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Thermodynamic Green functions in theory of superconductivity

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A general theory of superconductivity is formulated within the thermodynamic Green function method for various types of pairing mediated by phonons, spin fluctuations, and strong Coulomb correlations in the Hubbard and t-J models. A rigorous Dyson equation for matrix Green functions is derived in terms of a self-energy as a many-particle Green function. By applying the noncrossing approximation for the self-energy, a closed self-consistent system of equations is obtained, similar to the conventional Eliashberg equations. A brief discussion of superconductivity mediated by kinematic interaction with an estimation of a superconducting transition temperature in the Hubbard model is given.

Key words: *Green functions, theory of superconductivity, strong electron correlations*

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1. Introduction

Thermodynamic, retarded and advanced Green functions (GFs) introduced by Bogoliubov and Tyablikov in a seminal work [1] was used soon afterwards in the studies of superconductivity within the Bardeen-Cooper-Schrieffer (BCS) model [2] as discussed by Zubarev in his famous review on the double-time GF in the statistical physics [3]. At the same time, Zubarev formulated a theory of superconductivity for an electron-phonon system based on the equation of motion method for GFs [4]. The paper was submitted for publication only two months following the paper by Eliashberg [5] where the temperature diagram technique was used in describing superconductivity in electron-phonon systems. Zubarev formulation did not attract much attention in succeeding years, while the Eliashberg theory was frequently used and his formulation became known as the Eliashberg (or Migdal-Eliashberg) theory of superconductivity for electron-phonon systems. The real advantage of the Eliashberg formulation is that it permits to consider a strong coupling limit by using a skeleton diagram technique. In Zubarev formulation based on a subsequent differentiation of GFs over the same time, it is impossible to employ the skeleton diagram technique. However, by differentiating the GFs over two times, this problem can be easily solved and the Eliashberg type equations can be formulated in a very simple and transparent way for any model of electron-boson interaction as was shown by Vujičić et al. [6]. This method within the Mori-type projection technique was used later in order to study superconductivity in the t-J model [7–9] and the Hubbard model [10]. In those models written in terms of the Hubbard operators, the application of the diagram technique is rather involved and demands a summation of an infinite set of diagrams (see, e.g. [11]). A systematic investigation of superconductivity within the t-J model by the Hubbard operator diagram technique was performed by Izymov et al. [12,13].

In the present paper we give a general formulation of a theory of superconductivity by applying the equation of motion method to the thermodynamic GFs. We consider several models where superconducting pairing is mediated by electron-phonon and spin-fluctuation interactions, or by a kinematic interaction originating from strong Coulomb correlations, as in the Hubbard and t-J models. In our formulation the matrix self-energy operator, derived as a many-particle GF, is calculated in the noncrossing approximation (NCA), or equivalently, the self-consistent Born approximation (SCBA). In this approximation vertex corrections are neglected as in the

Migdal-Eliashberg theory. For the electron-phonon system the vertex corrections are small in the adiabatic approximation, as shown by Migdal. There are no small parameters for spin-fluctuation or kinematic interactions and vertex correction may be important in obtaining quantitative results. However, in the NCA the self-energy is calculated self-consistently enabling us to consider a strong coupling limit which plays an essential role both in renormalization of quasiparticle spectra and in superconducting pairing. Thus, this approach can be considered as the first reasonable approximation.

The paper is organized as follows. In the next section the Dyson equation is derived by using the equation of motion method for a general fermion-boson type interaction. A self-consistent system of equations for the Hubbard model and the t-J model are obtained in section 3 and an estimation of superconducting $T_{\rm c}$ is given in the weak coupling approximation. Conclusions are presented in section 4.

2. Eliashberg equations for fermion-boson models

2.1. Dyson equation

Let us consider a general model for electron interaction with phonons and spin fluctuations:

$$H = \sum_{p} \varepsilon(p) a^{\dagger}_{p} a_{p} + \sum_{p,p'} W(p,p')\, a^{\dagger}_{p}\, a_{p'}\,, \tag{1}$$

where $p = (\mathbf{p},\sigma)$ denotes the momentum $\mathbf{p}$ and the spin $\sigma = +(\uparrow), -(\downarrow)$ of an electron with the energy $\varepsilon(p) = \epsilon(p) - \mu$ measured from the chemical potential μ. The matrix element of the interaction has two contributions:

$$W(p,p') = \delta_{\sigma,\sigma'} V_{\rm ph}(\mathbf{p}-\mathbf{p}')\, \rho_{\mathbf{p}-\mathbf{p}'} + V_{\rm sf}(\mathbf{p}-\mathbf{p}') \sum_{\alpha} S^{\alpha}_{\mathbf{p}-\mathbf{p}'}\, \hat{\tau}^{\alpha}_{\sigma,\sigma'}\,. \tag{2}$$

The first term is electron scattering on lattice charge fluctuations $\rho_{\mathbf{q}}$ (phonons) and the second is scattering on spin fluctuations $S^{\alpha}_{\mathbf{q}}$ where $\hat{\tau}^{\alpha}_{\sigma,\sigma'}$ is the Pauli matrices. The scalar product of spin operators in (2) is convenient to write in the form $\sum_{\alpha} S^{\alpha}_{\mathbf{q}}\, \hat{\tau}^{\alpha} = S^{z}_{\mathbf{q}}\, \hat{\tau}^{z} + S^{+}_{\mathbf{q}}\, \hat{\tau}^{-} + S^{-}_{\mathbf{q}}\, \hat{\tau}^{+}$, where $S^{\pm}_{\mathbf{q}} = S^{x}_{\mathbf{q}} \pm {\rm i} S^{x}_{\mathbf{q}}$ and $\hat{\tau}^{\pm} = (1/2)(\hat{\tau}^{x} \pm {\rm i}\hat{\tau}^{y})$. In this notation the interaction with spin fluctuations reads

$$H_{\rm sf} = \sum_{\mathbf{p},\mathbf{p}'} V_{\rm sf}(\mathbf{q}) \left\{ S^{z}_{\mathbf{q}}\, (a^{\dagger}_{\mathbf{p}\uparrow}\, a_{\mathbf{p}'\uparrow} - a^{\dagger}_{\mathbf{p}\downarrow}\, a_{\mathbf{p}'\downarrow}) + S^{+}_{\mathbf{q}}\, a^{\dagger}_{\mathbf{p}\downarrow}\, a_{\mathbf{p}'\uparrow} + S^{-}_{\mathbf{q}}\, a^{\dagger}_{\mathbf{p}\uparrow}\, a_{\mathbf{p}'\downarrow} \right\}. \tag{3}$$

To discuss a singlet superconducting pairing within the model (1) we consider the matrix thermodynamic GF in Zubarev's notation [3]

$$\mathsf{G}_{\mathbf{p},\sigma}(t-t') = \langle\langle \Psi_{\mathbf{p},\sigma}(t) | \Psi^{\dagger}_{\mathbf{p},\sigma}(t') \rangle\rangle = \int_{-\infty}^{+\infty} \frac{{\rm d}\omega}{2\pi}\, \mathsf{G}_{\mathbf{p},\sigma}(\omega) {\rm e}^{-{\rm i}\omega(t-t')}, \tag{4}$$

in terms of the Nambu operators:

$$\Psi_{\mathbf{p},\sigma} = \begin{pmatrix} a_{\mathbf{p},\sigma} \\ a^{\dagger}_{-\mathbf{p}\bar{\sigma}} \end{pmatrix}, \qquad \Psi^{\dagger}_{\mathbf{p}\sigma} = \left(a^{\dagger}_{\mathbf{p},\sigma}\; a_{-\mathbf{p}\bar{\sigma}} \right), \tag{5}$$

where $\bar{\sigma} = -\sigma$. The Fourier transform of the matrix GF (4) can be written as

$$\mathsf{G}_{\mathbf{p}\sigma}(\omega) \equiv \begin{pmatrix} G^{11}_{p}(\omega) & G^{12}_{p}(\omega) \\ G^{21}_{p}(\omega) & G^{22}_{p}(\omega) \end{pmatrix} \equiv \begin{pmatrix} G_{\mathbf{p}\sigma}(\omega) & F_{\mathbf{p}\sigma}(\omega) \\ F^{\dagger}_{\mathbf{p}\sigma}(\omega) & -G_{-\mathbf{p}\bar{\sigma}}(-\omega) \end{pmatrix}, \tag{6}$$

where $G_{p}(\omega) \equiv G^{11}_{p}(\omega) = \langle\langle a_{\mathbf{p},\sigma} | a^{\dagger}_{\mathbf{p},\sigma} \rangle\rangle_{\omega}$ and $F_{\mathbf{p}\sigma}(\omega) \equiv G^{12}_{p}(\omega) = \langle\langle a_{\mathbf{p},\sigma} | a_{-\mathbf{p}\bar{\sigma}} \rangle\rangle_{\omega}$ are the normal and the anomalous components of the GF, respectively.

