Condensed Matter Physics, 2005, Vol. 8, No. 2(42), pp. 317–324

CONDENSED MATTER PHYSICS

Lagrangian vector field and Lagrangian formulation of partial differential equations

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Received November 4, 2004

In this paper we consider the Lagrangian formulation of a system of second order quasilinear partial differential equations. Specifically we construct a Lagrangian vector field such that the flows of the vector field satisfy the original system of partial differential equations.

Key words: *Euler-Lagrange equations, vector fields, quasi-linear partial differential equations*

PACS: *45.20.Jj, 02.30.Jr*

1. Introduction

Variational principle has played a fundamental role in the foundation of mathematics and physics. Minimization of an action integral by means of a Lagrange function gives rise to the set of Euler-Lagrange equations (ELs). Invariance of the action integral or ELs is intimately related to conservation laws [1]. It is well known in Lagrangian mechanics that the flows of the Lagrangian vector field satisfy ELs [2], where the set of ELs form a system of second order ordinary differential equations. In this paper we consider the Lagrangian formulation of a system of second order quasilinear partial differential equations in terms of ELs. Our objective is to construct a Lagrangian vector field X_{L} such that the flows of X_{L} satisfy ELs. We believe that this work is still lacking in the literature.

Let B_n be an open set in E_n with smooth boundary ∂B_n, and let $x = (x^1, x^2, \ldots, x^n)$ be the coordinate cover of B_n. Consider the following system of second order quasilinear PDEs:

$$F^{ij}_{ab}(x, f^c, \partial_k f^c)\partial_i\partial_j f^b + G_a(x, f^c, \partial_k f^c) = 0, \tag{1}$$

where $a, b, c \in [1, 2, \ldots, N]$ and $\{F^{ij}_{ab}, G_a\}$ are given functions. In (1) we have adopted the conventional summation notation for repeated indices. The solution set of (1) for the N functions f^a can be viewed as a map Φ from the base space B_n to the graph space $G = B_n \times R_N$ with coordinate cover $(x, u^a), 1 \leqslant a \leqslant N$. The map Φ is said to be a regular map if $\Phi^*\mu \neq 0$, where Φ^* is the pull back map of Φ, $\mu = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n$ is the volume element of B_n, and $\wedge$ is the exterior product of differential forms [3]. Next we imbed the graph space G in a larger space $M = G \times R_{nN}$ with coordinate cover (x, u^a, y_i^a). We furnish M with N contact 1-forms $w^a = \mathrm{d}u^a - y_i^a \mathrm{d}x^i$, where $w^a \wedge (\mathrm{d}w^a)^n \neq 0$, and $(\mathrm{d}w^a)^{n+1} = 0$. Here $(\centerdot)^m$ denotes the m^{th} exterior power. Then we extend the map Φ from B_n to M under the conditions $\Phi^*\mu \neq 0$ and $\Phi^* w^a = 0$. Since $\Phi^* w^a = \Phi^*(\mathrm{d}u^a) - \Phi^*(y_i^a)\Phi^*(\mathrm{d}x^i) = \mathrm{d}f^a - \Phi^*(y_i^a)\mathrm{d}x^i = 0$, we have $\Phi^*(y_i^a) = \partial_i f^a$, that is, the pull back of y_i^a from M to G is $\partial_i f^a$. In the following discussions we say that $\Phi : B_n \to M$ is a regular map if $\Phi^*\mu \neq 0$ and $\Phi^* w^a = 0$. From geometric point of view, a regular map Φ has the representation $\Phi \mid x^i = x^i, u^a = f^a(x), y_i^a = \partial_i f^a(x)$. It defines an n-dimensional section of M.

Based on the system of PDEs in (1), we define the system of n-forms $\{E_a\}$,

$$E_a = F_{ab}^{ij}(x, u^c, y_k^c)\mathrm{d}y_i^b \wedge \mu_j + G_a(x, u^c, y_k^c)\mu, \tag{2}$$

where $\mu_j = \partial_j \lrcorner \mu$, $\lrcorner$ is the inner multiplication between the partial differential operator ∂_j and the n-form μ, such that, $\partial_j \lrcorner \mathrm{d}x^i = \delta_j^i$. Then

$$\Phi^* E_a = \{F_{ab}^{ij}(x, f^c, \partial_k f^c)\partial_i \partial_j f^b + G_a(x, f^c, \partial_k f^c)\}\mu.$$

Hence regular maps Φ solve the system of PDEs (1) if and only if $\Phi^* E_a = 0$. Therefore, instead of considering the system of PDEs (1), we concentrate on the set of n-forms $\{E_a\}$ and consider regular maps Φ such that $\Phi^* E_a = 0$.

