Condensed Matter Physics 2008, Vol. 11, No 2(54), pp. 361–370

CONDENSED MATTER PHYSICS

Differential functional von Foerster equations with renewal

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Received January 31, 2008

Natural iterative methods converge to the exact solution of a differential-functional von Foerster-type equation which describes a single population dependent on its past time and state densities as well as on its total size. On the lateral boundary we impose a renewal condition.

Key words: *iterative method, differential functional, Hale operator*

PACS: *87.10.+e, 82.39.-k, 82.39.Rk*

1. Introduction

The paper is dedicated to the memory of Professor A. Lasota. Von Foerster and Volterra-Lotka equations arise in biology, medicine and chemistry, [1,7,11,12,14]. The independent variables x_j and the unknown function u stand for certain features and densities, respectively. It follows from this natural interpretation that $x_j \geqslant 0$ and $u \geqslant 0$. Von Foerster model is essentially nonlocal, because it contains the total size of population $\int u(t,x)\mathrm{d}x$.

Existence results for certain von Foerster type problems have been established by means of the Banach contraction principle, the Schauder fixed point theorem or iterative methods, see [2] and [3–5]. Nonlocal terms always cause huge problems. Satisfactory conditions for convergence of iterative methods were provided in [9], where (for the sake of simplicity) some boundary data were prescribed. This forced additional restrictions on the (tangential!) flow of bicharacteristics near the lateral boundary.

In the present paper we generalize the $L^\infty \cap L^1$-convergence results of [9] to the case of renewal boundary conditions with natural assumptions on the flow of bicharacteristics. An associate result to [9] on fast convergent quasi-linearization methods has been published in [8]. The renewal condition of the form

$$u(t,0) = \int_0^\infty k(t,x,y)\, u(t,x)\mathrm{d}x$$

is interpreted as giving birth to young individuals at the age 0 by mature individuals ($0 < x < +\infty$). In A. Lasota's investigations there were included some delay effects like $\int u(t-r,x)\mathrm{d}x$. Such cases are also available using the method of the present paper. Investigations of finite difference schemes on large meshes with strongly nonlocal functionals were analyzed in [15]. Since the renewal case is complex, we arrive at its reduction to the one with given data on the lateral boundary. In fact, the unknown renewal part is expressed by a Green function. This approach can be extended to many species models. There is also a difference between the renewal case and [9]: the renewal involves such coupling of the unknown function that excludes monotone iterations.

1.1. Formulation of the differential problem.

Let $\tau = (\tau_1,\ldots,\tau_n) \in \mathbf{R}_+^n$, $\tau_0 > 0$, where $\mathbf{R}_+ := [0,+\infty)$. Define

$$B \;=\; [-\tau_0, 0] \times [-\tau,\tau], \quad \text{where} \quad [-\tau,\tau] \;=\; [-\tau_1,\tau_1] \times \cdots \times [-\tau_n,\tau_n]$$

and

$$E_0 \;=\; [-\tau_0, 0] \times \mathbf{R}^n, \qquad E = [0,a] \times \mathbf{R}^n, \qquad (a>0), \qquad \partial E = \{(t,x) \in E:\; x_1 \cdot \cdots \cdot x_n = 0\}\,.$$

For each function w defined on $[-\tau_0, a]$, we have the Hale functional w_t (see [6]), which is the function defined on $[-\tau_0, 0]$ by

$$w_t(s) \;=\; w(t+s), \qquad (s \in [-\tau_0, 0]).$$

For each function u defined on $E_0 \cup E$, we similarly write a Hale-type functional $u_{(t,x)}$, defined on B by

$$u_{(t,x)}(s,y) \;=\; u(t+s, x+y) \qquad \text{for } (s,y) \in B.$$

Let $\Omega_0 \;=\; E \times C\left([-\tau_0, a], \mathbf{R}_+\right)$ and $\Omega \;=\; E \times C\left(B, \mathbf{R}_+\right) \times C\left([-\tau_0, a], \mathbf{R}_+\right)$. Take $v : E_0 \to \mathbf{R}_+$ and

$$c_j : \Omega_0 \to \mathbf{R}, \qquad \lambda : \Omega \to \mathbf{R} \qquad (j = 1, \ldots, n).$$