By using equations for the Heisenberg operators $\Psi_{\mathbf{p},\sigma}(t)$ we derive the first equation of motion for the GF (4) in the form:

$$\mathsf{G}_{\mathbf{p},\sigma}(\omega) = \mathsf{G}^{(0)}_{\mathbf{p},\sigma}(\omega) + \mathsf{G}^{(0)}_{\mathbf{p},\sigma}(\omega) \sum_{p'} \left\langle\left\langle W(p,p')\Psi_{\mathbf{p}',\sigma'} \mid \Psi^{\dagger}_{\mathbf{p},\sigma}\right\rangle\right\rangle_{\omega} , \tag{7}$$

where we introduced the zero-order GF

$$\mathsf{G}^{(0)}_{\mathbf{p}\sigma}(\omega) = (\omega\hat{\tau}_0 - \varepsilon(p)\hat{\tau}_3)^{-1}. \tag{8}$$

A conventional Pauli matrix representation for the (2×2) matrix GFs (6) will be used: $\hat{\tau}_0$ is the unity matrix, $\hat{\tau}_3 = \hat{\tau}^z$, $\hat{\tau}_1 = \hat{\tau}^x$. By differentiating over the second time t' the many-particle GF in (7) $\langle\langle W(p,p')\Psi_{\mathbf{p}',\sigma'} \mid \Psi^{\dagger}_{\mathbf{p},\sigma}(t')\rangle\rangle$ we get the second equation of motion for the GFs

$$\left\langle\left\langle W(p,p')\,\Psi_{\mathbf{p}',\sigma'} \mid \Psi^{\dagger}_{\mathbf{p},\sigma}\right\rangle\right\rangle_{\omega} = \left\langle\left\langle W(p,p')\,\Psi_{\mathbf{p}',\sigma'} \middle| \sum_{p''} \Psi^{\dagger}_{\mathbf{p}'',\sigma''} W^{\dagger}(p,p'')\right\rangle\right\rangle_{\omega} \mathsf{G}^{(0)}_{\mathbf{p},\sigma}(\omega) , \tag{9}$$

where we assumed that there is no spin ordering and therefore an average value of the interaction matrix vanishes: $\langle W(p,p')\rangle = 0$. By introducing the scattering matrix

$$\mathsf{T}_{\mathbf{p},\sigma}(\omega) = \sum_{p',p''} \left\langle\left\langle W(p,p')\Psi_{\mathbf{p}',\sigma'} \middle| \Psi^{\dagger}_{\mathbf{p}'',\sigma''} W^{\dagger}(p,p'')\right\rangle\right\rangle_{\omega} , \tag{10}$$

we can solve the system of equations (7), (9) in the form:

$$\mathsf{G}_{\mathbf{p},\sigma}(\omega) = \mathsf{G}^{(0)}_{\mathbf{p},\sigma}(\omega) + \mathsf{G}^{(0)}_{\mathbf{p},\sigma}(\omega)\,\mathsf{T}_{\mathbf{p},\sigma}(\omega)\mathsf{G}^{(0)}_{\mathbf{p},\sigma}(\omega)\,. \tag{11}$$

The self-energy operator $\mathbf{\Sigma}_{\mathbf{p},\sigma}(\omega)$ is defined by a *proper part* of the scattering matrix (10) which cannot be cut by the single-particle GF $\mathsf{G}^{(0)}_{\mathbf{p},\sigma}(\omega)$:

$$\mathsf{T}_{\mathbf{p},\sigma}(\omega) = \mathbf{\Sigma}_{\mathbf{p},\sigma}(\omega) + \mathbf{\Sigma}_{\mathbf{p},\sigma}(\omega)\mathsf{G}^{(0)}_{\mathbf{p},\sigma}(\omega)\mathsf{T}_{\mathbf{p},\sigma}(\omega). \tag{12}$$

This results in the Dyson equation for the matrix GF (6)

$$\mathsf{G}_{\mathbf{p},\sigma}(\omega) = \left\{\mathsf{G}^{(0)}_{\mathbf{p},\sigma}(\omega)^{-1} - \mathbf{\Sigma}_{\mathbf{p},\sigma}(\omega)\right\}^{-1}. \tag{13}$$

In comparison with the conventional diagram technique, where the self-energy in the Dyson equation is defined in terms of the full vertex and the full single-particle GF, in our approach the self-energy is given by an *exact* many-particle GF

$$\mathbf{\Sigma}_{\mathbf{p},\sigma}(\omega) = \begin{pmatrix} \Sigma^{11}_{\mathbf{p},\sigma}(\omega) & \Sigma^{12}_{\mathbf{p},\sigma}(\omega) \\ \Sigma^{21}_{\mathbf{p},\sigma}(\omega) & \Sigma^{22}_{\mathbf{p},\sigma}(\omega) \end{pmatrix} = \sum_{p',p''} \left\langle\left\langle W(p,p')\Psi_{\mathbf{p}',\sigma'} \mid \Psi^{\dagger}_{\mathbf{p}'',\sigma''} W^{\dagger}(p,p'')\right\rangle\right\rangle^{\text{proper}}_{\omega} , \tag{14}$$

which describes many-body inelastic scattering processes of electrons on charge and spin fluctuations.

2.2. Non-crossing approximation

To obtain a closed system of equations for the GF (13) and the self-energy (14) one should consider an approximation for the many-particle GF in (14). Let us consider the non-crossing approximation (NCA) which is also known as the self-consistent Born approximation (SCBA) or as the mode-coupling approximation (MCA). In the NCA, the propagation of the Fermi excitations described by operators $\Psi_{\mathbf{p},\sigma}$ and the Bose-like excitations described by operators $\rho_{\mathbf{q}}$ and $S^{\alpha}_{\mathbf{q}}$ in the matrix element of the interaction (2) in the many-particle GF in (14) are assumed to be independent of each other. This is given by a decoupling of the corresponding operators in the time-dependent correlation functions as follows

$$\begin{aligned}
&\left\langle W(p,p')(t)\Psi_{\mathbf{p}',\sigma'}(t)\middle|\Psi^{\dagger}_{\mathbf{p}'',\sigma''}W^{\dagger}(p,p'')\right\rangle \\
&\quad\simeq\ \left\langle W(p,p')(t)\mid W^{\dagger}(p,p'')\right\rangle\left\langle\Psi_{\mathbf{p}',\sigma'}(t)\mid\Psi^{\dagger}_{\mathbf{p}'',\sigma''}\right\rangle \\
&\quad=\ \delta_{\sigma,\sigma'}\delta_{\sigma,\sigma''}|V_{\mathrm{ph}}(\mathbf{p}-\mathbf{p}')|^2\left\langle\rho_{\mathbf{p}-\mathbf{p}'}(t)|\rho^{\dagger}_{\mathbf{p}-\mathbf{p}'}\right\rangle\left\langle\Psi_{\mathbf{p}',\sigma}(t)\mid\Psi^{\dagger}_{\mathbf{p}',\sigma}\right\rangle \\
&\quad+\ \delta_{\sigma',\sigma''}|V_{\mathrm{sf}}(\mathbf{p}-\mathbf{p}')|^2\sum_{\alpha}\left\langle S^{\alpha}_{\mathbf{p}-\mathbf{p}'}(t)|S^{\dagger\,\alpha}_{\mathbf{p}-\mathbf{p}'}\right\rangle\hat{\tau}_{\alpha}(\sigma\sigma')\hat{\tau}_{\alpha}(\sigma'\sigma)\left\langle\Psi_{\mathbf{p}',\sigma'}(t)\mid\Psi^{\dagger}_{\mathbf{p}',\sigma'}\right\rangle. \qquad (15)
\end{aligned}$$

To calculate the time-dependent correlation functions in (15) we use the spectral representation:

$$\langle\Psi_{\mathbf{p}',\sigma}(t)\mid\Psi^{\dagger}_{\mathbf{p}',\sigma}\rangle = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{1+\mathrm{e}^{-\beta\omega}}\,\mathrm{e}^{-\mathrm{i}\omega t}\left[-\frac{1}{\pi}\mathrm{Im}\left\langle\left\langle\Psi_{\mathbf{p}',\sigma}\mid\Psi^{\dagger}_{\mathbf{p}',\sigma}\right\rangle\right\rangle_{\omega+i\delta}\right], \qquad (16)$$

$$\langle B_{\mathbf{q}}(t)|B^{\dagger}_{\mathbf{q}}\rangle = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{1-\mathrm{e}^{-\beta\omega}}\,\mathrm{e}^{-\mathrm{i}\omega t}\left[-\frac{1}{\pi}\mathrm{Im}\,\langle\langle B_{\mathbf{q}}|B^{\dagger}_{\mathbf{q}}\rangle\rangle_{\omega+i\delta}\right], \qquad (17)$$

where for the Bose-like operators, $B_{\mathbf{q}}=\rho_{\mathbf{q}},\,S^{\alpha}_{\mathbf{q}}$, we use the retarded commutator GFs. Thus, we obtain the following results for the self-energy (14) in the NCA:

$$\Sigma_{\mathbf{p},\sigma}(\omega) = \frac{1}{N}\sum_{\mathbf{p}'}\int_{-\infty}^{+\infty}\mathrm{d}z\,K^{(+)}(\omega,z|\mathbf{p}-\mathbf{p}')\left[-\frac{1}{\pi}\mathrm{Im}G_{\mathbf{p}'\sigma}(z)\right], \qquad (18)$$

$$\Phi_{\mathbf{p},\sigma}(\omega) = \frac{1}{N}\sum_{\mathbf{p}'}\int_{-\infty}^{+\infty}\mathrm{d}zK^{(-)}(\omega,z|\mathbf{p}-\mathbf{p}')\left[-\frac{1}{\pi}\mathrm{Im}F_{\mathbf{p}'\sigma}(z)\right], \qquad (19)$$

where we introduced the normal $\Sigma_{\mathbf{p},\sigma}(\omega)=\Sigma^{11}_{\mathbf{p},\sigma}(\omega)=-\Sigma^{22}_{-\mathbf{p}\bar{\sigma}}(-\omega)$ and the anomalous $\Phi_{\mathbf{p},\sigma}(\omega)=\Sigma^{12}_{\mathbf{p},\sigma}(\omega)=\left(\Sigma^{21}_{\mathbf{p}\sigma}(\omega)\right)^{*}$ components of the self-energy (14). The latter defines the frequency dependent gap function. We emphasize that in the self-energy (18), (19) the spectral functions are defined by the imaginary parts of the full electronic GF (13) and the corresponding bosonic GFs. The kernel of the integral equations for the self-energy has the same form as in the Eliashberg theory:

$$K^{(\pm)}(\omega,z|\mathbf{q}) = \int_{-\infty}^{+\infty}\mathrm{d}\Omega\,\frac{\tanh(z/2T)+\coth(\Omega/2T)}{2\,(\omega-z-\Omega)}\,\lambda^{(\pm)}(\mathbf{q},\Omega)\,. \qquad (20)$$

