical basis for such calculations, is nonperturbative in its nature. It does not require that the critical exponents or any other universal quantities should be expanded in some space deviation, coupling constant, inverse number of order-parameter components, etc.

While investigating the critical phenomena, the RG allows, in principle, to work in any setting of interest provided that we are smart enough to apply the appropriate analytical or numerical tools. The ε-expansion has turned out to be a very useful tool in studying the qualitative features of systems in critical state even in its lowest-order approximations (see, e.g., [4]). Being pushed to higher orders, it allowed to produce very accurate numerical extrapolations for critical exponents of three-dimensional N-vector models [5]. An alternative calculational scheme used with comparative success is the so-called g-expansion within the massive field theory in fixed dimension put forward by Parisi [6]. This approach has been mainly applied directly in three dimensions to systems of different complexity [7–14] (for a recent review see [15]). It was used in calculating the exponents of ϕ^4 models in two dimensions in [8, 9], albeit with somewhat less accuracy. In [16], the massive theory approach was applied to disordered spin systems in two and three dimensions, while Yu. Holovatch and the present author analyzed the critical behavior of pure and disordered Ising systems in general dimensions $2 < d < 4$ [17].

A natural access to non-integer dimensions is also provided by the well-known large-N expansion [18–21]. In its general scope, it gives critical exponents as functions of d. These are valid in the whole range between the lower and upper critical dimensionalities, and can be handled analytically in lower-order approximations. The large-N expansion is capable of yielding information on dimensional dependencies that are hardly accessible by other means, for example, from the epsilon expansion. Unfortunately, it is very hard to obtain such results in higher orders in $1/N$, while short series expansions in $1/N$ usually fail to give very accurate numerical estimates for relatively small values of N, say $N = 3$. The convergence of truncated $1/N$ expansions has been analyzed on the basis of the field-theoretical approach at $d = 3$ in [22].

The knowledge of dimensional dependencies of critical exponents, or eventually also universal relations of critical amplitudes are of great theoretical and practical interest. The information of this kind broadens our fundamental knowledge about these main characteristics of the critical behavior. Moreover, the possibility to consider the results, which are not directly related to dimensional expansions in the vicinity of the upper or lower critical dimensions, provides us with useful checks of their correctness or with some new relevant insights.

Let us give several examples. At the beginning of 1980ies Newman and Riedel [23] used the exact Wegner-Houghton [24] RG equation and the scaling field method in the study of the pure and dilute Ising models in general dimensions from the interval $2.8 < d < 4$.

Pinn et al. [25] studied the RG in the hierarchical model in $2 < d < 4$. Some results of theirs have been given as tables illustrating the dimensional dependencies of critical indices calculated for some sets of discrete non-integer values of d. A contact with the ε-expansion has also been established.

Recently, Ballhausen et al. [26] calculated dimensional dependencies of IM critical exponents for $1 < d < 4$ using the first-order derivative expansion of the exact RG equation for the effective average action (for recent reviews see [27, 28]). Their results for $\nu(d)$ and $\eta(d)$ are very similar to those obtained in [29] and [17].

In a very recent paper [30], O'Dwyer and Osborn analyze the Polchinski version [31] of the exact RG equations [3, 24, 27] and its possible truncations and derivative expansions. This is done both for non-integer dimensions between two and four, and in the epsilon expansion. This combined approach gave the possibility to better understand the local potential approximation [32], the derivative expansions used in treatments of the Polchinski equation, and moreover, to suggest an alternative improved truncation of this equation.

Another story happened in the study of the critical behavior of $O(N) \times O(M)$ spin models [14, 33–36]. Apparently, certain controversies in the discussion of these references could be avoided if there were more well-established information related to the dimensional dependencies of RG functions, fixed points, and critical exponents.

Finally, let us note that there is a big continuous interest in the high-energy physics (HEP) literature in calculating explicit closed-form results for Feynman integrals in general dimensions [37–48].

We believe that the results showing the explicit dependencies of RG functions and physical observables on the space dimension d are of fundamental importance. The aim of the present paper is to make a small step in this direction. We return to the article [17] published some time ago together with Holovatch and calculate the closed-form expressions for the two-loop RG functions of the massive field theory, treated in that work numerically, in general dimensions $1 \leqslant d \leqslant 4$.

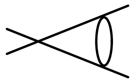
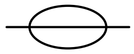
2. Two-loop RG functions and Feynman integrals

Let us consider the usual massive [6, 49–52] $O(N)$-symmetric ϕ^4 theory in general dimensions $d \leqslant 4$. To the second order in renormalized coupling constant u, the corresponding β and γ RG functions can be written as [16, 17, 52]

$$
\begin{aligned}
\beta(u) &= -(4-d)u\Big\{1 - u + \frac{8u^2}{(N+8)^2}\left[(5N+22)f(d) + (N+2)j(d)\right]\Big\} + O(u^4),\\
\gamma_\phi(u) &= -4(4-d)\frac{N+2}{(N+8)^2}\, j(d)\, u^2 + O(u^3),\\
\bar\gamma_{\phi^2}(u) &= (4-d)\left[\frac{N+2}{N+8}\, u - 12\frac{N+2}{(N+8)^2}\, f(d)\, u^2\right] + O(u^3).
\end{aligned}
$$

At the fixed point $u = u^*$, which is determined by the zero of the β function $\beta(u)$, the function γ_ϕ gives the value of the Fisher exponent η, while $\bar\gamma_{\phi^2}(u^*)$ leads to the combination $2 - \nu^{-1} - \eta$ where ν is the correlation length critical exponent.