Consider the differential-functional equation

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} c_j\left(t, x, z[u]_t\right) \frac{\partial u}{\partial x_j} \;=\; u(t,x)\, \lambda\left(t, x, u_{(t,x)}, z[u]_t\right), \tag{1}$$

where

$$z[u](t) := \int_{\mathbf{R}^n_+} u(t,y) \mathrm{d}y, \qquad t \in [-\tau_0, a], \tag{2}$$

with the initial conditions

$$u(t,x) = v(t,x), \qquad (t,x) \in E_0\,, \qquad x = (x_1, \ldots, x_n) \in \mathbf{R}^n_+\,, \tag{3}$$

and with the renewal condition

$$u(t,x) = \int_{\mathbf{R}^n_+} k(t,x,y)\, u(t,y) \mathrm{d}y, \qquad (t,x) \in \partial E, \qquad y \in \mathbf{R}^n_+\,, \tag{4}$$

where $k : \partial E \times \mathbf{R}^n_+ \to \mathbf{R}^n_+$. We are looking for Caratheodory's solutions to (1)–(4), see [2] and [10]. The functional dependence includes a possible delayed and integral dependence of the Volterra type. The Hale functional $z[u]_t$ takes into consideration the whole population within the time interval $[t-\tau_0, t]$, whereas the Hale-type functional $u_{(t,x)}$ shows the dependence on the density u locally in a left neighbourhood of (t,x). For simplicity we assume in the paper that $n=1$. Notice that it is possible to extend the result to the case $n>1$ with quite technical multiple integrals on the lateral boundary.

2. Bicharacteristics

First, for a given function $z \in C([-\tau_0, a], \mathbf{R}_+)$, consider the bicharacteristic equations for problem (1), (3):

$$\eta'(s) \;=\; c\left(s, \eta(s), z_s\right), \qquad \eta(t) \;=\; x. \tag{5}$$

Denote by $\eta = \eta[z](\cdot; t, x) = (\eta_1[z](\cdot; t,x), \ldots, \eta_n[z](\cdot; t, x))$ the bicharacteristic curve passing through $(t,x) \in E$, i.e., the solution to problem (5). We consider its maximal (left) existence domain to be an interval $[\alpha(t,x), t]$, where $\alpha(t,x) = 0$ or $0 < \alpha(t,x) \leqslant t$. This alternative splits E into two parts $E_0[z]$ and $E_+[z]$. Next, we consider the following equation

$$\frac{\mathrm{d}}{\mathrm{d}\, s} u(s, \eta[z](s;t,x)) = u(s, \eta[z](s;t,x))\, \lambda(s, \eta[z](s;t,x), u_{(s,\eta[z](s;t,x))}, z_s) \tag{6}$$

with the initial condition

$$u(0, \eta[z](0;t,x)) = v(0, \eta[z](0;t,x)) \text{ for } (t,x) \in E_0[z], \tag{7}$$

and (with the brief notation $\alpha = \alpha(t,x)$)

$$u(\alpha, \eta[z](\alpha;t,x)) = \tilde{v}(\alpha, \eta[z](\alpha;t,x)) \quad \text{for} \quad (t,x) \in E_+[z]. \tag{8}$$

In the latter equation (6) the existence of a suitable extension $\tilde{v}$ of v to the lateral boundary is complicated. We discuss this topic later on.

For any given function $z \in C([-\tau_0, a], \mathbf{R}_+)$, a solution of equation (6) along bicharacteristics (5) is a solution of (1). The initial conditions (3) and (7) or (8) correspond to each other.

Assume that:

(V0) $v \in CB\,(E_0, \mathbf{R}_+)$ (non-negative, bounded and continuous function);

(V1) $z[v] \in C\,([-\tau_0, 0], \mathbf{R}_+)$, where

$$z[v](t) \;=\; \int_{\mathbf{R}^n} v(t,x)\mathrm{d}x;$$

(V2) the function v satisfies the Lipschitz condition

$$|v(t,x) - v(t,\bar{x})| \leqslant L_v \|x - \bar{x}\| \quad \text{on} \quad E_0$$

with some constant $L_v > 0$;

(C0) $c_j : \Omega_0 \to \mathbf{R}_+$ are positive, continuous and

$$\|c(t,x,q) - c(t,\bar{x},\bar{q})\| \leqslant L_c\|x - \bar{x}\| + L_c^*\|q - \bar{q}\|.$$

A continuous function $\sigma : [0,a] \times \mathbf{R}_+ \to \mathbf{R}_+$ is said to be a *Perron comparison function* if $\sigma(t,0) \equiv 0$ and the differential problem $y' = \sigma(t,y)$, $y(0) = 0$ has the only zero solution. We call it *uniform* if σ, multiplied by any positive constant, is also a Perron comparison function. We call it *monotone* if σ is non-decreasing in the second variable.