The electron-electron interaction mediated by charge (phonons) and by spin fluctuations for the normal and the anomalous self-energy components is defined by the functions (see Appendix, (74), (75))

$$\lambda^{(\pm)}(\mathbf{q},\omega) = |V_{\mathrm{sf}}(\mathbf{q})|^2\left[-\frac{3}{\pi}\mathrm{Im}\langle\langle S^z_{\mathbf{q}}|S^z_{-\mathbf{q}}\rangle\rangle_{\omega}\right]\pm|V_{\mathrm{ph}}(\mathbf{q})|^2\left[-\frac{1}{\pi}\mathrm{Im}\langle\langle\rho_{\mathbf{q}}|\rho^{\dagger}_{\mathbf{q}}\rangle\rangle_{\omega}\right]. \qquad (21)$$

It is assumed that the dynamical spin susceptibility for the spin-fluctuation scattering in a paramagnetic state is isotropic and therefore

$$\chi^{\pm}(\mathbf{q},\omega) = 2\chi^{zz}(\mathbf{q},\omega) = -2\,\langle\langle S^z_{\mathbf{q}}|S^z_{-\mathbf{q}}\rangle\rangle_{\omega}\,.$$

The derived equations for the self-energy (18), (19) are equivalent to the Eliashberg equations [5] for phonon-mediated electron coupling and spin-fluctuation coupling considered within the temperature diagram technique (see, e.g., [14] where both the spin-singlet and spin-triplet pairings within the *s*-*d* model were studied for various types of magnetic ordering). In particular, to obtain only a single-phonon contribution to the self-energy one should consider a linear approximation for the dynamical structure factor of the lattice vibrations $S(\mathbf{q},\omega) = \langle\langle \rho_{\mathbf{q}} | \rho^{\dagger}_{\mathbf{q}} \rangle\rangle_{\omega}$ in (21). Effects of the long-range Coulomb interaction can be also considered within this method as described in [6].

The imaginary Matsubara frequency representation used in the temperature diagram technique follows from the equation

$$\frac{\tanh(z/2T) + \coth(\Omega/2T)}{2\,(\mathrm{i}\omega_n - z - \Omega)} = -T \sum_m \frac{1}{\mathrm{i}\omega_m - z}\, \frac{1}{\mathrm{i}(\omega_n - \omega_m) - \Omega}\,, \tag{22}$$

where $\mathrm{i}\omega_n = \mathrm{i}\pi T(2n+1)$. By using the spectral representation for the retarded GFs,

$$G_{\mathbf{p}'\sigma}(\mathrm{i}\omega_m) = -\frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}z}{\mathrm{i}\omega_m - z}\, \mathrm{Im}\, G_{\mathbf{p}'\sigma}(z), \tag{23}$$

we obtain the imaginary frequency representation for the self-energy (18), (19):

$$\Sigma^{11(12)}_{\mathbf{p},\sigma}(\mathrm{i}\omega_n) = -\frac{T}{N} \sum_{m,\mathbf{p}'} G^{11(12)}_{\mathbf{p}'\sigma}(\mathrm{i}\omega_m)\, \lambda^{(\pm)}\,(\mathbf{p} - \mathbf{p}', \mathrm{i}\omega_n - \mathrm{i}\omega_m)\,, \tag{24}$$

where $(+)$, $(-)$ in the interaction function (21) refer to the normal $\Sigma^{11}_p(\mathrm{i}\omega_n)$ and anomalous $\Sigma^{12}_p(\mathrm{i}\omega_n)$ components of the self-energy, respectively.

A formal solution of the Dyson equation (13) for the matrix GF can be written in the conventional Eliashberg form:

$$\mathsf{G}_{\mathbf{p},\sigma}(\omega) = \frac{\omega Z_p(\omega)\hat{\tau}_0 + (\varepsilon(p) + \xi_p(\omega))\hat{\tau}_3 + \Phi_{\mathbf{p}\sigma}(\omega)\hat{\tau}_1}{(\omega Z_p(\omega))^2 - (\varepsilon(p) + \xi_p(\omega))^2 - \mid \Phi_{\mathbf{p}\sigma}(\omega) \mid^2}\,, \tag{25}$$

where the odd and even in frequency ω self-energy components determine the Eliashberg functions

$$\omega(1 - Z_p(\omega)) = \frac{1}{2}[\Sigma_{\mathbf{p}\sigma}(\omega) - \Sigma_{\mathbf{p}\sigma}(-\omega)], \quad \xi_p(\omega) = \frac{1}{2}[\Sigma_{\mathbf{p}\sigma}(\omega) + \Sigma_{\mathbf{p}\sigma}(-\omega)]. \tag{26}$$

By writing the matrix self-energy (14) in terms of the Eliashberg functions

$$\mathbf{\Sigma}_{\mathbf{p},\sigma}(\omega) = \omega(1 - Z_p(\omega)\hat{\tau}_0 + \xi_p(\omega)\hat{\tau}_3 + \Phi_{\mathbf{p}\sigma}(\omega)\hat{\tau}_1, \tag{27}$$

we obtain an equivalent to (18), (19) self-consistent system of integral equations

$$\omega(1 - Z_{\mathbf{p},\sigma}(\omega)) \;=\; \frac{1}{N}\sum_{\mathbf{p}'} \int\limits_{-\infty}^{+\infty} \mathrm{d}\omega_1\, K^{(+)}(\omega,\omega_1|\mathbf{p} - \mathbf{p}') \left[-\frac{1}{\pi}\mathrm{Im}\, \frac{\omega_1\, Z_{p'}(\omega_1)}{D(p',\omega_1)}\right], \tag{28}$$

$$\xi_{\mathbf{p},\sigma}(\omega) \;=\; \frac{1}{N}\sum_{\mathbf{p}'} \int\limits_{-\infty}^{+\infty} \mathrm{d}\omega_1\, K^{(+)}(\omega,\omega_1|\mathbf{p} - \mathbf{p}') \left[-\frac{1}{\pi}\mathrm{Im}\, \frac{(\varepsilon(p') + \xi_{p'}(\omega_1))}{D(p',\omega_1)}\right], \tag{29}$$

$$\Phi_{\mathbf{p},\sigma}(\omega) \;=\; \frac{1}{N}\sum_{\mathbf{p}'} \int\limits_{-\infty}^{+\infty} \mathrm{d}\omega_1 K^{(-)}(\omega,\omega_1|\mathbf{p} - \mathbf{p}') \left[-\frac{1}{\pi}\mathrm{Im} \frac{\Phi_{\mathbf{p}',\sigma}(\omega_1)}{D(p',\omega_1)}\right], \tag{30}$$

where $D(p,\omega) = (\omega Z_p(\omega))^2 - (\varepsilon(p) + \xi_p(\omega))^2 - \mid \Phi_{\mathbf{p}\sigma}(\omega) \mid^2$. The imaginary frequency representation for these equations readily follows from (24), (26). Solution of the system of equations was considered in a number of papers and reviews (see, e.g., [15,16]).

Here we consider only a weak-coupling approximation (WCA) which results in the BCS-type equation for the gap function (30). In WCA the kernel of integral equation (20) is evaluated close to the Fermi energy for the energies $|\omega,\omega_1| \leqslant \Omega_b \ll \mu$ as follows

$$K^{(\pm)}(\omega,\omega_1|\mathbf{q}) \simeq -\frac{1}{2}\tanh\left(\frac{\omega_1}{2T}\right)\lambda^{(\pm)}(\mathbf{q})\,, \tag{31}$$

where the interaction is defined by the static susceptibility

$$\lambda^{(\pm)}(\mathbf{q}) = \int_{-\infty}^{+\infty} \frac{d\Omega}{\Omega}\, \lambda^{(\pm)}(\mathbf{q},\Omega) = 3\,|V_{\rm sf}(\mathbf{q})|^2\; \chi_{\rm sf}^{zz}(\mathbf{q}) \pm\, |V_{\rm ph}(\mathbf{q})|^2\, \chi_{\rm ph}(\mathbf{q}), \tag{32}$$

for spin-fluctuations, $\chi_{\rm sf}^{zz}(\mathbf{q}) = -{\rm Re}\langle\langle S^z_{\mathbf{q}}|S^z_{-\mathbf{q}}\rangle\rangle_{\omega=0} > 0$ and charge fluctuations $\chi_{\rm ph}(\mathbf{q}) = -{\rm Re}\langle\langle \rho_{\mathbf{q}}|\rho^{\dagger}_{\mathbf{q}}\rangle\rangle_{\omega=0} > 0$. In this approximation we have $Z_p = 1$ in (28) and $\xi_p \simeq 0$ in (29). Therefore, for the quasiparticle spectrum in (30) we can write

$$\left[-\frac{1}{\pi}{\rm Im}\frac{1}{D(\mathbf{p},\omega)}\right] = \frac{1}{2E_{\mathbf{p}}}\,\left[\delta(\omega - E_{\mathbf{p}}) - \delta(\omega + E_{\mathbf{p}})\right], \quad E_{\mathbf{p}} = \sqrt{\varepsilon(\mathbf{p})^2 + \mid \Phi_{\mathbf{p}\sigma}\mid^2}, \tag{33}$$

which results in the following equation for the gap function $\Phi_{\mathbf{p}\sigma} = \Phi_{\mathbf{p}\sigma}(0)$:

$$\Phi_{\mathbf{p},\sigma} = \frac{1}{N}\sum_{\mathbf{p}'=\mathbf{p}-\mathbf{q}} \left\{|V_{\rm ph}(\mathbf{q})|^2\,\chi_{\rm ph}(\mathbf{q}) - 3\,|V_{\rm sf}(\mathbf{q})|^2\;\chi_{\rm sf}^{zz}(\mathbf{q})\right\}\,\frac{\Phi_{\mathbf{p}',\sigma}}{2E_{\mathbf{p}'}}\,\tanh\frac{E_{\mathbf{p}'}}{2T}\,. \tag{34}$$

The integration over $\mathbf{p}'$ is restricted for the phonon contribution by $|\varepsilon(\mathbf{p}) - \varepsilon(\mathbf{p}')| < \Omega_{\rm ph}$ and for the spin-fluctuation contribution by $|\varepsilon(\mathbf{p}) - \varepsilon(\mathbf{p}')| < \Omega_{\rm sf}$ where $\Omega_{\rm ph(sf)}$ are the maximal frequency of phonon (spin-fluctuation) excitations. Though the spin-fluctuation in (34) gives a negative contribution to the pairing interaction, nevertheless it can result in a singlet superconducting pairing of the d-wave symmetry as we demonstrate in section 3.1.4.