The d-dependent functions $f(d) \equiv i(d) - 1/2$ and $j(d)$ are defined by the combinations of Feynman diagrams taken at zero external momenta,

$i(d) =$  $/$  ^{2} and $j(d) = \frac{\partial}{\partial q^2}$  $\Big|_{q^2=0}$ $/$  ^{2}. (1)

The massive two-loop integrals appearing in $i(d)$ and $j(d)$ are given by

$$I(m^2;d) = \iint \frac{\mathrm{d}^d k_1\, \mathrm{d}^d k_2}{(2\pi)^{2d}} \frac{1}{(k_1^2+m^2)^2(k_2^2+m^2)[(\boldsymbol{k}_1+\boldsymbol{k}_2)^2+m^2]}\,, \tag{2}$$

$$J(m^2,q^2;d) = \iint \frac{\mathrm{d}^d k_1\, \mathrm{d}^d k_2}{(2\pi)^{2d}} \frac{1}{(k_1^2+m^2)(k_2^2+m^2)[(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{q})^2+m^2]}\,. \tag{3}$$

The one-loop integral which appears in (1) and is traditionally used [7, 8, 16, 53] for normalization of the coupling constant is given by

$$D(m^2;d) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2+m^2)^2} = m^{d-4}(4\pi)^{-d/2}\Gamma(2-d/2)\,, \tag{4}$$

where $\Gamma(z)$ is a usual Euler's Gamma function (see e.g. [54]). This integral is absorbed, along with the symmetry factor $(N+8)/6$ into the normalization of the dimensionless renormalized coupling constant u. With this normalization, the RG β function starts with $-u+u^2+O(u^3)$ in the three-dimensional theory [7, 8]. On the other hand, since the integral $D(d)$ has a $1/\varepsilon$ pole as d approaches 4, in the epsilon expansion the fixed point value u^* will be given by $1+O(\varepsilon)$ as a consequence of such normalization.

It is easy to see that the following relations hold true for the integrals I and J,

$$I(m^2;d) = -\frac{1}{3}\frac{\partial}{\partial m^2} J(m^2,0;d) = -\frac{d-3}{3}\,(m^2)^{d-4}\,J(1,0;d), \tag{5}$$

where the second equality immediately follows from the scaling property $J(m^2,q^2;d) = (m^2)^{d-3}J(1,q^2/m^2;d)$ of the function J.

In [17], using the Feynman parametrization (see, e.g., [55]), $i(d)$ and $j(d)$ have been reduced to the double integrals over Feynman parameters and written down as

$$i(d) = \frac{\Gamma(\varepsilon)}{\Gamma^2(\varepsilon/2)} \int_0^1 \frac{x\,\mathrm{d}x}{[x(1-x)]^{1-\varepsilon/2}} \int_0^1 \frac{y^{\varepsilon/2}\,\mathrm{d}y}{[x(1-x)(1-y)+y]^{\varepsilon}}\,, \tag{6}$$

$$j(d) = -\frac{\Gamma(\varepsilon)}{\Gamma^2(\varepsilon/2)} \int_0^1 \frac{\mathrm{d}x}{[x(1-x)]^{-\varepsilon/2}} \int_0^1 \frac{y^{\varepsilon/2}(1-y)\,\mathrm{d}y}{[x(1-x)(1-y)+y]^{\varepsilon}}\,. \tag{7}$$

Here and further on, we parametrize the dimensional dependencies by the usual deviation from the upper critical dimension $\varepsilon = 4-d$. Generically, we do not assume that ε is infinitesimally small.

At the time of writing the paper [17], the following information was available. At $d=3$, the integrals i and j could be easily evaluated analytically as $i(3)=2/3$ and $j(3)=-2/27$. At $d=2$, their numerical values are [53] $i(2)=0.78130241\ldots$ and $j(2)=-0.11463575\ldots$. As $d\to 4$, both integrals $I(d)$ and $J(d)$ have poles in ε, namely $I(4-\varepsilon)\sim 1/\varepsilon^2$ and $J(4-\varepsilon)\sim 1/\varepsilon$. In the functions $i(4-\varepsilon)$ and $j(4-\varepsilon)$ these poles are compensated by division through $D^2(4-\varepsilon)$, so that their ε expansions start with[1] [52]

$$i(d) = \frac{1}{2} + \frac{\varepsilon}{4} + O(\varepsilon^2) \quad \text{and} \quad j(d) = -\frac{\varepsilon}{8} - \frac{\varepsilon^2}{8}\Big(\frac{3}{4}+I\Big) + O(\varepsilon^3), \tag{8}$$

[1] The similar formula given at p. 873 of [17] is not correct.

where

$$I = \int_0^1 \mathrm{d}x \left\{ \frac{1}{1-x(1-x)} + \frac{\ln[x(1-x)]}{[1-x(1-x)]^2} \right\}. \tag{9}$$

This integral has not been calculated in [52] since it is "renormalization-dependent" and disappears from the results for the critical exponents [52]. Numerically, $I = -1.5626048\ldots$. For non-integer d, the functions $f(d)$ and $j(d)$ have been tabulated in [17] using numerical computations in (6) and (7). The corresponding graphs have been plotted point by point for $2 \leqslant d \leqslant 4$. The aim of the following section is to analytically obtain the explicit expressions for these functions in general dimensions d.

3. Explicit calculations in general dimensions

Let us return to the integrals $i(d)$ and $j(d)$ given by (6) and (7) and consider the inner y integral from (6). Denoting for a while $z \equiv x(1-x)$ we write it as

$$I_f(z) = z^{-\varepsilon} \int_0^1 \mathrm{d}y\, y^{\varepsilon/2} [1 - y(z-1)/z]^{-\varepsilon}. \tag{10}$$

We recognize that here we have to deal with an integral representation of the incomplete beta function $B_t(a,b)$ (see, e.g., [54], entry 6.6.1). However, it is more convenient to express this function in terms of the Gauss hypergeometric function using the relation 6.6.8 of [54],

$$B_t(a,b) = a^{-1} x^a\, {}_2F_1(a, 1-b; a+1; t). \tag{11}$$

Thus we obtain

$$I_f(z) = \frac{2}{2+\varepsilon}\, z^{-\varepsilon}\, {}_2F_1\left[1+\varepsilon/2, \varepsilon; 2+\varepsilon/2; (z-1)/z\right]. \tag{12}$$

However, the argument of the resulting hypergeometric function is no good for further integration over x in the limits from 0 to 1 since the factor $1/z$ is singular at $x=0$ and $x=1$. The situation is substantially improved by using the three-term linear transformation formula 15.38 from [54] which reads

$$\begin{aligned} {}_2F_1(a,b;c;x) \quad &= \quad \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (1-x)^{-a}\, {}_2F_1\left[a, c-b; a-b+1; (1-x)^{-1}\right] \\ &+ \quad \text{a similar expression with } a \text{ and } b \text{ interchanged.} \end{aligned} \tag{13}$$