(Λ0) $\lambda : \Omega \to \mathbf{R}$ is continuous in (t,x,w,q) and

$$|\lambda(t,x,w,q) - \lambda(t,\bar{x},\bar{w},\bar{q})| \leqslant M_\lambda\, \sigma(t, \|x - \bar{x}\| + \|w - \bar{w}\| + \|q - \bar{q}\|),$$

where $\sigma : [0,a] \times \mathbf{R}_+ \to \mathbf{R}_+$ is a monotone, uniform Perron comparison function;

(Λ1) there exists a function $L_\lambda \in L^1([0,a], \mathbf{R}_+)$, such that

$$\lambda(t,x,w,q) \leqslant L_\lambda(t)$$

for $(t,x) \in E, w \in C(B, \mathbf{R}_+^m), q \in C([-\tau_0, a], \mathbf{R}_+^m)$.

Denote

$$W(t,x,w,q) = \lambda(t,x,w,q) + \,\mathrm{tr}\,\partial_x c(t,x,q)$$

for $(t,x) \in E, w \in C(B, \mathbf{R}_+), q \in C([-\tau_0, a], \mathbf{R}_+)$, where $\mathrm{tr}\,\partial_x c$ stands for the trace of the matrix $\partial_x c = [\partial_{x_k} c_j]_{j,k=1,\ldots,n}$.

(W0) There exists $M_W \in \mathbf{R}_+$, such that

$$|W(t,x,w,q) - W(t,\bar{x},\bar{w},\bar{q})| \leqslant M_W\;\sigma(t, \|x - \bar{x}\| + \|w - \bar{w}\| + \|q - \bar{q}\|),$$

where $\sigma : [0,a] \times \mathbf{R}_+ \to \mathbf{R}_+$ is a monotone, uniform Perron comparison function.

(W1) There exists a function $L_W \in L^1([0,a], \mathbf{R}_+)$, such that

$$W(t,x,w,q) \leqslant L_W(t)$$

for $(t,x) \in E, w \in C(B, \mathbf{R}_+), q \in C([-\tau_0, a], \mathbf{R}_+)$.

(K0) The kernel $k : \partial E \times \mathbf{R}_+^n \to \mathbf{R}_+^n$ is bounded and continuous, and $a \cdot \|k\|_\infty \cdot \|c\| < 1$.

The latter condition states that the length of the interval $[0,a]$ is sufficiently small. This assumption is superfluous if $k(t,x,y) = 0$.

Lemma 2.1 *If the conditions (V0), $(\Lambda 1)$, (K0) are satisfied, then any solution u of equation (6) has the estimate*

$$0 \leqslant u(t,x) \leqslant \|v\|_\infty \exp\left(\int_0^t L_\lambda(s)\mathrm{d}s\right) \quad on \quad E_0[z]$$

and

$$0 \leqslant u(t,x) \leqslant \|v\|_\infty \exp\left(\int_0^t (L_\lambda(s)+L_W(s))\mathrm{d}s\right)\cdot \frac{\|k\|_\infty}{1-a\cdot\|k\|_\infty\cdot\|c\|_\infty} \quad on\ E_+[z].$$

Proof. The first inequality is standard. The second one will be explained in the following subsection. □

2.1. The fixed point equation.

Let

$$\begin{aligned} Z(t) \quad = \quad & \max_{-\tau_0\leqslant s\leqslant 0}\|v(s,\cdot)\|_1 \exp\left(\int_0^t L_W(s)\mathrm{d}s\right) \\ & +t\cdot\|c\|_\infty\cdot\|v\|_\infty \exp\left(\int_0^t (L_\lambda(s)+L_W(s))\mathrm{d}s\right)\cdot\frac{\|k\|_\infty}{1-a\cdot\|k\|_\infty\cdot\|c\|_\infty} \end{aligned} \tag{9}$$

where we put $L_W(s)=0$ for $s\in[-\tau_0,0]$, and

$$\mathcal{Z}=\{z\in C([-\tau_0,a],\mathbf{R}_+):\ z(t)\leqslant Z(t)\}. \tag{10}$$