3. Superconductivity in strongly correlated systems

In recent years, in connection with studies of high-temperature superconductivity in cuprates, a pairing theory in strongly correlated systems was investigated by many authors (for a review see [17]). As it becomes evident, the AFM spin fluctuations in cuprates play a major role in superconducting pairing as originally has been proposed by Anderson [18]. Here we briefly discuss a pairing theory developed within the GFs method for an effective p-d Hubbard model [10] and the t-J model [9].

3.1. Effective Hubbard model

3.1.1. Dyson Equation

To discuss superconducting pairing in cuprates, instead of the original Hubbard model [19] we start from a two-band p-d model for CuO_2 layer [20]. This can be reduced within the cell-cluster perturbation theory [21–23] to an effective two-band Hubbard model with the lower Hubbard subband (LHB) occupied by one-hole Cu d-like states and the upper Hubbard subband (UHB) occupied by two-hole p-d singlet states as given below

$$H = E_1 \sum_{i,\sigma} X_i^{\sigma\sigma} + E_2 \sum_i X_i^{22} + \sum_{i\neq j,\sigma}\left\{t_{ij}^{11} X_i^{\sigma 0} X_j^{0\sigma} + t_{ij}^{22} X_i^{2\sigma} X_j^{\sigma 2} + 2\sigma t_{ij}^{12}(X_i^{2\bar\sigma} X_j^{0\sigma} + {\rm H.c.})\right\}, \tag{35}$$

where $X_i^{nm} = |in\rangle\langle im|$ are the Hubbard operators (HOs) for the four states $n,m = |0\rangle$, $|\sigma\rangle$, $|2\rangle = |\uparrow\downarrow\rangle$, $\sigma = \pm 1/2 = (\uparrow,\downarrow)$, $\bar\sigma = -\sigma$. Here $E_1 = \epsilon_d - \mu$ and $E_2 = 2E_1 + \Delta$ where μ is the chemical potential and $\Delta = \epsilon_p - \epsilon_d$ is the charge transfer energy (see [21]). The superscripts 2 and 1 refer to the singlet and one-hole subbands, respectively. The hopping integrals are given by $t_{ij}^{\alpha\beta} = K_{\alpha\beta}\, 2t\nu_{ij}$ where t is the p-d hybridization parameter and ν_{ij} are estimated as: $\nu_1 = \nu_{j\; j\pm a_{x/y}} \simeq -0.14$, $\nu_2 = \nu_{j\; j\pm a_x \pm a_y} \simeq -0.02$. The coefficients $K_{\alpha\beta} < 1$, e.g., for the singlet subbands we have $t_{\rm eff} \simeq K_{22} 2t\nu_1 \simeq 0.14t$ and the bandwidth $W = 8t_{\rm eff}$. Since the ratio $\Delta/W \simeq 2$, the Hubbard model (35) corresponds to the strong correlation limit. The HOs in (35) obey the completeness relation

$$X_i^{00} + X_i^{\uparrow\uparrow} + X_i^{\downarrow\downarrow} + X_i^{22} = 1, \tag{36}$$

which rigorously preserves the constraint of no double occupancy of any quantum state $|in\rangle$ at each lattice site i. The HOs have the following multiplication rules $X_i^{\alpha\beta}X_i^{\gamma\delta} = \delta_{\beta\gamma}X_i^{\alpha\delta}$ and obey the commutation relations

$$\left[X_i^{\alpha\beta}, X_j^{\gamma\delta}\right]_{\pm} = \delta_{ij}\left(\delta_{\beta\gamma}X_i^{\alpha\delta} \pm \delta_{\delta\alpha}X_i^{\gamma\beta}\right). \tag{37}$$

In (37) the upper sign stands for the case when both HOs are Fermi-like ones (as, e. g., $X_i^{0\sigma}$) and the lower sign for the Bose-like ones, as the spin or charge density.

To discuss the superconducting pairing within the model Hamiltonian (35), we introduce the four-component Nambu operators $\hat{X}_{i\sigma}$ and $\hat{X}_{i\sigma}^{\dagger}$ and define the 4×4 matrix GF

$$\tilde{G}_{ij\sigma}(t-t') = \left\langle\left\langle \hat{X}_{i\sigma}(t)\,|\,\hat{X}_{j\sigma}^{\dagger}(t')\right\rangle\right\rangle, \qquad \tilde{G}_{ij\sigma}(\omega) = \begin{pmatrix} \hat{G}_{ij\sigma}(\omega) & \hat{F}_{ij\sigma}(\omega) \\ \hat{F}_{ij\sigma}^{\dagger}(\omega) & -\hat{G}_{ji\bar{\sigma}}(-\omega)\end{pmatrix}, \tag{38}$$

where $\hat{X}_{i\sigma}^{\dagger} = (X_i^{2\sigma}\ X_i^{\bar{\sigma}0}\ X_i^{\bar{\sigma}2}\ X_i^{0\sigma})$. Due to two-band character of the model (35), the normal $\hat{G}_{ij\sigma}$ and anomalous $\hat{F}_{ij\sigma}$ GFs are 2×2 matrices.

To calculate the GF (38) we use the equation of motion method as in section 2.1. Differentiation with respect to time t of the GF (38) and the use of the Fourier transform as in (6) result in the following equation

$$\omega\tilde{G}_{ij\sigma}(\omega) = \delta_{ij}\tilde{\chi} + \left\langle\left\langle \hat{Z}_{i\sigma}\,|\,\hat{X}_{j\sigma}^{\dagger}\right\rangle\right\rangle_{\omega}, \tag{39}$$

where $\hat{Z}_{i\sigma} = [\hat{X}_{i\sigma}, H]$. For a paramagnetic state, the matrix $\tilde{\chi} = \langle\{\hat{X}_{i\sigma}, \hat{X}_{i\sigma}^{\dagger}\}\rangle = \tau_0 \times \begin{pmatrix}\chi_2 & 0\\ 0 & \chi_1\end{pmatrix}$, where $\chi_2 = \langle X_i^{22} + X_i^{\sigma\sigma}\rangle = n/2$ and $\chi_1 = \langle X_i^{00} + X_i^{\bar{\sigma}\bar{\sigma}}\rangle = 1-\chi_2$ depend only on the occupation number of holes:

$$n = \langle N_i\rangle = \sum_{\sigma}\langle X_i^{\sigma\sigma}\rangle + 2\langle X_i^{22}\rangle. \tag{40}$$

It is important to point out that contrary to the spin-fermion model (1), in the Hubbard model there is no dynamical interaction of electrons with spin- or charge fluctuations. The nonfermionic commutation relations (37) for the HOs generate these interactions as has been pointed out already by Hubbard [19]. For instance, the equation of motion for the HO $X_i^{\sigma 2}$ reads

$$\begin{aligned} Z_i^{\sigma 2} &= [X_i^{\sigma 2}, H] = (E_1+\Delta)X_i^{\sigma 2} + \sum_{l\neq i,\sigma'}\left(t_{il}^{22}B_{i\sigma\sigma'}^{22}X_l^{\sigma' 2} - 2\sigma t_{il}^{21}B_{i\sigma\sigma'}^{21}X_l^{0\bar{\sigma}'}\right) \\ &\quad - \sum_{l\neq i}X_i^{02}\left(t_{il}^{11}X_l^{\sigma 0} + 2\sigma t_{il}^{21}X_l^{2\bar{\sigma}}\right), \end{aligned} \tag{41}$$

where $B_{i\sigma\sigma'}^{\alpha\beta}$ are Bose-like operators describing the number (charge) and spin fluctuations:

$$B_{i\sigma\sigma'}^{22} = \left(X_i^{22} + X_i^{\sigma\sigma}\right)\delta_{\sigma'\sigma} + X_i^{\sigma\bar{\sigma}}\delta_{\sigma'\bar{\sigma}} = \left(\frac{1}{2}N_i + S_i^z\right)\delta_{\sigma'\sigma} + S_i^{\sigma}\delta_{\sigma'\bar{\sigma}}, \tag{42}$$

$$B_{i\sigma\sigma'}^{21} = \left(\frac{1}{2}N_i + S_i^z\right)\delta_{\sigma'\sigma} - S_i^{\sigma}\delta_{\sigma'\bar{\sigma}}, \quad S_i^{\sigma} = S_i^{\pm}. \tag{43}$$

To separate a mean-field type contribution to the quasiparticle energy in the equation of motion (39), we employ a Mori-type projection technique by writing the operator $\hat{Z}_{i\sigma}$ as a sum of a linear part and an irreducible part orthogonal to it, $\hat{Z}_{i\sigma}^{(\mathrm{ir})}$, which originates from the inelastic QP scattering:

$$\hat{Z}_{i\sigma} = [\hat{X}_{i\sigma}, H] = \sum_{l}\tilde{E}_{il\sigma}\hat{X}_{l\sigma} + \hat{Z}_{i\sigma}^{(\mathrm{ir})}. \tag{44}$$

The orthogonality condition $\langle\{\hat{Z}_{i\sigma}^{(\mathrm{ir})}, \hat{X}_{j\sigma}^{\dagger}\}\rangle = 0$ provides the definition of the the frequency matrix:

$$\tilde{E}_{ij\sigma} = \tilde{\mathcal{A}}_{ij\sigma}\tilde{\chi}^{-1}, \quad \tilde{\mathcal{A}}_{ij\sigma} = \left\langle\left\{[\hat{X}_{i\sigma}, H], \hat{X}_{j\sigma}^{\dagger}\right\}\right\rangle. \tag{45}$$

The frequency matrix (45) defines the zero-order GF in the generalized MFA. In the $(\mathbf{q},\omega)$-representation, its expression is given by

$$\tilde{G}^0_\sigma(\mathbf{q},\omega) = \Big(\omega\tilde{\tau}_0 - \tilde{E}_\sigma(\mathbf{q})\Big)^{-1}\tilde{\chi}\,, \tag{46}$$

where $\tilde{\tau}_0$ is the 4×4 unity matrix.

