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Self-gravitational system. New approach

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The field theory approach to statistical description of the system of gravitational interacting particles is proposed in order to describe spatially inhomogeneous structures. A nonperturbutive calculation of the partition function is demonstrated for such a system. Spatially inhomogeneous system's state – cluster is considered. The spatial distribution function, cluster's size and the conditions of phase transition to the collapsed phase are determined exactly in this approach.

Key words: *gravitational interaction, cluster, spatially inhomogeneous distribution, collapse, soliton solution*

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The statistical description of the of interacting particles has attracted a permanent attention. A few model systems of interacting particles are known, as far as the partition function can be exactly evaluated, at least, in the thermodynamic limit. The gravitation system does not have an exact solution so far. The problem of mean-field thermodynamics of self-gravitational system lies in the possible collapse in this system. An important point, which emerges from these studies and which is quite obvious is the non-extensiveness of the usual thermodynamic function in the thermodynamic limit, when the number of particle $N \to \infty$. But the example of scaling consideration suggests an extensive homogeneous mean field in thermodynamic limit when the $N \to \infty$ [1]. The formation of the spatial inhomogeneous distribution of the particle and field distribution which accompanies the gravitational interaction requires another approach which can describe the cluster formation which is related as collapsing states. In this paper the developed approach [3–5] suggests a statistical description of gravitational interacting particles of the system with regard to cluster formation. Systems with spatially inhomogeneous particle distributions are described in terms of various approaches. Within this approach, special methods [3–5] have been proposed concerning the selection of states with thermodynamically stable spatially inhomogeneous particle distributions. When describing a wide range of systems of interacting particles with regard to the type of statistics but neglecting the quantum correlations, so that the interaction is treated in the classical manner, we can write the Hamiltonian of the system as given by [2,3,10,12]

$$H(n) = \sum_{s} \varepsilon_s n_s - \frac{1}{2} \sum_{s,s'} W_{ss'} n_s n_{s'} \,, \tag{1}$$

where ε_s is the additive part of the particle energy in the state s which is equal to the kinetic energy in most cases, $W_{ss'}$ are attraction energies for the particles in the states s and s'. The macroscopic states of the system are described by a set of occupation numbers n_s. Index s labels an individual particle state; it can also correspond to a fixed site of the Ising lattice [10], whose explicit form is irrelevant in the continuum approximation. This expression for the Hamiltonian also holds for the model of substitution and interstitial solid solutions with two atom species present [2]. It is clear that to calculate the partition function is a rather involved problem even in the case of the Ising model with Hamiltonian equation (1). Let us select the states that bring the dominant contribution

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to it and take into account the probable spatial inhomogeneity of the particle distribution in the system.

The partition function for the grand canonical ensemble of a system of interacting particles with Hamiltonian equation (1) is given by

$$Z_N = \sum_{\{n\}} \exp\left(-\beta H(n)\right) = \sum_{\{n\}} \exp\left\{-\beta\left[\sum_s \varepsilon_s n_s - \frac{1}{2}\sum_{s,s'} W_{ss'} n_s n_{s'}\right]\right\}, \quad (2)$$

where $\sum_{\{n\}}$ implies the summation over all probable distributions $\{n_s\}$ and $\beta \equiv (kT)^{-1} \equiv \theta^{-1}$ is the inverse temperature. In order to perform formal summation in equation (2) we introduce additional field variables making use of the known results of the theory of Gaussian integrals [6,11], i.e.,

$$\exp\left\{\frac{1}{2\theta}\nu^2 \sum_{s,s'} \omega_{ss'} n_s n_{s'}\right\} = \int_{-\infty}^{\infty} D\varphi \exp\left\{\nu \sum_s n_s \varphi_s - \frac{\theta}{2}\sum_{s,s'} \omega_{ss'}^{-1} \varphi_s \varphi_{s'}\right\},$$

where $D\varphi = \prod_s \mathrm{d}\varphi_s \left(\sqrt{\det 2\pi\beta\omega_{ss'}}\right)^{-1}$ and $\omega_{ss'}^{-1}$ is the inverse of the interaction matrix that satisfies the condition $\omega_{ss''}^{-1}\omega_{s''s'} = \delta_{ss'}$; we have $\nu^2 = \pm 1$ depending on the character of interaction and the sign of the potential energy. Within the context of (3), the partition function of a system of interacting particles may be rewritten as

$$Z = \int_{-\infty}^{\infty} D\varphi \sum_{\{n_s\}} \exp\left\{\sum_s \left(\varphi_s - \beta\varepsilon_s\right) n_s - \frac{1}{2\beta}\sum_{s,s'} \left(W_{ss'}^{-1} \varphi_s \varphi_{s'}\right)\right\}. \quad (3)$$