Consider the operator $\mathcal{T}\colon\mathcal{Z}\to\mathcal{Z}$ given by the formula

$$\mathcal{T}[z](t)=\int_{\mathbf{R}^n_+} u[z](t,x)\mathrm{d}x \qquad \text{for} \quad \mathrm{t}\geqslant 0, \tag{11}$$

where $u=u[z]\in C^1(B,\mathbf{R}_+)$ is the solution of (6)–(7) with the initial condition $u[z](t,x)=v(t,x)$ on E_0. The function $u=u[z]$ has the following integral representation

$$u[z](t,x)=v(\alpha,\eta(\alpha))\exp\left(\int_\alpha^t \lambda\left(s,\eta(s),u_{(s,\eta(s))},z_s\right)\mathrm{d}s\right), \tag{12}$$

where $\eta(s)=\eta[z](s;t,x)$ and $\alpha=\alpha(t,x)$ (α depends on z). By Lemma 2.1, we write (11) in the following way

$$\mathcal{T}[z](t)=\int_{\mathbf{R}^n_+} v(\alpha,\eta(\alpha))\exp\left(\int_\alpha^t \lambda\left(s,\eta(s),u_{(s,\eta(s))},z_s\right)\mathrm{d}s\right)\mathrm{d}x \tag{13}$$

for $t\geqslant 0$. The bicharacteristics admit the group property:

$$y=\eta[z](0;t,x)\Longleftrightarrow \eta[z](s;t,x)=\eta[z](s;0,y),$$

that is: any bicharacteristic curve passing through the points $(0,y)$ and (t,x) has the same value at $s\in[0,a]$.

If we change the variables $y=\eta[z](0;t,x)$, then, by the Liouville theorem, the Jacobian $J=\det\left[\dfrac{\partial c}{\partial x}\right]$ is given for $\alpha=\alpha(t,x)$ by the formula

$$J(\alpha;t,x)=\exp\left(-\int_\alpha^t \mathrm{tr}\,\partial_x c(s,\eta_i[z](s;0,y),z_s)\mathrm{d}s\right).$$

Hence (13) can be written in the form

$$\begin{aligned}\mathcal{T}[z](t) &= \int_{\mathbf{R}_+^n} v(0,y)\exp\left(\int_0^t W(s,\eta(s),u_{(s,\eta(s))},z_s)\mathrm{d}s\right)\mathrm{d}y \\ &\quad + \int_{S_t} u(\alpha,\eta(\alpha;t,x))\,\exp\left(\int_\alpha^t \lambda(s,\eta(s),u_{(s,\eta(s))},z_s)\mathrm{d}s\right)\mathrm{d}x, \end{aligned} \tag{14}$$

where $\eta(s)=\eta[z](s;0,y)$, $\alpha=\alpha(t,x)$ and S_t is the set of $x\in\mathbf{R}_+^n$ such that $\alpha(t,x)>0$.

Lemma 2.2 *If the conditions (V0), (V1), (W1), (K0) are satisfied, then*

$$0\leqslant \mathcal{T}[z](t)\leqslant Z(t)<+\infty \quad \text{for} \quad t\in[0,a],$$

where Z is given by (9).

Proof. This assertion follows directly from (14) and Assumptions (V0), (V1) and (W1). □

The respective fixed point equation for bicharaceristics $\eta=\eta[z]$ has the form

$$\eta(s;t,x)=x-\int_s^t c(\zeta,\eta(\zeta;t,x),z_\zeta)\mathrm{d}\zeta. \tag{15}$$

Lemma 2.3 *If Assumption (C0) is satisfied and $z,\bar z\in\mathcal{Z}$, then*

$$\|\eta[z](s;t,x)-\eta[\bar z](s;t,x)\|\leqslant \int_s^t L_c^*\|z_\zeta-\bar z_\zeta\|\mathrm{e}^{L_c(\zeta-s)}\mathrm{d}\zeta.$$

Lemma 2.4 *Under the Assumptions (V0), (V1), (C0), (K0), for each $z\in\mathcal{Z}$, there exists the unique continuous function $\tilde v:\partial E\to\mathbf{R}_+$ which satisfies the renewal condition.*