Differentiation of the many-particle GF (39) with respect to the second time t' and the use of the same projection procedure as in (44) result in the Dyson equation for the GF (38). In $(\mathbf{q},\omega)$-representation, the Dyson equation reads

$$\Big(\tilde{G}_\sigma(\mathbf{q},\omega)\Big)^{-1} = \Big(\tilde{G}^0_\sigma(\mathbf{q},\omega)\Big)^{-1} - \tilde{\Sigma}_\sigma(\mathbf{q},\omega). \tag{47}$$

The self-energy operator $\tilde{\Sigma}_\sigma(\mathbf{q},\omega)$ is defined by the *proper* part of the scattering matrix as described in previous section that has no parts connected by the single-particle zero-order GF (46):

$$\tilde{\Sigma}_\sigma(\mathbf{q},\omega) = \tilde{\chi}^{-1}\left\langle\!\left\langle \hat{Z}^{(\mathrm{ir})}_{\mathbf{q}\sigma}\,|\,\hat{Z}^{(\mathrm{ir})\dagger}_{\mathbf{q}\sigma}\right\rangle\!\right\rangle^{(\mathrm{prop})}_{\omega}\tilde{\chi}^{-1}. \tag{48}$$

The equations (46)–(48) provide an exact representation for the GF (38). However, to calculate it one has to use approximations for the self-energy matrix (48) which describes the finite lifetime effects (inelastic scattering of electrons on spin and charge fluctuations).

3.1.2. Mean-Field Approximation

In the MFA the electronic spectrum and superconducting pairing are described by the zero-order GF in (46). By applying the commutation relations for the HOs we get for the frequency matrix (45):

$$\tilde{\mathcal{A}}_{ij\sigma} = \begin{pmatrix} \hat{\omega}_{ij\sigma} & \hat{\Delta}_{ij\sigma} \\ \hat{\Delta}^*_{ji\sigma} & -\hat{\omega}_{ji\bar{\sigma}} \end{pmatrix}, \tag{49}$$

where $\hat{\omega}_{ij\sigma}$ and $\hat{\Delta}_{ij\sigma}$ are 2×2 matrices for the normal and anomalous components, respectively. The normal component determines quasiparticle spectra of the model in the normal state which have been studied in detail in [21]. The anomalous component defines the gap functions for the singlet and one-hole subbands, respectively, $(i\neq j)$:

$$\Delta^{22}_{ij\sigma} = -2\sigma t^{12}_{ij}\left\langle X^{02}_i N_j\right\rangle, \qquad \Delta^{11}_{ij\sigma} = -2\sigma t^{12}_{ij}\left\langle (2-N_j)X^{02}_i\right\rangle, \tag{50}$$

where the number operator is $N_i = \sum_\sigma X^{\sigma\sigma}_i + 2X^{22}_i$. Using the definitions of the Fermi annihilation operators: $c_{i\sigma} = X^{0\sigma}_i + 2\sigma X^{\bar{\sigma}2}_i$, we can write the anomalous average in (50) as $\langle c_{i\downarrow}c_{i\uparrow}N_j\rangle = \langle X^{0\downarrow}_i X^{\downarrow 2}_i N_j\rangle = \langle X^{02}_i N_j\rangle$ since other products of HOs vanish according to the multiplication rule: $X^{\alpha\gamma}_i X^{\lambda\beta}_i = \delta_{\gamma,\lambda}X^{\alpha\beta}_i$. Therefore the anomalous correlation functions describe the pairing at one lattice site but in different Hubbard subbands.

The same anomalous correlation functions were obtained in MFA for the original Hubbard model in [24–26]. To calculate the anomalous correlation function $\langle c_{i\downarrow}c_{i\uparrow}N_j\rangle$ in [24,26] the Roth procedure was applied based on a decoupling of the operators on the same lattice site in the time-dependent correlation function: $\langle c_{i\downarrow}(t)|c_{i\uparrow}(t')N_j(t')\rangle$. However, the decoupling of the HOs on the same lattice site is not unique (as has been really observed in [24,26]) and turns out to be unreliable. To escape uncontrollable decoupling, in [25] kinematical restrictions imposed on the correlation functions for the HOs were used which, however, also have not produced a unique solution for superconducting equations.

To overcome this problem, we calculate the correlation function $\langle X^{02}_i N_j\rangle$ directly from the equation of motion for the corresponding commutator GF $L_{ij}(t-t') = \langle\langle X^{02}_i(t)\mid N_j(t')\rangle\rangle$ which can be solved without *any decoupling*. This results in the following representation for the correlation function at sites $i\neq j$ for the singlet subband in the case of hole doping [10]:

$$\langle X^{02}_i N_j\rangle = -\frac{1}{\Delta}\sum_{m\neq i,\sigma} 2\sigma t^{12}_{im}\langle X^{\sigma 2}_i X^{\bar{\sigma}2}_m N_j\rangle \simeq -\frac{4t^{12}_{ij}}{\Delta}2\sigma\,\langle X^{\sigma 2}_i X^{\bar{\sigma}2}_j\rangle. \tag{51}$$

The last equation is obtained in the two-site approximation, $m = j$, usually applied to the t-J model. The identity for the HOs, $X_j^{\bar\sigma 2} N_j = 2X_j^{\bar\sigma 2}$ was used as well. This finally enables us to write the gap function in (50) in the case of hole doping as follows

$$\Delta_{ij\sigma}^{22} = -2\sigma\, t_{ij}^{12} \left\langle X_i^{02} N_j \right\rangle = J_{ij} \left\langle X_i^{\sigma 2} X_j^{\bar\sigma 2} \right\rangle . \tag{52}$$

The obtained gap equation determines the exchange pairing as in the t-J model with the exchange energy $J_{ij} = 4\,(t_{ij}^{12})^2/\Delta$. In the case of electron doping, an analogous calculation for the anomalous correlation function of the one-hole subband $\langle (2-N_j) X_i^{02} \rangle$ gives for the gap function $\Delta_{ij\sigma}^{11} = J_{ij}\,\langle X_i^{0\bar\sigma} X_j^{0\sigma} \rangle$. We may therefore conclude that the anomalous contributions to the zero-order GF (46) can be described as conventional anomalous pairs in one of the two Hubbard subbands. Their pairing in MFA is mediated by the exchange interaction which has been studied in the t-J model (see, e.g., [7,9]).

3.1.3. Self-Energy

The self-energy matrix (48) can be written in the form

$$\tilde\Sigma_{ij\sigma}(\omega) = \tilde\chi^{-1} \begin{pmatrix} \hat M_{ij\sigma}(\omega) & \hat\Phi_{ij\sigma}(\omega) \\ \hat\Phi_{ij\sigma}^{\dagger}(\omega) & -\hat M_{ij\bar\sigma}(-\omega) \end{pmatrix} \tilde\chi^{-1} , \tag{53}$$

where the 2×2 matrices $\hat M$ and $\hat\Phi$ denote the normal and anomalous contributions to the self-energy, respectively. The self-energy (53) is calculated below in NCA as in section 2.2. This is given by the decoupling of the corresponding operators in the time-dependent correlation functions for lattice sites $(1 \neq 1', 2 \neq 2')$ as follows

$$\langle B_{1'}(t) X_1(t) B_{2'}(t') X_2(t') \rangle \simeq \langle X_1(t) X_2(t') \rangle \langle B_{1'}(t) B_{2'}(t') \rangle. \tag{54}$$

Using the spectral representation for these correlation functions as in (16), (17), we get a closed system of equations for the GF (38) and the self-energy components (53) which is similar to the system of equation for the fermion-boson model in section 2.2.

Below we consider explicitly only the self-energy for the singlet subband (UHB) which is relevant for hole-doped curates. The normal, $M_\sigma^{22}(\mathbf{q},\omega)$, and anomalous, $\Phi_\sigma^{22}(\mathbf{q},\omega)$, diagonal components of the self-energy in the SCBA approximation read:

$$M_\sigma^{22}(\mathbf{q},\omega) = \frac{1}{N}\sum_{\mathbf{k}} \int_{-\infty}^{+\infty} \mathrm{d}\omega_1 K^{(+)}(\omega,\omega_1|\mathbf{k},\mathbf{q}-\mathbf{k}) \left\{ -\frac{1}{\pi} \mathrm{Im}\left[K_{22}^2 G_\sigma^{22}(\mathbf{k},\omega_1) + K_{12}^2 G_\sigma^{11}(\mathbf{k},\omega_1) \right] \right\} , \tag{55}$$

$$\Phi_\sigma^{22}(\mathbf{q},\omega) = \frac{1}{N}\sum_{\mathbf{k}} \int_{-\infty}^{+\infty} \mathrm{d}\omega_1 K^{(-)}(\omega,\omega_1|\mathbf{k},\mathbf{q}-\mathbf{k}) \left\{ -\frac{1}{\pi} \mathrm{Im}\left[K_{22}^2 F_\sigma^{22}(\mathbf{k},\omega_1) - K_{12}^2 F_\sigma^{11}(\mathbf{k},\omega_1) \right] \right\} . \tag{56}$$