We fix the total number of particles $N = \sum\limits_s n_s$ (this is equivalent to the consideration of the canonical ensemble). The procedure can be carried out in several ways [10], we use the well known formula $(2\pi \mathrm{i})^{-1} \oint \mathrm{d}\xi \xi^{\sum\limits_s n_s - N - 1} = 1$.

Then the partition function of a system with a fixed number of particles may be written as given by [11]

$$Z_N = \frac{1}{2\pi}\oint \mathrm{d}\xi \int_{-\infty}^{\infty} D\varphi \exp\left\{-\sum_{s,s'}\left(W_{ss'}^{-1}\varphi_s\varphi_{s'}\right) - (N+1)\ln\xi\right\} \prod_s \sum_{\{n_s\}} \left[\xi \exp\left(\varphi_s - \beta\varepsilon_s\right)\right]^{n_s}. \quad (4)$$

Now we can carry out the summation over the occupation numbers n_s, then the partition function reduces to

$$Z_N = \frac{1}{2\pi}\oint \mathrm{d}\xi \int_{-\infty}^{\infty} D\varphi \exp(-S\left(\varphi, \xi\right)), \quad (5)$$

where

$$S\left(\varphi, \xi\right) = \frac{1}{2\beta}\sum_{s,s'}\left(W_{ss'}^{-1}\varphi_s\varphi_{s'}\right) + \delta\sum_s \ln\left(1 - \delta\xi e^{\varphi - \beta\varepsilon_s}\right) + (N+1)\ln\xi \quad (6)$$

and $\delta = \pm 1$ for various statistics (the upper and lower signs correspond to Bose and Fermi statistics, respectively). Presentation of the partition function in terms of a functional integral over the additional fields corresponds to the construction of an equilibrium sequence of alternative probable states treated with regard to their weights. This presentation of the partition function enables us to make use of the efficient methods developed in the quantum field theory and to formulate the principle of state selection without imposing additional restrictions and fixing the order of the perturbation theory. Thus we can take into account the states associated with spatially inhomogeneous particle distributions. To do this, it is sufficient to treat $S\left(\varphi, \xi\right)$ as a variable functional depending on the distribution of the fields φ and ψ and the chemical potential analog ξ. We employ the saddle point method to find the asymptotic value of the partition function Z_N for $N \to \infty$; the dominant contribution is given by the states which satisfy the extremum condition for the functional. The solutions which correspond to the finite action $S(\varphi, \xi)$ as the volume of the system tends to infinity, may be interpreted as a thermodynamically stable particle distribution. It depends on the saddle point solutions of $\delta S/\delta\xi = \delta S/\delta\varphi = 0$, whether this distribution is spatially inhomogeneous or not. The above set of equations in principle solves the many-particle problem of selection of states with the dominant contribution in the partition function of the system of interacting particles. The normalization condition provides a possibility to regard the $\rho_s = (\xi_s e^{\varphi_s})/(1 - \delta\xi_s e^{\varphi_s})$ as the generalized one-particle distribution function with regard to the interaction in terms of additional fields. It is clear that for zero fields, the partition function reduces to the standard Bose or Fermi distributions. The inhomogeneous behavior of the fields provides inhomogeneity of the particle distribution which, in this approach, is associated with the nature and intensity of interaction. It is easy to show that the character of the distribution function behaviour also depends on the interaction. The representation thus proposed makes it possible to extend the Bose-condensation concept to the coordinate space treatment. The cluster formation is associated with the accumulation of particles within a finite spatial region and is reflected in the behavior of the fields and the chemical potential. In order to specialize the results to be obtained, we use the continuum approximation in what follows. In the continuum limiting case, δ takes a continuous set of values within the volume occupied by the system. Integration over the momenta and coordinates should be performed with regard to the cell volume $(2\pi h)^3$ in the space of individual states. The inverse matrix $\omega_{ss'}^{-1}$ for the interaction, $\omega_{ss'} = \omega(|r_s - r_{s'}|)$, in the continuum case should be treated in the operator sense [3,11], i.e.,