Proof. The renewal condition (4) for $(t,x)\in\partial E$ can be rewritten as follows

$$u(t,x)=\int_{\mathbf{R}_+^n\setminus S_t} k(t,x,y)\,u(t,y)+\int_{S_t} k(t,x,y)\,u(t,y)\mathrm{d}y.$$

The first term is a bounded operator of v, see (12). From (12), the second term is equal to

$$\int_{S_t} k(t,x,y)\,v(\alpha,\eta(\alpha))\exp\left(\int_\alpha^t \lambda\left(s,\eta(s),u_{(s,\eta(s))},z_s\right)\mathrm{d}s\right)\mathrm{d}y,$$

where $\eta(s)=\eta[z](s;t,y)$ and $\alpha=\alpha(t,y)$. By (K0) the second term is small (has the norm less than 1). □

The above lemma explains the estimate of Lemma 2.1. The next statement is crucial in our paper.

Lemma 2.5 *Under all previous assumptions, any solution z of the fixed point equation for (14) has the representation*

$$z(t)=\int_{R_+^n} v(0,y)\,G(t,y;z)\mathrm{d}y,$$

where the Green function G has the same estimate as the first kernel in (14), multiplied by some constant.

Proof. The assertion follows from a Neumann series expansion for u aided by the renewal condition. □

3. The iterative method.

Define the iterative method by $z^{(k+1)} = \mathcal{T}[z^{(k)}]$ with an arbitrary function $z^{(0)} \in \mathcal{Z}$, where the class $\mathcal{Z}$ is defined by (10). We prove its uniform convergence under natural assumptions on the given functions. The algorithm splits into three stages:

1. finding bicharacteristics $\eta^{(k)} = \eta[z^{(k)}]$, given by (15),
2. finding $u^{(k)} = u[z^{(k)}]$ as a solution of (12),
3. calculating $z^{(k+1)} = \mathcal{T}[z^{(k)}]$ by means of (13) or (14). In this way there are given the integral equations

$$\begin{aligned}
\eta^{(k)}(s;t,x) &= x - \int_s^t c(\zeta, \eta^{(k)}(\zeta;t,x), z_\zeta^{(k)})\mathrm{d}\zeta\,, \\
u^{(k)}(t,x) &= v(\alpha, \eta^{(k)}(\alpha;t,x)) \exp\left(\int_\alpha^t \lambda\left(Q^{(k)}(s)\right)\mathrm{d}s\right), \\
z^{(k+1)}(t) &= \int_{\mathbf{R}^n} v(0,y) \exp\left(\int_0^t W\left(R^{(k)}(s)\right)\mathrm{d}s\right)\mathrm{d}y \\
&\quad + \int_{S_t} u(\alpha, \eta^{(k)}(\alpha;t,x)) \exp\left(\int_\alpha^t \lambda\left(Q^{(k)}(s)\right)\mathrm{d}s\right)
\end{aligned}$$

where $\alpha = \alpha(t,x)$ (depends on $z^{(k)}$) and

$$Q^{(k)}(s) = \left(s, \eta^{(k)}(s;t,x), u^{(k)}_{(s,\eta^{(k)}(s;t,x))}, z_s^{(k)}\right), \qquad R^{(k)}(s) = \left(s, \eta^{(k)}(s;0,y), u^{(k)}_{(s,\eta^{(k)}(s;0,y))}, z_s^{(k)}\right).$$

Theorem 3.1 *If $z^{(0)} \in \mathcal{Z}$ and Assumptions $(V0)$–$(V2)$, $(C0)$, $(\Lambda 0)$, $(\Lambda 1)$, $(W0)$, $(W1)$, $(K0)$ are satisfied and there are $K \in \mathbf{R}_+$, $\theta \in (0,1]$ such that*

$$\sigma(t,r) \leqslant K\, t^{\theta-1} p\, r^{1-1/p} \quad \textit{for } p \geqslant 2, \tag{16}$$

then the iterative method $z^{(k+1)} = \mathcal{T}[z^{(k)}]$ is well defined in $\mathcal{Z}$ and uniformly converges to the unique fixed point $z = \mathcal{T}[z]$, locally, that is: on a sufficiently small $[0,a]$.