The kernel of the integral equations for the self-energy is defined by the equation similar to (20)

$$K^{(\pm)}(\omega,\omega_1|\mathbf{k},\mathbf{q}-\mathbf{k}) = \int_{-\infty}^{+\infty} \mathrm{d}\Omega\, \frac{\tanh(\omega_1/2T) + \coth(\Omega/2T)}{2\,(\omega-\omega_1-\Omega)}\, \lambda^{(\pm)}(\mathbf{k},\mathbf{q}-\mathbf{k},\Omega) , \tag{57}$$

where the interaction function reads

$$\lambda^{(\pm)}(\mathbf{k},\mathbf{q}-\mathbf{k},\Omega) = |t(\mathbf{k})|^2 \left[\frac{1}{\pi}\, \mathrm{Im}\, \chi_{sc}^{(\pm)}(\mathbf{q}-\mathbf{k},\Omega) \right] . \tag{58}$$

The kinematic interaction is defined by hopping matrix elements for the nearest, $t\,\nu_1$, and the second, $t\,\nu_2$, neighbors and is given by $t(\mathbf{k}) = 8t\,[\nu_1\gamma(\mathbf{k}) + \nu_2\gamma'(\mathbf{k})]$, where $\gamma(\mathbf{k}) = (1/2)(\cos k_x +$

$\cos k_y)$ and $\gamma'(\mathbf{k}) = \cos k_x \cos k_y$. The pairing interaction is mediated by the spin-charge fluctuations which are determined by the corresponding dynamical susceptibilities

$$\chi_{sc}^{(\pm)}\mathbf{q},\omega) = \chi_s(\mathbf{q},\omega) \pm \chi_c(\mathbf{q},\omega) = -\left\{\langle\langle \mathbf{S_q}|\mathbf{S_{-q}}\rangle\rangle_\omega \pm \frac{1}{4}\langle\langle \delta N_\mathbf{q}|\delta N_{-\mathbf{q}}\rangle\rangle_\omega\right\}. \tag{59}$$

They arise from the correlation functions $\langle B_{1'}(t)B_{2'}(t')\rangle$ for the Bose-like operators (42), (43) in (54). As we see, the obtained equations for the self-energy (55), (56) are quite similar for the spin-fermion model (1) apart from the origin of the interaction: in the Hubbard model it originates from the kinematical interaction proportional to the hopping matrix elements, while in the spin-fermion model it has a dynamical character with independent coupling constants.

3.1.4. Solution of the gap equation in WCA

Let us consider the gap equation (56) for a hole doped case, $n > 1$, when the chemical potential is in the singlet subband $\mu \simeq \Delta$. For energies $|\omega, \omega_1|$ close to the Fermi energy we can use the weak coupling approximation (31) to calculate of the contribution from the same subband (the first term) in (56). The contribution from another subband (the second term) is rather small since the one-hole subband lies below the FS at the energy of the order $\Delta \gg W$. Neglecting this contribution and taking into account the contribution from the exchange interaction in MFA (52) we arrive at the following equation for the superconducting gap in the singlet subband:

$$\Phi^{22}(\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{k}}[J(\mathbf{k}-\mathbf{q}) - \lambda(\mathbf{k},\mathbf{q}-\mathbf{k})]\frac{\Phi^{22}(\mathbf{k})}{2E_2(\mathbf{k})}\tanh\frac{E_2(\mathbf{k})}{2T}, \tag{60}$$

where the interaction $\lambda(\mathbf{k},\mathbf{q}-\mathbf{k}) = |K_{22}\, t(\mathbf{k})|^2\, \chi(\mathbf{q}-\mathbf{k},\omega=0) > 0$ is determined by the static correlation function as in WCA (32) . The quasiparticle energy in the singlet band is given by $E_2(\mathbf{k}) = [\varepsilon(\mathbf{k})^2 + \Phi^{22}(\mathbf{k})^2]$ where $\varepsilon(\mathbf{k})$ is the quasiparticle energy in the normal state in the singlet subband [21]. Similar considerations hold true for an electron doped system, $n \leqslant 1$, when the chemical potential lies in the one-hole band, $\mu \simeq 0$. In that case, the WCA equation for the gap $\Phi^{11}(\mathbf{q})$ is quite similar to (60).

To solve the gap equation (60) we consider only antiferromagnetic (AFM) spin-fluctuation contribution which is modelled by the following static susceptibility:

$$\chi_s(\mathbf{q},0) = \frac{\chi_0}{1+\xi^2[1+\gamma(\mathbf{q})]}, \qquad \gamma(\mathbf{q}) = \frac{1}{2}(\cos q_x + \cos q_y), \tag{61}$$

where ξ is the AFM correlation length. The susceptibility $\chi_s(\mathbf{q}=\mathbf{Q})$ at the AFM wave-vector $\mathbf{Q} = (\pi,\pi)$ is equal to the constant $\chi_0 = 3(2-n)/(2\pi\omega_s C_1)$ where $\omega_s \leqslant J$ is a characteristic spin-fluctuation energy. The constant is not a free parameter but is determined from the normalization condition: $(1/N)\sum_i\langle \mathbf{S_i S_i}\rangle = (3/4)(1-|1-n|)$ which gives $C_1 = (1/N)\sum_\mathbf{q}\{1+\xi^2[1+\gamma(\mathbf{q})]\}^{-1}$.

Let us estimate the superconducting transition temperature T_c by solving the gap equation (60) for a model d-wave gap function $\Phi^{22}(\mathbf{q}) = \varphi_d\,(\cos q_x - \cos q_y) \equiv \varphi_d\,\eta(\mathbf{q})$ in the standard logarithmic approximation in the limit of weak coupling. By taking into account that the spin susceptibility (61) peaks sharply at the AFM wave-vector $\mathbf{Q}$ for large ξ, we obtain the following equation for T_c:

$$1 = \frac{1}{N}\sum_{\mathbf{k}}\left[J\,\eta(\mathbf{k})^2 + \lambda_s\,(4\gamma(\mathbf{k}))^2\eta(\mathbf{k})^2\right]\frac{1}{2\varepsilon(\mathbf{k})}\tanh\frac{\varepsilon(\mathbf{k})}{2T_c}, \tag{62}$$

where $\lambda_s \simeq t_{\rm eff}^2/\omega_s$. As we observe, the spin-fluctuation interaction λ_s gives a positive contribution to the d-wave gap. Now we should take into account that for the exchange interaction in (62) mediated by the interband hopping with large energy transfer $\Delta \gg W$ the retardation effects are negligible. This results in coupling of all electrons in a broad energy shell of the order of the bandwidth W and high T_c [8]:

$$T_c \simeq \sqrt{\mu(W-\mu)}\exp(-1/\lambda_{\rm ex}), \tag{63}$$

where $\lambda_{\rm ex} \simeq J\, N(\delta)$ is an effective coupling constant for the exchange interaction J and the average density $N(\delta)$ of electronic states for doping δ. The spin-fluctuation pairing in (62) is effective only in a narrow region $\pm\omega_s$ close to the Fermi energy and therefore produces a much lower $T_{\rm c}$. By taking into account both contributions we can write the following estimation for $T_{\rm c}$:

$$T_{\rm c} \simeq \omega_s \exp(-\frac{1}{\tilde{\lambda}_{\rm sf}}), \qquad \tilde{\lambda}_{\rm sf} = \lambda_{\rm sf} + \frac{\lambda_{\rm ex}}{1-\lambda_{\rm ex}\ln(\mu/\omega_s)}\,, \tag{64}$$

where $\lambda_{\rm sf} \simeq \lambda_s\, N(E_F)$ is the coupling constant for the spin-fluctuation pairing. By taking for estimation $\mu = W/2 \simeq 0.35$ eV, $\omega_s \simeq J \simeq 0.13$ eV and a weak coupling: $\lambda_{\rm sf} \simeq \lambda_{\rm ex} = 0.2$, we get $\tilde{\lambda}_{\rm sf} \simeq 0.2 + 0.25 = 0.45$ and $T_{\rm c} \simeq 160$ K, while only the spin-fluctuation pairing gives $T_{\rm c}^0 \simeq \omega_s \exp(-1/\lambda_{\rm sf}) \simeq 10$ K. Results of a direct numerical solution of the gap equation (60) for the superconducting transition temperature $T_c(\delta)$ and for $\mathbf{k}$-dependence of the gap function $\Phi^{22}(\mathbf{k})$ are presented in [10] which qualitatively agree with experiments in cuprate superconductors.

3.2. t-J model

Now we compare the results for the original two-band p-d model for CuO_2 layer (35) with the calculations for the t-J in [9]. In that paper, a full self-consistent numerical solution for the normal and anomalous GF in the Dyson equation was performed allowing for finite life-time effects caused by the imaginary parts of the self-energy operators which were neglected in the above calculations in WCA for the Hubbard model.

In the limit of strong correlations the interband hopping in the model (35) can be excluded by perturbation theory which results in the effective t-J model

$$H_{t-J} = -\sum_{i\neq j,\sigma} t_{ij} X_i^{\sigma 0} X_j^{0\sigma} - \mu \sum_{i\sigma} X_i^{\sigma\sigma} + \frac{1}{4}\sum_{i\neq j,\sigma} J_{ij}\left(X_i^{\sigma\bar\sigma} X_j^{\bar\sigma\sigma} - X_i^{\sigma\sigma} X_j^{\bar\sigma\bar\sigma}\right), \tag{65}$$

where only the lower Hubbard subband is considered with the hopping energy $t_{ij} = -t_{ij}^{11}$. Exclusion of the interband hopping results in the instantaneous exchange interaction $J_{ij} = 4\,(t_{ij}^{12})^2/\Delta$. The superconducting pairing within the model (65) can be studied by considering the matrix GF for the lower Hubbard subband in terms of the Nambu operators: $\Psi_{i\sigma}$ and $\Psi_{i\sigma}^{+} = (X_i^{\sigma 0}\; X_i^{0\bar\sigma})$:

$$\hat{G}_{ij,\sigma}(t-t') = \langle\langle \Psi_{i\sigma}(t) | \Psi_{j\sigma}^{+}(t')\rangle\rangle, \qquad \hat{G}_{ij\sigma}(\omega) = Q\begin{pmatrix} G_{ij\sigma}^{11}(\omega) & G_{ij\sigma}^{12}(\omega) \\ G_{ij\sigma}^{21}(\omega) & G_{ij\sigma}^{22}(\omega)\end{pmatrix}. \tag{66}$$

Here we introduced the Hubbard factor $Q = 1 - n/2$ which depends on the average number of electrons $n = \sum_\sigma \langle X_i^{\sigma\sigma}\rangle$.