After some algebraic transformations we obtain for the inner y integral from (6)

$$I_f(z) = \frac{2}{2-\varepsilon}\left[-\frac{\varepsilon}{2}\frac{\Gamma^2(\varepsilon/2)}{\Gamma(\varepsilon)} z^{1-\varepsilon/2}(1-z)^{-1-\varepsilon/2} + {}_2F_1(\varepsilon, 1; \varepsilon/2; z)\right]. \tag{14}$$

An essential simplification occurred in the first term due to the appearance of a hypergeometric function with equal nominator and denominator parameters (see, e.g., [54], entry 15.1.8):

$${}_2F_1(a,b;b;z) = \sum_{k\geqslant 0} \frac{(a)_k}{k!} z^k = (1-z)^{-a}, \tag{15}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\ldots(a+k-1)$ is the Pochhammer symbol.

Let us substitute the first term of (14) into the outer x integral remaining in (6). Recalling the short-hand notation $z = x(1-x)$ we see that the factor $z^{1-\varepsilon/2}$ is exactly canceled by the analogous term in the denominator and, apart from the numerical constants, we remain with

$$\int_0^1 \frac{x\,\mathrm{d}x}{(1-x+x^2)^{1+\varepsilon/2}} = \int_0^{1/2} \frac{\mathrm{d}t}{(\frac{3}{4}+t^2)^{1+\varepsilon/2}} = \frac{2^{1+\varepsilon}}{3^{1+\varepsilon/2}}\, {}_2F_1\Big(1+\frac{\varepsilon}{2}, \frac{1}{2}; \frac{3}{2}; -\frac{1}{3}\Big). \tag{16}$$

The last equality can be obtained by simply expanding the denominator of the t integral via (15) and integrating the resulting series expansion term by term.

The second part of the square brackets in (14) can be handled in a similar fashion. A straightforward calculation employing the Gauss series representation of ${}_2F_1$ yields

$$\int_0^1 \frac{x\,\mathrm{d}x}{[x(1-x)]^{1-\varepsilon/2}}\,{}_2F_1\Big[\varepsilon,1;\frac{\varepsilon}{2};x(1-x)\Big] = \frac{\Gamma^2(\varepsilon/2)}{2\Gamma(\varepsilon)}\,{}_2F_1\Big(\varepsilon,1;\frac{1}{2}+\frac{\varepsilon}{2};\frac{1}{4}\Big)\,. \tag{17}$$

Now we notice that the hypergeometric function with the argument $-1/3$ can be expressed in terms of a similar function with the argument $1/4$ using the linear transformation formula (see, e.g., [54], entry 15.3.4)

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\Big(a,c-b;c;\frac{z}{z-1}\Big)\,. \tag{18}$$

Thus, we get from (6) and (16) along with (18) and (17) our first explicit result

$$i(d) = \frac{1}{2-\varepsilon}\left[{}_2F_1\Big(\varepsilon,1;\frac{1}{2}+\frac{\varepsilon}{2};\frac{1}{4}\Big) - \frac{\varepsilon}{2}\,{}_2F_1\Big(1+\frac{\varepsilon}{2},1;\frac{3}{2};\frac{1}{4}\Big)\right]\,. \tag{19}$$

As $\varepsilon \to 0$, the first hypergeometric function in square brackets reduces to 1, the second term vanishes due to the explicit factor ε, and $i(d) = 1/2 + O(\varepsilon)$, as it should (see (8)). Despite the overall factor $1/(2-\varepsilon)$, there is no singularity at $d=2$ since the combination in square brackets vanishes at $\varepsilon = 2$. In appendix A we derive the explicit result for $i(2)$ in terms of the generalized hypergeometric function ${}_3F_2$.

Successive application of Gauss' relations for contiguous functions ${}_2F_1$ given by entries 7.3.1.16 and 7.3.1.18 of [56] to the second hypergeometric function from (19) leads to an elegant representation

$$i(d) = \frac{1}{2-\varepsilon}\left[2+{}_2F_1\Big(\varepsilon,1;\frac{1}{2}+\frac{\varepsilon}{2};\frac{1}{4}\Big) - 2\,{}_2F_1\Big(\frac{\varepsilon}{2},1;\frac{1}{2};\frac{1}{4}\Big)\right]\,. \tag{20}$$

Here an interesting symmetry of the nominator and denominator parameters in both functions ${}_2F_1$ is observed, while they differ from one another by $\varepsilon/2$.

Let us consider the second function $j(d)$. The calculation of the double integral (7) is somewhat more involved but the experience of the above calculation allows us to be brief.

While considering the inner integration $I_f(z)$ in (10) we have seen that it is useful to get its result in terms of a Gauss hypergeometric function of the argument $z \equiv x(1-x)$, rather than $(z-1)/z$ which is suggested directly by the integrals over y in (6)–(7) and (10). This can be achieved directly at the level of the integral representation. Indeed, changing the initial integration variable via $y = 1/(t+1)$ we obtain for the inner integral $I_j(z)$ from (7)

$$\begin{aligned} I_j(z) \;&=\; \int_0^\infty \frac{t\,\mathrm{d}t}{(t+1)^{3-\varepsilon/2}}\,(1+zt)^{-\varepsilon} = \frac{4}{(4-\varepsilon)(2-\varepsilon)}\,{}_2F_1(\varepsilon,2;\varepsilon/2;z) \\ &\quad - \frac{\varepsilon}{(4-\varepsilon)(2-\varepsilon)}\,\frac{\Gamma^2(\varepsilon/2)}{\Gamma(\varepsilon)}\,z^{1-\varepsilon/2}(1-z)^{-2-\varepsilon/2}\,[4-\varepsilon-2(1-\varepsilon)z]\,. \end{aligned} \tag{21}$$