Remark 3.1 *The technical condition (16) is fulfilled in the Lipschitz case ($\sigma(t,r) = L\,r$) as well as the simplest nonlinear Perron comparison functions such as $\sigma(t,r) = L\,r\ln(1+1/r)$. Its formulation also includes weak singularities, e.g. $\sigma(t,r) = t^{-1/2}L\,r$ and $\sigma(t,r) = t^{-1/2}L\,r\ln(1+1/r)$.*

Proof. (of Theorem 3.1) Denote

$$\Delta z^{(k)} = z^{(k+1)} - z^{(k)},\ \Delta\eta^{(k)} = \eta^{(k+1)} - \eta^{(k)},\ \Delta u^{(k)} = u^{(k+1)} - u^{(k)}.$$

Then we have the estimates

$$\begin{aligned}
\|\Delta\eta^{(k)}(s;t,x)\| &\leqslant \int_s^t L_c^* \|\Delta z_\zeta^{(k)}\| \mathrm{e}^{L_c(\zeta-s)}\mathrm{d}\zeta\,, \\
|\Delta u^{(k)}(t,x)| &\leqslant C\|\Delta\eta^{(k)}(0;t,x)\| \exp\left(\int_0^t L_\lambda(s)\mathrm{d}s\right) \\
&\quad + C\exp\left(\int_0^t L_\lambda(s)\mathrm{d}s\right)\int_0^t M_\lambda \sigma\left(s, P^{(k)}(s;t,x)\right)\mathrm{d}s, \\
|\Delta z^{(k+1)}(t)| &\leqslant C\int_0^t M_W \sigma\left(s, P^{(k)}(s;t,x)\right)\mathrm{d}s,
\end{aligned}$$

where $P^{(k)}(s;t,x) = \|\Delta\eta^{(k)}(s;t,x)\| + \|\Delta u^{(k)}\|_s + \|\Delta z^{(k)}\|_s$. In the above estimate the constant C is generic (depending on the data).

Denote $\hat{L}_\lambda = \int_0^a L_\lambda(s)\mathrm{d}s$ and

$$\Psi^{(k)}(s,t) = \bar{\psi}^{(k)}(s) + \psi^{(k)}(s) + \int_s^t L_c^* \mathrm{e}^{L_c a} \psi^{(k)}(\zeta)\mathrm{d}\zeta\,.$$

Consider the comparison equations

$$\bar{\psi}^{(k)}(t) = C\int_0^t L_c^* \mathrm{e}^{L_c a + \hat{L}_\lambda} \psi^{(k)}(s)\mathrm{d}s + C\,e^{\hat{L}_\lambda}\int_0^t M_\lambda \sigma\left(s, \Psi^{(k)}(s,t)\right)\mathrm{d}s, \tag{17}$$

$$\psi^{(k+1)}(t) = C\int_0^t M_W \sigma\left(s, \Psi^{(k)}(s,t)\right)\mathrm{d}s \tag{18}$$

with $\psi^{(0)}(t) = Z(t)$ and

$$\begin{aligned}\bar{\psi}^{(0)}(t) &= C\,\exp\left(\int_0^t L_W(s)\mathrm{d}s\right) + C\int_0^t L_c^* \mathrm{e}^{L_c a + \hat{L}_\lambda} Z(s)\mathrm{d}s \\ &+ C\mathrm{e}^{\hat{L}_\lambda}\int_0^t M_\lambda \sigma\left(s, \bar{\psi}^{(0)}(s) + Z(s) + \int_s^t L_c^* \mathrm{e}^{L_c a} Z(\zeta)\mathrm{d}\zeta\right)\mathrm{d}s.\end{aligned} \tag{19}$$

The remaining part of the proof is split into several auxiliary lemmas.

Lemma 3.1 *Under the assumptions of Theorem 3.1 there is $a_0 \in (0,a]$ such that $|\Delta u^{(k)}(t,x)| \leqslant \bar{\psi}^{(k)}(t)$, $|\Delta z^{(k)}(t)| \leqslant \psi^{(k)}(t)$,*

$$\|\Delta\eta^{(k)}(s;t,x)\| \leqslant \int_s^t L_c^* \mathrm{e}^{L_c a} \psi^{(k)}(\zeta)\mathrm{d}\zeta$$

on $[0,a_0] \times \mathbf{R}_+$, and the sequences $\{\psi^{(k)}\}$, $\{\bar{\psi}^{(k)}\}$ are non-decreasing in k.