$$\omega_{rr'}^{-1} = \delta_{rr'} \widehat{L_{r'}} \, , \tag{7}$$

where $\widehat{L_{r'}}$ is the operator for which the Green function is given by the interaction potential. For the interactions associated with Coulomb or Newton potentials, the inverse operator is given by

$$\widehat{L_{r'}} = -\frac{1}{4\pi g^2} \triangle_{r'} \, , \tag{8}$$

where g^2 is the interaction constant. The inverse operator can be found for a restricted number of realistic interactions.

Let us consider a system of particles whose interactions consist of gravitational attraction and hard sphere repulsion. For the Newtonian attraction the inverse operator (8) has no screening. In the case of hard sphere interaction, the inverse operator can be described by expression $U_{r,r'}^{-1} = U_0 \delta_{rr'}$.

Using the results of the hard sphere model we get an expression for action, for Boltzman statistics, that [4,5]:

$$S = \int_{V_0} \mathrm{d}V \left\{ \left\{ \frac{1}{2r_m} (\nabla\varphi)^2 - \frac{\xi}{\lambda^3} e^{\varphi} \right\} + \frac{\xi}{\lambda^3} V_0 + (N+1) \ln \xi \right\}, \tag{9}$$

where the constants $\lambda = \left[(\beta h^2)/(2\pi m)\right]^{1/2}$, $r_m = 2\pi G m^2 \beta$, $V_0 \approx 2 v_0 N$ is the volume which will be occupied by the particles if they are collected close to each other, v_0 is the volume of one particle and the integration is carried out over the whole space except for the volume occupied by particles. An expression analogous to this one was obtained in [7]. However,the authors did not fix the number of particles and disregarded the particle repulsion. The result of this article can be supplemented with the solution that allows for inhomogeneous particle distributions. Let us introduce a dimensionless quality $r = R/r_m$ and denote $r_m^3/\lambda^3 = \alpha^2$, then rewrite equation (9) in terms of the new variable $\sigma = \exp 0.5\varphi$ to obtain

$$S = \int \mathrm{d}\widetilde{V} \left\{ \left(\frac{1}{\sigma} \frac{\mathrm{d}\sigma}{\mathrm{d}r} \right)^2 - \xi \alpha^2 \sigma^2 \right\} + \frac{\xi}{\lambda^3} V_0 + (N+1) \ln \xi. \tag{10}$$

An extremum of the effective action is realized by the solution of the equation which represents the saddle-point relation:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}r^2} - \frac{1}{\sigma}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right)^2 + \xi\alpha^2\sigma^2 = 0 \tag{11}$$

with the first integral of motion

$$\left(\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right)^2 + \xi\alpha^2\sigma^2 = \Delta^2. \tag{12}$$

Thus the extremum is given by

$$\widetilde{\sigma} = \frac{\Delta}{\sqrt{\xi}\alpha}\frac{1}{\cosh\Delta\left(r - r'\right)}\,, \tag{13}$$

where r' is the soliton center coordinate. This solution describes a spatially inhomogeneous distribution of the field φ and hence, within the context of the distribution function definition, $\rho = m\xi\lambda^{-1}e^{\varphi} = m\xi\lambda^{-1}\sigma^2$, the particle distribution and thus may be interpreted as a finite-size cluster. Thus the introduction of ansaz $\varphi = \ln\sigma^2$ enabled us to find the solution of the nonlinear equation $0.5(\mathrm{d}^2\varphi)/(\mathrm{d}r^2) + \xi\alpha^2\exp\varphi = 0$ which satisfies the extremum of action

$$\int r^2\mathrm{d}r\left\{(2r_m)^{-1}\left(\nabla\varphi\right)^2 - \xi\lambda^{-3}e^{\varphi}\right\}.$$