The two terms of $I_j(z)$ are similar to that generated by the inner y integration in $i(d)$ (see (14)). They can be handled in the same manner inside the external integral over x in (7). According to this decomposition of $I_j(z)$, we obtain two different contributions to the function $j(d)$. We can write

$$j(d) = -\frac{\varepsilon}{(4-\varepsilon)(2-\varepsilon)}\,[j_1(d)+j_2(d)] \tag{22}$$

where

$$j_1(d) = \frac{1-\varepsilon}{3(1+\varepsilon)}\left[{}_2F_1\Big(\varepsilon,1;\frac{3+\varepsilon}{2};\frac{1}{4}\Big) + \frac{4(4+\varepsilon)}{3(1-\varepsilon)} - \frac{4+\varepsilon}{3(3+\varepsilon)}\,{}_2F_1\Big(1+\varepsilon,1;\frac{5+\varepsilon}{2};\frac{1}{4}\Big)\right],$$

$$j_2(d) = \frac{2^{1+\varepsilon}}{3^{\varepsilon/2}}\left[(1-\varepsilon){}_2F_1\Big(\frac{\varepsilon}{2},\frac{1}{2};\frac{3}{2};-\frac{1}{3}\Big) + 2\varepsilon\,{}_2F_1\Big(1+\frac{\varepsilon}{2},\frac{1}{2};\frac{3}{2};-\frac{1}{3}\Big)\right.$$
$$\left. -\frac{8}{9}(2+\varepsilon)\;{}_2F_1\Big(2+\frac{\varepsilon}{2},\frac{1}{2};\frac{3}{2};-\frac{1}{3}\Big)\right].$$

Similarly as before, using the linear transformation (18), we can reduce all the above hypergeometric functions to that with the argument $1/4$. Further, the repeated use of the relations of contiguous functions leaves us only with the same two ${}_2F_1$ functions which appeared in $i(d)$. Moreover, they build up just the combination defining that function. Finally we arrive at a simple expression of $j(d)$ in terms of $i(d)$:

$$j(d) = \frac{1}{3d}\Big[4-(4+d)i(d)\Big]. \tag{23}$$

We recall that two equivalent explicit expressions for the function $i(d)$ are given in (19) and (20).

4. Some results in two dimensions

While the integrals $i(d)$ and $j(d)$ are smooth functions of d in the range of our interest, their calculation at $d=2$ requires some care. In order to calculate $i(2)$ from (19) or (20), we have to introduce a small deviation α from $d=2$ and consider the limits $\alpha\to 0$ of these expressions. Details of the calculation can be found in appendix A.

We obtain the explicit expression for $i(d)$ at $d=2$ in terms of the generalized hypergeometric function ${}_3F_2$:

$$i(2) = \frac{4}{3\sqrt{3}}\,{}_3F_2\Big(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{1}{4}\Big). \tag{24}$$

Using the relation (23) it is easy to obtain the value of $j(2)$:

$$j(2) = -\frac{4}{3\sqrt{3}}\,{}_3F_2\Big(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{1}{4}\Big) + \frac{2}{3}. \tag{25}$$

Searching for some other representations of $i(2)$ we have found the relation (details are given in appendix A)

$${}_3F_2\Big(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{1}{4}\Big) = \mathrm{Cl}_2(\pi/3). \tag{26}$$

Here $\mathrm{Cl}_2(\pi/3) = 1.0149417\ldots$ is the maximum value of the Clausen's function (for more information on this and related functions see [57])

$$\mathrm{Cl}_2(\theta) = -\int_0^\theta \mathrm{d}\theta\ln\Big|2\sin\frac{\theta}{2}\Big| = \sum_{n\geqslant 1}\frac{\sin n\theta}{n^2}. \tag{27}$$

Apparently, the identity (26) does not appear explicitly in the mathematical literature. It supplements the very similar formula

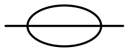
$${}_3F_2\Big(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};-\frac{1}{4}\Big) = \frac{\pi^2}{10} \tag{28}$$

from mathematical tables [56], entry 7.4.6.1.

5. What can we learn from the HEP literature?

As well as in the condensed-matter physics, the calculation of multi-loop Feynman integrals is an important issue in the high-energy physics literature. Although the Feynman diagrams in four space-time dimensions are of main concern in the HEP theory, very often they are calculated by using the dimensional regularization in arbitrary space-time dimension d. It is recognized (see e.g. [37]) that working in general d dimensions is often much easier than directly at $d = 4$.

Actually, the condensed-matter and high-energy physicists are often doing much the same work by pursuing physically different goals. This is, perhaps, most pronounced just in calculations of Feynman integrals. Unfortunately, it frequently happens that the authors of these two communities do not pay enough attention to the achievements gained by one another. The aim of the present section is to quote some HEP results related to calculations of the preceding sections. In doing this we show several useful implications of establishing such contacts. By briefly reviewing some HEP references we would like to draw attention of the cond-mat readers to them, as we feel that these are not well known and appreciated in the cond-mat literature.

Some of the results obtained in [37] are directly related to ours. In particular, the authors of this reference calculate the two-point function $J_3(m^2, q^2) \sim$ . By using the Mellin-Barnes contour integral representation for massive propagators [58] they derive for $J_3(m^2, q^2)$ an infinite series over linear combinations of ${}_3F_2$ functions[2].

The first two coefficients of the small-q^2 expansion of $J_3(1, q^2)$ are simply proportional to our functions $i(d)$ and $j(d)$. This can be seen by noticing (see (3)–(4)) that $J(m^2, q^2)/D^2(m^2) = (\varepsilon^2/4) J_3(m^2, q^2)$ and taking into account the relation (5). Thus we get, first,

$$i(d) = \frac{1}{2-\varepsilon}\,\frac{2}{3}\,C(0) = \frac{1}{2-\varepsilon}\,\frac{2}{3}\left[3^{\frac{1}{2}-\frac{\varepsilon}{2}}\,\frac{\pi\Gamma(\varepsilon)}{\Gamma^2(\varepsilon/2)} + \frac{3}{2}(1-\varepsilon)\,{}_2F_1\Big(\frac{\varepsilon}{2}, 1; \frac{3}{2}; \frac{1}{4}\Big)\right] \tag{29}$$

where we quote the equation (33) from [37] with the usual replacement $\varepsilon \to \varepsilon/2$. Analogous expressions can be also found in [40], equation (4.38), and [59], equation (A.11).