By applying the projection technique as described above we get the Dyson equation which can be written in the Eliashberg notation similar to (25) as

$$\hat{G}_\sigma(\mathbf{q},\omega) = Q\frac{\omega Z_\sigma(\mathbf{q},\omega)\hat\tau_0 + (\varepsilon(\mathbf{q}) + \xi_\sigma(\mathbf{q},\omega))\hat\tau_3 + \Phi_\sigma(\mathbf{q},\omega)\hat\tau_1}{(\omega Z_\sigma(\mathbf{q},\omega))^2 - (\varepsilon(\mathbf{q}) + \xi_\sigma(\mathbf{q},\omega))^2 - \mid \Phi_\sigma(\mathbf{q},\omega)\mid^2}\,. \tag{67}$$

The electron dispersion $\varepsilon(\mathbf{q})$ in the normal state in the MFA is calculated within the projection technique as discussed above (for details see [9]). The frequency-dependent functions $Z_\sigma(\mathbf{q},\omega))$, $\xi_\sigma(\mathbf{q},\omega)$ are defined as in (28)–(30). The self-energy is calculated in the noncrossing approximation (54) as in the Hubbard model:

$$\Sigma_\sigma^{11(12)}(\mathbf{q},\omega) = \frac{1}{N}\sum_{\mathbf{k}}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1 K^{(\pm)}(\omega,\omega_1|\mathbf{k},\mathbf{q}-\mathbf{k})\left[-\frac{1}{\pi}\,\mathrm{Im}G_\sigma^{11(12)}(\mathbf{k},\omega_1)\right]. \tag{68}$$

The kernel of the integral equation $K^{(\pm)}(\omega,\omega_1|\mathbf{k},\mathbf{q}-\mathbf{k})$ is defined by the same equation (57) as in the Hubbard model where the interaction function reads

$$\lambda^{(\pm)}(\mathbf{k},\mathbf{q}-\mathbf{k},\Omega) = \left| t(\mathbf{k}) - \frac{1}{2}J(\mathbf{q}-\mathbf{k})\right|^2 \left[\frac{1}{\pi}\,\mathrm{Im}\,\chi_{sc}^{(\pm)}(\mathbf{q}-\mathbf{k},\Omega)\right]. \tag{69}$$

The electron-electron interaction is caused by the same spin-charge dynamical susceptibility (59) as in the Hubbard model. Taking into account the mean-field contribution to the gap mediated by exchange interaction we obtain the following gap equation:

$$\Phi_\sigma(\mathbf{q},\omega) = \Delta_\sigma(\mathbf{q}) + \Sigma^{12}_\sigma(\mathbf{q},\omega)\,, \quad \Delta_\sigma(\mathbf{q}) = \frac{1}{NQ}\sum_{\mathbf{k}} J(\mathbf{q}-\mathbf{k})\langle X^{0\bar\sigma}_{-\mathbf{k}} X^{0\sigma}_{\mathbf{k}}\rangle\,. \tag{70}$$

As we see, the equation for the self-energy (68) is similar to (56) obtained for the Hubbard model if we disregard in the latter the small contribution from the second subband $\propto F^{11}_\sigma(\mathbf{k},\omega_1)$ as discussed above. However, contrary to the gap equation (60) in the WCA for the Hubbard model, equation (70) for the t-J model preserves the frequency-dependent self-energy contribution $\Sigma^{12}_\sigma(\mathbf{q},\omega)$ (68). Moreover, in [9] for the t-J model a full self-consistent solution for the normal GF $G^{11}_\sigma(\mathbf{q},\omega)$ in equation (67) and the corresponding self-energy $\Sigma^{11}_\sigma(\mathbf{q},\omega)$, equation (68), was performed.

Numerical calculations in [9] have demonstrated that quasiparticle-like peaks emerge only in the vicinity of the Fermi level, while an anomalous, non Fermi-liquid behavior for the self-energy $\mathrm{Im}\Sigma^{11}_\sigma(\mathbf{q},\omega+\mathrm{i}\delta)\propto\omega$ reveals close to the Fermi level. The occupation number $N(\mathbf{q}) = (1/Q)\langle X^{\sigma\sigma}_{\mathbf{q}}\rangle$ reveals a small jump at the Fermi level which is generic for strongly correlated systems. The superconducting T_c was calculated from a linearized gap equation which was solved by direct diagonalization in $(\mathbf{q},\omega_n)$-space:

$$\begin{aligned}\Phi_\sigma(\mathbf{q},i\omega_n) &= \frac{T}{N}\sum_{\mathbf{k}}\sum_{m}\{J(\mathbf{q}-\mathbf{k}) + \lambda^{(-)}(\mathbf{k},\mathbf{q}-\mathbf{k}\mid \mathrm{i}\omega_n - \mathrm{i}\omega_m)\}\\ &\quad\times G^{11}_\sigma(\mathbf{k},\mathrm{i}\omega_m)G^{11}_{\bar\sigma}(\mathbf{k},-\mathrm{i}\omega_m)\Phi_\sigma(\mathbf{k},\mathrm{i}\omega_m)\end{aligned} \tag{71}$$

for the Matsubara frequencies. The doping dependence of superconducting $T_\mathrm{c}(\delta)$ and $\Phi_\sigma(\mathbf{q},\mathrm{i}\omega_n)$ were calculated which unambiguously demonstrated the d-wave character of superconducting pairing (for details see [9]). By comparing the $T_\mathrm{c}(\delta)$ dependence for the Hubbard model with $T_\mathrm{c}^{\max}\sim 280$ K, and for the t-J model with $T_\mathrm{c}^{\max}\sim 180$ K, we observe a strong reduction of $T_\mathrm{c}^{\max}$ in the latter model due to a large contribution from the $\mathrm{Im}\Sigma^{11}_\sigma(\mathbf{q},\omega)$ being taken into account.

4. Conclusions

In the present paper a theory of superconducting pairing within the general fermion-boson model (1) with electron-phonon or electron-spin-fluctuation interactions, or within the Hubbard model (35) and the t-J model (65) with strong-electron correlations is presented. By employing the equation of motion method for the thermodynamic double-time GFs [1,3] with differentiation the GFs over two times, t and t', we easily obtained the self-consistent system for the matrix GFs and the self-energies in the noncrossing approximation. The latter is equivalent to the Migdal-Eliasberg approximation and exactly reproduces the results of the diagram technique.

It is important to point out that the investigations of models with strong electron correlations provide a microscopic theory for superconducting pairing mediated by the AFM exchange interaction and spin-fluctuation scattering induced by the kinematic interaction, characteristic of systems with strong correlations. These mechanisms of superconducting pairing are absent in the fermionic models (for a discussion, see Anderson [27]) and are generic for cuprates. The singlet $d_{x^2-y^2}$-wave superconducting pairing was proved both for the original two-band p-d Hubbard model and for the reduced effective one-band t-J model. Therefore, we believe that the proposed magnetic mechanism of superconducting pairing is a relevant mechanism of high-temperature superconductivity in copper-oxide materials.

Appendix

Let us consider more in detail the NCA for the normal and anomalous components of the self-energy (14) which are given by the following many-particle GFs:

$$\Sigma_{\mathbf{p},\sigma}(\omega) = \sum_{\mathbf{p}',\mathbf{p}''} \langle\langle A_\sigma(\mathbf{p},\mathbf{p}') | A_\sigma^\dagger(\mathbf{p},\mathbf{p}'')\rangle\rangle_\omega \,, \tag{72}$$

$$\Phi_{\mathbf{p},\sigma}(\omega) = - \sum_{\mathbf{p}',\mathbf{p}''} \langle\langle A_\sigma(\mathbf{p},\mathbf{p}') | A_{\bar\sigma}(-\mathbf{p},-\mathbf{p}'')\rangle\rangle_\omega \,, \tag{73}$$

where

$$A_\sigma(\mathbf{p},\mathbf{p}') = \delta_{\sigma,\sigma'}\, V_{\rm ph}(\mathbf{p}-\mathbf{p}')\, \rho_{\mathbf{p}-\mathbf{p}'} a_{\mathbf{p}'\sigma} + V_{\rm sf}(\mathbf{p}-\mathbf{p}')$$
$$\times \Big\{ S^z_{\mathbf{p}-\mathbf{p}'}\, a_{\mathbf{p}'\sigma}\, (\delta_{\sigma,\uparrow} - \delta_{\sigma,\downarrow}) + (S^+_{\mathbf{p}-\mathbf{p}'}\, \delta_{\sigma,\downarrow} + S^-_{\mathbf{p}-\mathbf{p}'} \delta_{\sigma,\uparrow}) a_{\mathbf{p}'\bar\sigma} \Big\} .$$