Lemma 3.2 *Under the assumptions of Theorem 3.1 the estimate*

$$\int_0^t \sigma(s, As^l + Bt^{l+1})\mathrm{d}s \leqslant t^{l+\theta-l/p}\,p\,K\theta^{-1}\left[\frac{A}{\theta+l} + \frac{B\,a}{\theta}\right]^{1-1/p}$$

holds.

Lemma 3.3 *Under the assumptions of Theorem 3.1 the sequences $\{\psi^{(k)}\}$ and $\{\bar{\psi}^{(k)}\}$ tend uniformly to 0 as $k \to +\infty$.*

Proof. With some constants M, M^* and c_a, dependent on the data, the equation

$$\hat{\psi}(t) = M\int_0^t \hat{\psi}(s)\mathrm{d}s + M^*\int_0^t \sigma\left(s, \hat{\psi}(s) + c_a\int_s^t \hat{\psi}(\zeta)\mathrm{d}\zeta\right)\mathrm{d}s \tag{20}$$

describes a comparison function $\hat{\psi}$ with respect to $\psi + \bar{\psi}$, where

$$\psi(t) = \lim_{k\to\infty}\psi^{(k)}(t), \qquad \bar{\psi}(t) = \lim_{k\to\infty}\bar{\psi}^{(k)}(t).$$

One can prove by induction on $k = 0,1,\ldots$ that $\hat{\psi}(t) \leqslant \hat{C}_k t^{\theta/2}$ and $\hat{C}_k a^{\theta/2} \to 0$ as $k \to +\infty$, provided that the interval $[0,a]$ is sufficiently small. Take an arbitrary $\hat{C}_0$ which estimates $\hat{\psi}(t)$. Applying Lemma 3.2 with $p=2$ to (20), we get

$$\hat{\psi}(t) \leqslant Mt\hat{C}_0 + M^* t^\theta 2K\theta^{-1}\left[\frac{\hat{C}_0\,(1+c_a a)}{\theta}\right]^{1/2} \leqslant t^{\theta/2}\hat{C}_1,$$

where

$$\hat{C}_1 = Ma^{1-\theta/2}\hat{C}_0 + a^{\theta/2}2K\theta^{-1}\left[\frac{\hat{C}_0\,(1+c_a a)}{\theta}\right]^{1/2}.$$

Suppose that the desired estimate holds for some $k \geqslant 1$. Applying Lemma 3.2 with $p = 2k$ to (20), we get

$$\hat{\psi}(t) \leqslant M\frac{t^{1+k\theta/2}}{1+k\theta/2}\hat{C}_k + M^* t^{(k+1)\theta/2}2K\theta^{-1}\left[\frac{\hat{C}_k}{\theta+k\theta/2} + \frac{c_a\hat{C}_k}{\theta\,(1+k\theta/2)}\right]^{1-1/(2k)},$$

hence $\hat{\psi}(t) \leqslant t^{(k+1)\theta/2}\hat{C}_{k+1}$, where

$$\hat{C}_{k+1} = M\hat{C}_k\frac{a^{1-\theta/2}}{1+k\theta/2} + M^*\,2K\theta^{-1}\left[\frac{\hat{C}_k}{\theta+k\theta/2} + \frac{c_a\hat{C}_k}{\theta\,(1+k\theta/2)}\right]^{1-1/(2k)}.$$

The constants $\hat{C}_k$ have an upper estimate of the form AQ^k, thus $\hat{\psi}(t) \equiv 0$ in a neighbourhood of 0 (because $\hat{\psi}(t) \leqslant AQ^k t^{k\theta/2}$).