The last expression for $i(d)$ is equivalent to that found in (19) and (20). A direct comparison between (29) and (19)–(20) can be done by applying the contiguous relations for hypergeometric functions with $\varepsilon/2$ in nominator parameters. This leads to an interesting and non-trivial relation for involved Gauss' functions differing in $O(\varepsilon)$ as $\varepsilon \to 0$,

$${}_2F_1\Big(\varepsilon, 1; \frac{1}{2} + \frac{\varepsilon}{2}; \frac{1}{4}\Big) + {}_2F_1\Big(\frac{\varepsilon}{2}, 1; \frac{1}{2}; \frac{1}{4}\Big) - 2 = 3^{-\frac{1}{2}-\frac{\varepsilon}{2}}\,\frac{2\pi\Gamma(\varepsilon)}{\Gamma^2(\varepsilon/2)}\,. \tag{30}$$

The second check is given by the relation

$$j(d) = \frac{4}{d(d-2)(d-4)}\,C(1) \tag{31}$$

where $C(1)$ is the small-q^2 expansion coefficient appearing in equation (31) of [37]. This coefficient is expressible in terms of $C(0)$ by the recurrence relation (37) of [37],

$$C(1) = \frac{d-3}{3}\Big[d - 2 - \frac{d+4}{6}\,C(0)\Big]. \tag{32}$$

Now, taking into account the first equality of (29), we reproduce from (31)–(32) the previously obtained expression (23) for $j(d)$ in terms of $i(d)$. Thus, the formula (23) is a consequence of recurrent relations that hold true for the small-q^2 expansion coefficients of the Feynman integral $J(m^2, q^2; d)$. In turn, these recurrences follow from the differential equation[3] satisfied by the function $J(m^2, q^2; d)$ [37].

[2] Quite recently an explicit expression for $J_3(m^2, q^2)$ has been found [43] in terms of Gauss and Appell hypergeometric functions.

[3] For a recent review on applications of differential equations in calculations of Feymnan integrals see [44].

A full ε expansion of the functions $i(d)$ and $j(d)$ can be obtained with the help of equation (4.16) from [40]. Using it we can write

$$i(d) = \frac{1}{2-\varepsilon}\left\{1 + 3^{-\frac{1}{2}-\frac{\varepsilon}{2}}\,\frac{2\pi\Gamma(\varepsilon)}{\Gamma^2(\varepsilon/2)} - \frac{\varepsilon}{2}\,3^{\frac{1}{2}-\frac{\varepsilon}{2}}\sum_{j\geqslant 0}\frac{\varepsilon^j}{j!}\left[\mathrm{Ls}_{j+1}\left(\frac{2\pi}{3}\right) - \mathrm{Ls}_{j+1}(\pi)\right]\right\}, \tag{33}$$

where

$$\mathrm{Ls}_j(\theta) = -\int\limits_0^\theta \mathrm{d}\theta \ln^{j-1}\left|2\sin\frac{\theta}{2}\right| \tag{34}$$

are the log-sine functions (see [40, 57]). Per definition,

$$\mathrm{Ls}_1(\theta) = -\theta \quad \text{and} \quad \mathrm{Ls}_2(\theta) = \mathrm{Cl}_2(\theta), \tag{35}$$

where $\mathrm{Cl}_2(\theta)$ is the Clausen's function. This special function appeared readily in section 4, in the calculation of the integrals $i(d)$ and $j(d)$ in two dimensions. With the first two terms from the sum, we get the ε expansions

$$i(d) = \frac{1}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon^2}{8} - \frac{\varepsilon^2}{2\sqrt{3}}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + O(\varepsilon^3) \tag{36}$$

and

$$j(d) = -\frac{\varepsilon}{8} - \frac{3\varepsilon^2}{32} + \frac{\varepsilon^2}{3\sqrt{3}}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + O(\varepsilon^3)\,. \tag{37}$$

In deriving these we took into account that $\mathrm{Cl}_2(2\pi/3)$=$2/3\cdot\mathrm{Cl}_2(\pi/3)$, $\mathrm{Cl}_2(\pi)$=0, and used the relation (23) between $i(d)$ and $j(d)$. The non-trivial $O(\varepsilon^2)$ term appearing in curly brackets of (33) agrees with that of equation (33) in [37].

We see that a very special transcendental constant $\mathrm{Cl}_2(\pi/3)$, which is the maximum value of the Clausen's integral $\mathrm{Cl}_2(\theta)$, appears both in the epsilon expansion of $i(d)$ near $d = 4$ and in the explicit expression of $i(d)$ at $d = 2$. This is quite natural because the Feynman integrals of the type considered here obey certain general relations that connect their values in different space dimensions [38, 42, 43, 60].

A simple formula relating the values of Feynman integrals associated with the zero-momentum "sunrise" diagram  with arbitrary masses on the lines at $d = 4-\varepsilon$ and $d = 2-\varepsilon$ has been found in [60]. In the particular equal-mass case with $m^2 = 1$, this formula reads, in our notation,

$$J(1,0;4-\varepsilon) = \frac{3\pi^2}{(1-\varepsilon)(2-\varepsilon)}\left[J(1,0;2-\varepsilon) - \pi^{2-\varepsilon}\Gamma^2\left(\frac{\varepsilon}{2}\right)\right]. \tag{38}$$

Note that the pole terms of the Laurent expansion of $J(4-\varepsilon)$ are contained only in the last, trivial term, and the value $J(2)$ directly enters the finite part of $J(4-\varepsilon)$ (cf. [42], sections 3–4). This feature directly maps onto the connection between the functions $i(4-\varepsilon)$ and $i(2-\varepsilon)$, which can be written as

$$i(4-\varepsilon) = \frac{1}{2-\varepsilon}\left[1 - \frac{3\varepsilon^2}{4(1+\varepsilon)}\,i(2-\varepsilon)\right]. \tag{39}$$

It is straightforward to check that the last relation is indeed satisfied by the function $i(d)$ given, for instance, by (29).