For the time-dependent correlation function corresponding to the normal many-particle GF in (72) we get the following result:

$$\sum_{\mathbf{p}',\mathbf{p}''} \big\langle A_\sigma(\mathbf{p},\mathbf{p}')(t) | A_\sigma^\dagger(\mathbf{p},\mathbf{p}'')\big\rangle = \sum_{\mathbf{p}',\mathbf{p}''} V_{\rm ph}(\mathbf{p}-\mathbf{p}')\, V_{\rm ph}(\mathbf{p}''-\mathbf{p}) \langle \rho_{\mathbf{p}-\mathbf{p}'}(t)\, a_{\mathbf{p}'\sigma}(t) |\, a^\dagger_{\mathbf{p}''\sigma}\, \rho^\dagger_{\mathbf{p}-\mathbf{p}''}\rangle$$
$$+ \sum_{\mathbf{p}',\mathbf{p}''} V_{\rm sf}(\mathbf{p}-\mathbf{p}')\, V_{\rm sf}(\mathbf{p}''-\mathbf{p}) \Big\langle \Big\{ S^z_{\mathbf{p}-\mathbf{p}'}(t)\, a_{\mathbf{p}'\sigma}(t)\, (\delta_{\sigma,\uparrow} - \delta_{\sigma,\downarrow}) + (S^+_{\mathbf{p}-\mathbf{p}'}(t)\, \delta_{\sigma,\downarrow}$$
$$+ S^-_{\mathbf{p}-\mathbf{p}'}(t) \delta_{\sigma,\uparrow}) a_{\mathbf{p}'\bar\sigma}(t) \Big\} \Big| \Big\{ S^z_{\mathbf{p}''-\mathbf{p}}\, a^\dagger_{\mathbf{p}''\sigma}\, (\delta_{\sigma,\uparrow} - \delta_{\sigma,\downarrow}) + (S^-_{\mathbf{p}''-\mathbf{p}}\, \delta_{\sigma,\downarrow} + S^+_{\mathbf{p}''-\mathbf{p}} \delta_{\sigma,\uparrow}) a^\dagger_{\mathbf{p}''\bar\sigma} \Big\} \Big\rangle$$
$$\simeq \sum_{\mathbf{p}'} |V_{\rm ph}(\mathbf{p}-\mathbf{p}')|^2\, \langle a_{\mathbf{p}'\sigma}(t) |\, a^\dagger_{\mathbf{p}'\sigma}\rangle\, \langle \rho_{\mathbf{p}-\mathbf{p}'}(t)\,|\, \rho_{\mathbf{p}'-\mathbf{p}}\rangle + \sum_{\mathbf{p}'} |V_{\rm sf}(\mathbf{p}-\mathbf{p}')|^2\, \Big\{ \Big\langle a_{\mathbf{p}'\sigma}(t) |\, a^\dagger_{\mathbf{p}'\sigma}\Big\rangle$$
$$\times (\delta_{\sigma,\uparrow} + \delta_{\sigma,\downarrow})\, \langle S^z_{\mathbf{p}-\mathbf{p}'}\,(t)\, | S^z_{\mathbf{p}'-\mathbf{p}}\rangle + \langle a_{\mathbf{p}'\bar\sigma}\,(t)\,|\, a^\dagger_{\mathbf{p}'\bar\sigma}\rangle$$
$$\times \Big(\delta_{\sigma,\downarrow}\, \langle S^+_{\mathbf{p}-\mathbf{p}'}\,(t)\, | S^-_{\mathbf{p}'-\mathbf{p}}\rangle + \delta_{\sigma,\uparrow} \langle S^-_{\mathbf{p}-\mathbf{p}'}\,(t)\, | S^+_{\mathbf{p}'-\mathbf{p}}\rangle \Big) \Big\}$$
$$= \sum_{\mathbf{p}'} \{ |V_{\rm ph}(\mathbf{p}-\mathbf{p}')|^2\, \langle \rho_{\mathbf{p}-\mathbf{p}'}(t) | \rho_{\mathbf{p}'-\mathbf{p}}\rangle + \; 3\, |V_{\rm sf}(\mathbf{p}-\mathbf{p}')|^2\, \langle S^z_{\mathbf{p}-\mathbf{p}'}(t) | S^z_{\mathbf{p}'-\mathbf{p}}\rangle \}\, \langle a_{\mathbf{p}'\sigma}(t) |\, a^\dagger_{\mathbf{p}'\sigma}\rangle, \tag{74}$$

where we took into account that in a paramagnetic state $\langle a_{\mathbf{p}'\sigma}(t) |\, a^\dagger_{\mathbf{p}'\sigma}\rangle = \langle a_{\mathbf{p}'\bar\sigma}(t) |\, a^\dagger_{\mathbf{p}'\bar\sigma}\rangle$ and $\langle S^+_{\mathbf{q}}(t) | S^-_{-\mathbf{q}}\rangle = \langle S^-_{\mathbf{q}}(t) | S^+_{-\mathbf{q}}\rangle = 2 \langle S^z_{\mathbf{q}}(t) | S^z_{-\mathbf{q}}\rangle$.

For the anomalous time-dependent correlation function in (73) the NCA gives

$$\sum_{\mathbf{p}',\mathbf{p}''} \langle A_\sigma(\mathbf{p},\mathbf{p}')(t) | A_{\bar\sigma}(-\mathbf{p},-\mathbf{p}'')\rangle = \sum_{\mathbf{p}',\mathbf{p}''} V_{\rm ph}(\mathbf{p}-\mathbf{p}')\, V_{\rm ph}(-\mathbf{p}''+\mathbf{p})$$
$$\times \langle \rho_{\mathbf{p}-\mathbf{p}'}(t)\, a_{\mathbf{p}'\sigma}(t) | a_{-\mathbf{p}''\bar\sigma}\, \rho_{-\mathbf{p}+\mathbf{p}''}\rangle + \sum_{\mathbf{p}',\mathbf{p}''} V_{\rm sf}(\mathbf{p}-\mathbf{p}')\, V_{\rm sf}(\mathbf{p}''-\mathbf{p})$$
$$\times \Big\langle \Big\{ S^z_{\mathbf{p}-\mathbf{p}'}(t)\, (\delta_{\sigma,\uparrow} - \delta_{\sigma,\downarrow})\, a_{\mathbf{p}'\sigma}(t) + (S^+_{\mathbf{p}-\mathbf{p}'}(t)\, \delta_{\sigma,\downarrow} + S^-_{\mathbf{p}-\mathbf{p}'}(t) \delta_{\sigma,\uparrow}) a_{\mathbf{p}'\bar\sigma}(t) \Big\} \Big|$$
$$\times \Big\{ S^z_{\mathbf{p}''-\mathbf{p}}\, (\delta_{\sigma,\downarrow} - \delta_{\sigma,\uparrow})\, a_{-\mathbf{p}''\bar\sigma} + (S^+_{\mathbf{p}''-\mathbf{p}} \delta_{\sigma,\uparrow} + S^-_{\mathbf{p}''-\mathbf{p}}\, \delta_{\sigma,\downarrow}) a_{-\mathbf{p}''\sigma} \Big\} \Big\rangle$$
$$\simeq \sum_{\mathbf{p}'} |V_{\rm ph}(\mathbf{p}-\mathbf{p}')|^2\, \langle a_{\mathbf{p}'\sigma}(t) |\, a_{-\mathbf{p}'\bar\sigma}\rangle\, \langle \rho_{\mathbf{p}-\mathbf{p}'}(t)\,|\, \rho_{\mathbf{p}'-\mathbf{p}}\rangle$$

$$+ \sum_{\mathbf{p}'} |V_{\rm sf}(\mathbf{p}-\mathbf{p}')|^2 \, \{ \langle a_{\mathbf{p}'\sigma}(t)|\, a_{-\mathbf{p}'\bar\sigma}\rangle (-\delta_{\sigma,\uparrow} - \delta_{\sigma,\downarrow}) \langle S^z_{\mathbf{p}-\mathbf{p}'}(t)|S^z_{\mathbf{p}'-\mathbf{p}}\rangle$$

$$+ \langle a_{\mathbf{p}'\bar\sigma}(t)|a_{-\mathbf{p}'\sigma}\rangle (\delta_{\sigma,\downarrow}\, \langle S^+_{\mathbf{p}-\mathbf{p}'}(t)|S^-_{\mathbf{p}'-\mathbf{p}}\rangle + \delta_{\sigma,\uparrow}\langle S^-_{\mathbf{p}-\mathbf{p}'}(t)|S^+_{\mathbf{p}'-\mathbf{p}}\rangle)\}$$

$$= \sum_{\mathbf{p}'} \{ |V_{\rm ph}(\mathbf{p}-\mathbf{p}')|^2 \langle \rho_{\mathbf{p}-\mathbf{p}'}(t)|\rho_{\mathbf{p}'-\mathbf{p}}\rangle - 3|V_{\rm sf}(\mathbf{p}-\mathbf{p}')|^2 \langle S^z_{\mathbf{p}-\mathbf{p}'}(t)|S^z_{\mathbf{p}'-\mathbf{p}}\rangle \} \langle a_{\mathbf{p}'\sigma}(t)|a_{-\mathbf{p}'\bar\sigma}\rangle, \quad (75)$$

where we took into account that for the anomalous correlation functions we have the relations: $\langle a_{\mathbf{p}'\bar\sigma}(t)|\, a_{-\mathbf{p}'\sigma}\rangle = \langle a_{-\mathbf{p}'\bar\sigma}(t)|\, a_{\mathbf{p}'\sigma}\rangle = -\langle a_{\mathbf{p}'\sigma}(t)|\, a_{-\mathbf{p}'\bar\sigma}\rangle$. Thus, a corresponding spin-fluctuation contribution to the anomalous self-energy has a sign opposite to that in the normal self-energy. Taking into account a negative sign in the definition of the anomalous self-energy (73), we obtain the effective interaction (21) for the normal and anomalous components with equal signs for the spin-scattering contribution and opposite signs for charge-scattering contribution.

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Термодинамічні функції Гріна в теорії надпровідності

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Формулюється загальна теорія надпровідності в рамках методу термодинамічних функцій Гріна для різних типів спарювань через фонони, спінові флуктуації та сильні кулонівські кореляції у моделі Хаббарда та t-J моделі. Точне рівняння Дайсона для матриці функцій Гріна отримано через власну енергію як багаточастинкову функцію Гріна. Застосовуючи неперехресне наближення для власної енергії, отримано замкнуту самоузгоджену систему рівнянь, подібну до звичайних рівнянь Еліашберга. Коротко обговорено надпровідність завдяки кінематичній взаємодії та оцінено температуру переходу у надпровідний стан в моделі Хаббарда.

Ключові слова: *функції Гріна, теорія надпровідності, сильні електронні кореляції*

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