Within the context of the first integral (14), the action can be rewritten in the form

$$S = 4\pi\int_{r_0}^{d} r^2\mathrm{d}r\left\{\Delta^2 - 2\xi\alpha^2\sigma^2\right\} + \frac{\xi}{\lambda^3}V_0 + (N+1)\ln\xi\,. \tag{14}$$

In our interpretation, any soliton solution corresponds to spatial inhomogeneous particle distribution as finite-size cluster. It depends on the interaction parameter, chemical potential ξ and temperature at which the solution is realized. This corresponds to the solution with asymptotic $\sigma = 1, \varphi = 0$ for $r = d$ and $\sigma = 0$ as $r \to \infty$. In our model under consideration, the soliton solution is associated with the case when the particles in the inhomogeneous formation of the size d are present and the particles at infinity are absent. It is not difficult to notice in the case $\Delta d > 1$, that the action equation (10) has no minima. It follows that

$$\int_{r_0}^{d} -\left(2\xi\alpha^2\sigma^2\right)r^2\mathrm{d}r = \int_{r_0}^{d} -\left(2\xi\alpha^2\exp(-\Delta r)r^2\right)\mathrm{d}r \to 0 \quad \text{for} \quad \Delta \to \infty,$$

and action

$$S = \frac{\Delta^2}{r_m^3}\left(V - V_0\right) + \xi\alpha^2\frac{V_0}{r_m^3} + (N+1)\ln\xi$$

has no minima.

When the $\triangle d \leqslant 1$ we can decompose $1/\cosh x \approx 1 - x^2/2$ in power series of $x \equiv \Delta d \ll 1$ and find Δ^2 from the asymptotics $1 = \Delta^2/(\xi\alpha^2)\left[1 - \Delta^2 d^2\right] \Longrightarrow \Delta^2 \approx \xi\alpha^2 + \xi^2 d^2\alpha^4$. Thus, assuming that $V \gg V_0$, we have the result:

$$S = -\frac{V - 2V_0}{\lambda^3}\xi + (N+1)\ln\xi - \frac{V - V_0}{\lambda^3}\xi^2 d^2\alpha^2. \tag{15}$$

Then the free energy can be expressed through this action $F = kTS(\widetilde{\xi})$ where $\widetilde{\xi} \approx \lambda^3 N(V - V_0)^{-1}$ has been found from the "saddle point" equation $\partial S/\partial\xi = 0$. Minimizing equation (15) by the size of cluster $d = D/r_m$ we obtain the optimum radius of the cluster [5]:

$$d_0^2 = \frac{V}{4Nr_m^3}\left(1 - \frac{V_0}{V}\right). \tag{16}$$

The decrease of the cluster's size with the increase of the number of particles in the system N is connected with a closer packing of the particles in the cluster due to the increase of the gravitational energy. The rising of the cluster's size with temperature is connected with less close packing of the particles due to the resistance of thermal motion energy to gravitational energy. Such a situation is realized due to the long-range attraction $(1/R)$ of gravitational interaction.

It is easy to see from equation (11) that this equation has soliton solution equation (13) whenever any thermodynamical conditions $\xi\alpha^2 \equiv N r_m^3 V^{-1}$ take place. This means that gravitating gas is always in the collapsed state. Such a situation takes place in the limits $N \to \infty$ and $V \to \infty$, but $\frac{V}{N}$ is fixed. Let us suppose that our system has a terminated volume and consists of sufficiently large but terminal number of particles neglecting its own volume V_0. The termination of the number of particles means that the function equation (13) is normalized as

$$r_m^3 \int \rho(r)\mathrm{d}^3 r = mN \quad \text{or} \quad \int \sigma^2 \mathrm{d}^3 r = \frac{V}{r_m^3}\,. \tag{17}$$

Then, the Lagrangian in the action equation (10) should be replaced by the Lagrangian:

$$\left(\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right)^2 - (\xi\alpha^2 - \chi)\sigma^2, \tag{18}$$

where χ is Lagrange indefinite multiplier which appears due to the normalization equation (17). The corresponding action has a minimum in relation to the solution of the equation:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}r^2} - \frac{1}{\sigma}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right)^2 + (\xi\alpha^2 - \chi)\sigma^2 = 0. \tag{19}$$

The multiplier χ is a function of N such that $\lim_{N\to\infty}\chi = 0$ because equation (19) should be turned into equation (11) in the limit $N \to \infty$ (or $V \to \infty$).