Let us return to the epsilon expansions (36)–(37). They can be compared with that quoted in (8)–(9). In (36) we see the $O(\varepsilon^2)$ term of $i(d)$, which is missing in (8). By comparing the $O(\varepsilon^2)$ terms of $j(d)$ we can make the identification

$$I = \int\limits_0^1 \frac{\mathrm{d}x}{1-x(1-x)}\left\{1 + \frac{\ln[x(1-x)]}{[1-x(1-x)]}\right\} = -\frac{8}{3\sqrt{3}}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right). \tag{40}$$

An attempt to calculate this integral with the help of Mathematica [61] yields an expression in terms of $\psi'(x)$ where $\psi(x)$ is the psi function, the logarithmic derivative of the Gamma function (see e.g. [62]). Thus we get the chain of equalities

$$ {}_3F_2\Big(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{1}{4}\Big) = \mathrm{Cl}_2\Big(\frac{\pi}{3}\Big) = \frac{1}{2\sqrt{3}}\,\psi'\Big(\frac{1}{3}\Big) - \frac{\pi^2}{3\sqrt{3}}\,. \tag{41} $$

The rightmost connection can be also read off from (A.13) of [13].

6. Concluding remarks

Let us return to the two-loop RG functions defined at the beginning of the Section 2. We see that they can be expressed in terms of a single function of $d = 4 - \varepsilon$, say

$$ i(d) = \frac{1}{2-\varepsilon}\left[4 - 3\,{}_2F_1\Big(\frac{\varepsilon}{2},1;\frac{1}{2};\frac{1}{4}\Big) + 3^{-\frac{1}{2}-\frac{\varepsilon}{2}}\,\frac{2\pi\Gamma(\varepsilon)}{\Gamma^2(\varepsilon/2)}\right], \tag{42} $$

which contains only one non-trivial Gauss' hypergeometric function. For example, the function $\gamma_\phi(u)$ is thus given by

$$ \gamma_\phi(u) = -4\,\frac{4-d}{3d}\,\frac{N+2}{(N+8)^2}\,\Big[4 - (4+d)i(d)\Big]\,u^2 + O(u^3). \tag{43} $$

Similar expressions can be also constructed for the remaining RG functions $\beta(u)$ and $\bar{\gamma}_{\phi^2}(u)$.

Everywhere the dependence on space dimension d is given explicitly in a simple parametric form. Obviously, the convenience of such expressions is much better compared to representations in terms of multiple integrals as that in (6)–(7). In principle, this kind of formulas could be used in considering some dimensional expansions not tied to $d = 4$.

Analytical two-loop results are interesting in view of availability of numerical tables of three-loop diagrams of the massive field theory in d dimensions [63]. It could be thought of extending explicit calculations to that order.

Finally, they are also interesting in their own right, as any mathematical results derived in closed form. Moreover, in the course of the present work we were able to find out some interesting mathematical relations given in equations (26), (41) and (30).

A. Explicit results in two dimensions

In order to derive the values of the functions $i(d)$ and $j(d)$ at $d = 2$ we have to take $\varepsilon = 2 - \alpha$ and consider the corresponding limits with $\alpha \to 0$. Thus we write $i(d)$ in (19) as

$$ i(d) = i_1(\alpha) + i_2(\alpha) \tag{44} $$

with

$$ i_1(\alpha) = \frac{1}{\alpha}\left[{}_2F_1\Big(2-\alpha,1;\frac{3}{2}-\frac{\alpha}{2};\frac{1}{4}\Big) - {}_2F_1\Big(2-\frac{\alpha}{2},1;\frac{3}{2};\frac{1}{4}\Big)\right] \tag{45} $$

and

$$ i_2(\alpha) = \frac{1}{2}\,{}_2F_1\Big(2-\frac{\alpha}{2},1;\frac{3}{2};\frac{1}{4}\Big)\,. \tag{46} $$

There exists a simple $\alpha \to 0$ limit of $i_2(\alpha)$,

$$ i_2(0) = \frac{1}{3} + \frac{2\pi}{9\sqrt{3}}\,, \tag{47} $$

while each of the hypergeometric functions in $i_1(\alpha)$ has to be expanded to $O(\alpha)$ in order to give the finite $i_1(0)$. To this end, we found it useful to transform these functions via the linear transformation formula ([54], entry 15.3.3)

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1\Big(c-a,c-b;c;z\Big)\,. \tag{48} $$

Thus we get

$$i_1(\alpha)=\Big(\frac{3}{4}\Big)^{-3/2+\alpha/2}\frac{1}{\alpha}\left[{}_2F_1\Big(-\frac{1}{2}+\frac{\alpha}{2},\frac{1}{2}-\frac{\alpha}{2};\frac{3}{2}-\frac{\alpha}{2};\frac{1}{4}\Big)-{}_2F_1\Big(-\frac{1}{2}+\frac{\alpha}{2},\frac{1}{2};\frac{3}{2};\frac{1}{4}\Big)\right].$$

Here the both Gauss functions are of the type (11), and hence can be represented in terms of simple integrals appearing in (10). We have

$$i_1(\alpha)=\Big(\frac{3}{4}\Big)^{-3/2+\alpha/2}\frac{1}{2\alpha}\left[(1-\alpha)I_1-I_2\right]=\Big(\frac{3}{4}\Big)^{-3/2+\alpha/2}\frac{1}{2\alpha}\left[(I_1-I_2)-\alpha I_1\right] \tag{49}$$

where

$$I_1=\int_0^1 \mathrm{d}t\, t^{-\frac{1}{2}-\frac{\alpha}{2}}(1-t/4)^{\frac{1}{2}-\frac{\alpha}{2}} \quad \text{and} \quad I_2=\int_0^1 \mathrm{d}t\, t^{-\frac{1}{2}}(1-t/4)^{\frac{1}{2}-\frac{\alpha}{2}}. \tag{50}$$

At $\alpha=0$, the integrals I_1 and I_2 are both equal to

$$I_0=\frac{\pi}{3}+\frac{\sqrt{3}}{2}\,, \tag{51}$$

while the difference between them I_1-I_2 is of the order of α:

$$I_1-I_2=\int_0^1\frac{\mathrm{d}t}{\sqrt{t}}(1-t/4)^{\frac{1}{2}-\frac{\alpha}{2}}(t^{-\frac{\alpha}{2}}-1)=-\frac{\alpha}{2}\int_0^1\frac{\mathrm{d}t}{\sqrt{t}}(1-t/4)^{\frac{1}{2}}\ln t+O(\alpha^2). \tag{52}$$