Lemma 3.4 *Under the assumptions of Theorem 3.1 the sequences $\{z^{(k)}\}$, $\{u^{(k)}\}$, $\{\eta^{(k)}\}$ tend uniformly to $z, u[z], \eta[z]$ such that $z = \mathcal{T}[z]$.*

Proof. We intend to find the following estimates

$$\psi^{(k)}(t) \leqslant C_k t^{l_k}, \quad \bar{\psi}^{(k)}(t) \leqslant \bar{C}_k t^{l_k},$$

where the series $\sum_k C_k t^{l_k}$ is convergent in a neighbourhood of 0. The assertion is seen if we replace the comparison equations (17)–(18) by the inequalities

$$\bar{C}_k t^{l_k} \geqslant C\int_0^t L_c^* \mathrm{e}^{L_c a+\hat{L}_\lambda} C_k s^{l_k}\mathrm{d}s + C\mathrm{e}^{\hat{L}_\lambda}\int_0^t M_\lambda \sigma\left(s, (C_k+\bar{C}_k)s^{l_k} + L_c^*\mathrm{e}^{L_c a}C_k t^{l_k+1}/(l_k+1)\right)\mathrm{d}s,$$

$$C_{k+1}t^{l_{k+1}} \geqslant C\int_0^t M_W \sigma\left(s, (C_k+\bar{C}_k)s^{l_k} + L_c^*\mathrm{e}^{L_c a}C_k t^{l_k+1}/(l_k+1)\right)\mathrm{d}s$$

with suitable $C_0 t^{l_0} \geqslant Z(a)$ and with some

$$\bar{C}_0 \;\geqslant\; a\,C\,L_c^*\mathrm{e}^{L_c a+\hat{L}_\lambda}Z(a) + C\mathrm{e}^{\hat{L}_\lambda}\,M_\lambda\int_0^a \sigma\left(s, \bar{C}_0 + Z(a) + a\,L_c^*\mathrm{e}^{L_c a}Z(a)\right)\mathrm{d}s.$$

If we put

$$l_0 = 0, \qquad p_0 = 2/\theta, \qquad l_k = k\theta/2, \qquad p_k = 4\,k \qquad \text{for } k = 1,2,\ldots$$

and exploit Lemma 3.2, then $C_k, \bar{C}_k$ can be defined as making the series $\sum_k C_k t^{l_k}$ convergent, hence the series $\psi^{(0)} + \psi^{(2)} + \ldots$ uniformly converges, and $z^{(k)}$ has a limit, which is continuous.

Corollary 3.1 *If Assumptions $(V0)$–$(V2)$, $(C0)$, $(\Lambda 0)$, $(\Lambda 1)$, $(W0)$ and $(W1)$ are satisfied, then there exists a unique solution of problem (1)–(3), locally with respect to t.*

3.1. The Lipschitz case.

Suppose that Assumptions $(V0)$–$(V2)$, $(C0)$, $(\Lambda 1)$ and $(W1)$, (K0) formulated in Section 2, are valid. We modify some assumptions on the functions λ and W as follows:

$(\tilde{\Lambda}0)$ $\lambda : \Omega \to \mathbf{R}$ is continuous in (t,x,w,q) and there exists a function $L \in L^1([0,a],\mathbf{R}_+)$, such that

$$|\lambda(t,x,w,q) - \lambda(t,\bar{x},\bar{w},\bar{q})| \leqslant L(t)(\|x-\bar{x}\| + \|w-\bar{w}\| + \|q-\bar{q}\|);$$

($\tilde{W}$0) $W : \Omega \to \mathbf{R}$ and there exists a function $L \in L^1([0,a], \mathbf{R}_+)$, such that

$$|W(t,x,w,q) - W(t,\bar{x},\bar{w},\bar{q})| \leqslant L(t)(\|x-\bar{x}\| + \|w-\bar{w}\| + \|q-\bar{q}\|).$$

Using the same notation as in the proof of Theorem 3.1, we have the estimates of the form

$$\psi^{(k)}(t) \leqslant \frac{Q^{k+1}\left(\int_0^t \Delta(s)\mathrm{d}s\right)^k}{k\,!}, \quad t \in [0,a],$$

where Q is a generic constant. Hence the sequences $\{\psi^{(k)}\}$ and $\{\bar{\psi}^{(k)}\}$ tend uniformly to 0 as $k \to +\infty$.

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Диференціальні функціональні рівняння фон Фьорстера з відновленням

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Отримано 31 січня 2008 р.

Природні ітеративні методи збігаються до точного розв'язку диференціально-функціонального рівняння типу фон Фьорстера, що описує популяцію, залежну від своїх минулих густин станів і від загального розміру. На бічній границі ми накладаємо умову відновлення.

Ключові слова: *ітеративний метод, диференціальний функціонал, оператор Гейла*

PACS: *87.10.+e, 82.39.-k, 82.39.Rk*