If $\xi\alpha^2 - \chi > 0$, then solution (13) is realized. If this condition is not satisfied, then other spatial distributions (homogeneous) take place. The temperature or concentration at which the equality is reached

$$r_m^3 N (V\chi)^{-1} = 1 \tag{20}$$

is the point of transition between homogeneous and inhomogeneous distributions – i.e., collapse.

Let us assume two actions *on the saddle point*. One of them is the action of a collapsed gas equation (14) and the other one is the action corresponding to a spatially homogeneous distribution. In the case of collapses the action equation (14) is smaller than the action of spatially homogeneous distribution [5] (absolute minimum) and they are equal in the point of the collapse:

$$\Delta^2 \frac{V}{r_m^3} - 2\xi\alpha^2 \frac{V}{r_m^3} + N\ln\xi = -\frac{V}{\lambda^3} + N\ln\xi\,, \tag{21}$$

whence it follows $r_m^3 N V^{-1} = \Delta^2$. Comparing this equality with equation (20) we can find $\chi = \Delta^2$. Let us integrate equation (13) over all space and use the normalization equation (17). We find that $\Delta = \pi^3/(3N)$. Then, at temperatures or concentrations at which inequality

$$\frac{r_m^3 N^3}{V} \geqslant \left(\frac{\pi^3}{3}\right)^2 \tag{22}$$

is executed, the gravitating gas is in a collapsed state. This process is connected with the increase of the gravitation energy at the concentration increasing and with the decrease of the thermal energy at the temperature decreasing.

If $r_m^3 N^3 / V\chi < 1$, then we have spatial distribution $\sigma = \frac{\Delta}{\sqrt{\xi}\alpha}/\sinh(\Delta r)$. In this case the inequality is executed:

$$\Delta^2 \frac{V}{r_m^3} + 2\xi\alpha^2 \frac{V}{r_m^3} + N\ln\xi = \Delta^2 \frac{V}{r_m^3} + 2N + N\ln\xi > -\frac{V}{\lambda^3} + N\ln\xi = -N + N\ln\xi. \tag{23}$$

Thus, the homogeneous spatial distribution is realized at this condition.

The equality equation (22) determines such a volume, starting with which a further compression is accompanied by the collapse:

$$V_m = \frac{36}{\pi^3}\left(\frac{Nm^2G}{kT}\right)^3 . \tag{24}$$

If we assume that $V_m = V_0$, then we can determine the critical temperature as the highest temperature at which the gas cannot collapse at any volume:

$$kT_c \sim Nm^2GV_0^{-1/3} . \tag{25}$$

Thus, the limitation of the number of particles or the volume of the system brings about the transition from spatial homogeneous distribution to spatial inhomogeneous distribution – i.e., a collapse. In case of $N \to \infty$, the gravitating gas is always in a collapsed state. The calculation of the own volume of the particles gives a critical point on the P–V diagram.

The proposed statistical approach is based on the self-consistent field approximation for the case of a spatially inhomogeneous particle distribution. This makes the essence of selecting the states that bring the dominant contribution to the thermodynamic potential of the system. The procedure reduces to the calculation of the free energy in terms of the slowest variable accompanied by the averaging over all probable fluctuations. The averaged effect of fluctuations forms the effective potential for a slow variable whereas the nonlinearity thus obtained provides spatial inhomogeneity of the particle distribution.

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Самогравітаційна система. Новий підхід

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Запропонований польовий підхід до статистичного опису системи гравітаційно взаємодіючих частинок для того щоб описати просторово неоднорідні структури. Продемонстровано непертурбаційне обчислення статистичної суми для такої системи. Розглянута просторово неоднорідна структура – кластер. В рамках цього підходу точно знайдені просторова функція розподілу, розмір кластера та умова фазового переходу в колапсуючий стан.

Ключові слова: *гравітаційна взаємодія, кластер, просторово неоднорідний розподіл, колапс, солітонний розв'язок*

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