Using Mathematica [61] we get

$$I_1-I_2=\alpha\left[\frac{\pi}{6}+\frac{\sqrt{3}}{4}+{}_3F_2\Big(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{1}{4}\Big)\right]+O(\alpha^2). \tag{53}$$

Combining (49), (53), and (51) we obtain

$$i_1(0)=-\frac{2\pi}{9\sqrt{3}}-\frac{1}{3}+\frac{4}{3\sqrt{3}}\,{}_3F_2\Big(\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{1}{4}\Big)\,, \tag{54}$$

and finally, together with $i_2(0)$ from (47), we arrive at $i(2)$ given in equation (24) of the main text.

We did not find in mathematical tables any summation formula for the generalized hypergeometric function ${}_3F_2$ appearing in the last two equations. So we looked for the series representations of this function

$$\sum_{k\geqslant 0}\frac{(1/2)_k}{k!(2k+1)^2}\frac{1}{4^k}=4\sum_{k\geqslant 0}\frac{(2k)!}{(k!)^2(2k+1)^2}\Big(\frac{1}{4}\Big)^{2k+1}=\sum_{k\geqslant 0}\frac{(2k-1)!!}{(2k)!!(2k+1)^2}\frac{1}{4^k}\,. \tag{55}$$

The middle version of the series appears, indeed, as a partial case of the formula 5.2.13.7 in [62] expressed in terms of the function $\arcsin(2x)$ and its derivative. But, unfortunately, a simple numerical check at $x=1/4$ revealed the incorrectness of this entry.

In order to find the right answer, we considered the series expansion

$$g(x)\equiv\sum_{k\geqslant 0}\frac{(1/2)_k}{k!(2k+1)^2}\frac{x^{2k+1}}{4^k}\,, \tag{56}$$

such that $g(1)={}_3F_2(1/2,1/2,1/2;3/2,3/2;1/4)$ and $g(0)=0$. Its derivative with respect to x, multiplied by x, is

$$x\,g'(x)=\sum_{k\geqslant 0}\frac{(1/2)_k}{k!(2k+1)}\frac{x^{2k+1}}{4^k}\,. \tag{57}$$

Differentiating once again we get

$$[x\,g'(x)]' = \sum_{k\geqslant 0} \frac{(1/2)_k}{k!}\frac{x^{2k}}{4^k} = \frac{1}{\sqrt{1-x^2/4}}\,. \tag{58}$$

Integrating back leads us to

$$x\,g'(x) = 2\arcsin\left(\frac{x}{2}\right) \tag{59}$$

and

$$g(x) = \int \mathrm{d}x \frac{2}{x}\arcsin\left(\frac{x}{2}\right) = \mathrm{Cl}_2\big(2\arcsin\left(\frac{x}{2}\right)\big) + \arcsin\left(\frac{x}{2}\right)\,\ln x, \tag{60}$$

where $\mathrm{Cl}_2(z)$ is the Clausen function [57]. In deriving the last equality we used the entry 1.7.4.13 from [62] and the fact that the integration constants in the both last indefinite integrations vanish. Here we also observe that the neighboring entry 1.7.4.12 of [62] involves a series expansion of the same type as the rightmost one in (55), giving, along with 1.7.4.11, the corrected version of the false formula 5.2.13.7 in the same reference.

Now, recalling that at $x = 1$, the function $g(x)$ reproduces the hypergeometric series ${}_3F_2$ from (54), we come to the relation (26) given in the main text.

References

1. Fisher M.E., Rev. Mod. Phys., 1974, **46**, No. 4, 597–616.
2. Wilson K.G., Fisher M.E., Phys. Rev. Lett., 1972, **28**, No. 4, 240–243, 1972.
3. Wilson K.G., Kogut J., Phys. Rep. C, 1974, **12**, No. 2, 75–200.
4. C. Domb, Green M.S., eds., Phase Transitions and Critical Phenomena, vol. 6. Academic Press, London, 1976.
5. Kleinert H., Schulte-Frohlinde V., Critical Properties of ϕ^4-Theories. World Scientific, Singapore, 2000.
6. Parisi G., J. Stat. Phys., 1980, **23**, No. 1, 49–81.
7. Baker G.A., Jr., Nickel B.G., Green M.S., Meiron D.I., Phys. Rev. Lett., 1976, **36**, No. 23, 1351–1354.
8. Le Guillou J.C., Zinn-Justin J., Phys. Rev. Lett., 1977, **39**, No. 2, 95–98.
9. Baker G.A., Jr., Nickel B.G., Meiron D.I., Phys. Rev. B, 1978, **17**, No. 3, 1365–1374.
10. Le Guillou J.C., Zinn-Justin J., Phys. Rev. B, 1980, **21**, No. 9, 3976–3998.
11. Bervillier C., Shpot M., Phys. Rev. B, 1992, **46**, No. 2, 955–968.
12. Diehl H.W., Shpot M., Nucl. Phys. B, 1998, **528**, No. 3, 595–647.
13. Pelissetto A., Vicari E., Phys. Rev. B, 2000, **62**, No. 10, 6393–6409.
14. Pelissetto A., Rossi P., Vicari E., Nucl. Phys. B, 2001, **607**, No. 3, 605–634.
15. Pelissetto A., Vicari E., Phys. Rep., 2002, **368**, 549–727.
16. Jug G., Phys. Rev. B, 1983, **27**, No. 1, 609–612.
17. Holovatch Yu., Shpot M., J. Stat. Phys., 1992, **66**, No. 3/4, 867–883.
18. Abe R., Hikami S., Progr. Theor. Phys., 1973, **49**, No. 2, 442–452.
19. Ma S., Phys. Rev. A, 1973, **7**, No. 6, 2172–2187.
20. Ma S., J. Math. Phys., 1974, **15**, No. 11, 1866–1891.
21. Moshe M., Zinn-Justin J., Phys. Rep., 2003, **385**, 69–228.
22. Antonenko S.A., Sokolov A.I., Phys. Rev. E, 1995, **51**, No. 3, 1894–1898.
23. Newman K.E., Riedel E.K., Phys. Rev. B, 1982, **25**, No. 1, 264–280.
24. Wegner F.J., Houghton A., Phys. Rev. A, 1973, **8**, No. 1, 401–412.
25. Pinn K., Pordt A., Wieczerkowski C., J. Stat. Phys., 1994, **77**, No. 5/6, 1572–9613.
26. Ballhausen H., Berges J., Wetterich C., Phys. Lett. B, 2004, **582**, No. 2, 144–150.
27. Bagnuls C., Bervillier C., Phys. Rep., 2001, **348**, No. 1–2, 91–157.
28. Berges J., Tetradis N., Wetterich C., Phys. Rep., 2002, **363**, No. 4–6, 223–386.
29. Le Guillou J.C., Zinn-Justin J., J. de Physique, 1987, **48**, No. 1, 19–25.
30. O'Dwyer J., Osborn H., Ann. Phys., 2008, **323**, No. 8, 1859–1898.
31. Polchinski J., Nucl. Phys. B, 1984, **231**, No. 2, 269–295.
32. Hasenfratz A., Hasenfratz P., Nucl. Phys. B, 1986, **270**, 687–701.
33. Delamotte B., Holovatch Yu., Ivaneyko D., Mouhanna D., Tissier M. Preprint arXiv:cond-mat/0609285v1, 2006.
34. Pelissetto A., Vicari E. Preprint arXiv:cond-mat/0610113v1, 2006.

35. Delamotte B., Holovatch Yu., Ivaneyko D., Mouhanna D., Tissier M., Preprint arXiv:cond-mat/0610613v1, 2006.
36. Delamotte B., Holovatch Yu., Ivaneyko D., Mouhanna D., Tissier M., J. Stat. Mech., 2008, **2008**, No. 03, P03014.
37. Broadhurst D.J., Fleischer J., Tarasov O.V., Z. Phys. C, 1993, **60**, No. 2, 287–301.
38. Tarasov O.V., Phys. Rev. D, 1996, **54**, No. 10, 6479–6490.
39. Davydychev A.I., Delbourgo R., J. Math. Phys., 1998, **39**, No. 9, 4299–4334.
40. Davydychev A.I., Kalmykov M.Yu., Nucl. Phys. B, 2001, **605**, No. 1-3, 266–318.
41. Fleischer J., Jegerlehner F., Tarasov O.V., Nucl. Phys. B, 2003, **672**, No. 1-2, 303–328.
42. Laporta S., Remiddi E., Nucl. Phys. B, 2005, **704**, No. 1–2, 349–386.
43. Tarasov O.V., Phys. Lett. B, 2006, **638**, No. 2-3, 195–201.
44. Argeri M., Mastrolia P., Int. J. Mod. Phys. A, 2007, **22**, No. 24, 4375–4436.
45. Groote S., Körner J.G., Pivovarov A.A., Ann. Phys., 2007, **322**, No. 10, 2374–2445.
46. Shpot M.A., J. Math. Phys., 2007, **48**, No. 12, 123512–1–13.
47. Tarasov O.V., Phys. Lett. B, 2008, **670**, No. 1, 67–72.
48. Kalmykov M.Yu., Kniehl B.A., Nucl. Phys. B, 2009, **809**, No. 3, 365–405.
49. Callan C.G., Phys. Rev. D, 1970, **2**, No. 8, 1541–1547.
50. Symanzik K., Comm. Math. Phys., 1970, **18**, No. 3, 227–246.
51. Symanzik K., Comm. Math. Phys., 1971, **23**, No. 1, 48–86.
52. Brezin E., Le Guillou J.C., Zinn-Justin J., Phys. Rev. D, 1973, **8**, No. 2, 434–440.
53. Nickel B.G., Meiron D.I., Baker G.A., Jr., Compilation of 2-pt and 4-pt Graphs for Continuum Spin Models. Guelph University Report, 1977.
54. Abramowitz M., Stegun I.A., Handbook of Mathematical Functions. National Bureau of Standards, New York, 1972.
55. Amit D.J., Field Theory, the Renormalization Group, and Critical Phenomena, 2nd ed. World Scientific, Singapore, 1984.
56. Prudnikov A., Brychkov Yu.A., Marichev O.I., Integrals and Series. More Special Functions, vol. 3. Gordon and Breach, New York, 1990.
57. Lewin L., Polylogarithms and Associated Functions. Elsevier, New York, 1981.
58. Boos E.E., Davydychev A.I., Teor. Mat. Fiz., 1991, **89**, 56–72 [Sov. Phys. Theor. Math. Phys., 1992, **89**, 1052–1064].
59. Pelissetto A., Vicari E., Nucl. Phys. B, 2000, **575**, No. 3, 579–598.
60. Davydychev A.I., Tausk J.B., Phys. Rev. D, 1996, **53**, No. 12, 7381–7384.
61. Mathematica 4.1, a product of Wolfram Research.
62. Prudnikov A., Brychkov Yu.A., Marichev O.I., Integrals and Series. Elementary Functions, vol. 1. Gordon and Breach, New York, 1986.
63. Holovatch Yu., Krokhmal's'kii T., J. Math. Phys., 1994, **35**, No. 8, 3866–3880.

Двопетлеві РГ-функції масивної теорії поля ϕ^4 у довільних вимірностях простору

М.А. Шпот

Інститут фізики конденсованих систем НАН України, 79011 Львів, вул. Свєнціцького, 1

Двопетлеві інтеграли Фейнмана масивної теорії поля типу ϕ^4_d розраховані в явному вигляді для довільних вимірностей простору d. Відповідні ренормгрупові функції записані у компактному вигляді у термінах Гаусових гіпергеометричних функцій. Продемонстровано ряд цікавих та корисних співвідношень для цих інтегралів, а також для деяких спеціальних математичних функцій та констант.

Ключові слова: *інтеграли Фейнмана, спеціальні функції, ренормалізаційна група, теорія поля*