2. Lagrangian formulation of PDEs

The solution set $\{f^a(x)\}$ forms the states of the system of PDEs (1). Suppose the system has a smooth Lagrangian function $L(x, f^a(x), \partial_i f^a(x))$. Define the action integral $A[f^a]$ of the system by

$$A[f^a] = \int_{B_n} L(x, f^a(x), \partial_i f^a(x))\mu. \tag{3}$$

Lift the Lagrangian function$L$ from the graph space G to M so that $L(x, u^a, y_i^a)$ is defined on M. Then $\Phi^* L(x, u^a, y_i^a) = L(x, f^a(x), \partial_i f^a(x))$. We also define the Euler-Lagrange n-forms $E_a(L)$ by

$$E_a(L) = L_{u^a}\mu - (\mathrm{d}L_{y_i^a}) \wedge \mu_i\,, \tag{4a}$$

which can be rewritten as

$$E_a(L) = L_{u^a}\mu - \left\{(L_{y_i^a u^b})\mathrm{d}u^b + (L_{y_i^a y_j^b})\mathrm{d}y_j^b\right\} \wedge \mu_i\,. \tag{4b}$$

Here $L_{y_i^a u^b} = \partial^2 L/(\partial u^b \partial y_i^a)$, $L_{y_i^a y_j^b} = \partial^2 L/(\partial y_j^b \partial y_i^a)$. Assume that L is independent of x explicitly. Further assume that the values of $\{f^a(x)\}$ are specified on the boundary ∂B_n. It can be proved that a regular map $\Phi : B_n \to M$ stationarizes the action integral $A[f^a]$ if and only if $\Phi^* E_a(L) = 0$ at every interior point of B_n [4]. Hence $\Phi^* E_a(L) = 0$ yields the Euler-Lagrange equations

$$\frac{\partial L}{\partial f_a} = D_i \left[\frac{\partial L}{\partial(\partial_i f^a)} \right], \qquad 1 \leqslant a \leqslant N, \tag{5}$$

where
$$D_i = \left(\partial_i f^b\right) \frac{\partial}{\partial f^b} + \left(\partial_i \partial_j f^b\right) \frac{\partial}{\partial\left(\partial_j f^b\right)}$$

is the total derivative operator.

Now we consider the existence as well as the construction of the Lagrangian function L on M. To this end, we let $V_a = G_a, \mathrm{d}V_a^i = -F_{ab}^{ij}\mathrm{d}y_j^b$, and $Q_a = V_a\mu - \mathrm{d}V_a^i \wedge \mu_i$. The system $\{Q_a\}$ is said to admit a variational principle if and only if there exists a Lagrangian function $L(x, u^a, y_i^a)$ defined on M such that $V_a = L_{u^a}$ and $V_a^i = L_{y_i^a}$. Set $Q = V_a\mathrm{d}u^a + V_a^i\mathrm{d}y_i^a$. Then $Q = \mathrm{d}L$ and $Q \wedge \mu = \mathrm{d}(L\mu)$ is a closed $(n+1)$-form. On the other hand, if $Q \wedge \mu$ is a closed $(n+1)$-form, it can be proved that the system $\{Q_a\}$ admits a variational principle [5]. Thus we first define $Q = V_a\mathrm{d}u^a + V_a^i\mathrm{d}y_i^a$ and determine whether or not $Q \wedge \mu$ is a closed $(n+1)$-form. Suppose the answer is affirmative. The Lagrange function L can be written as $L = V_a u^a + V_a^i y_i^a$. Then $E_a(L) = Q_a$.

On the other hand, suppose the system $\{Q_a\}$ does not admit a variational principle. We can extend the space M to a larger space $\hat{M} = M \times E_{N+nN}$ with coordinate cover $(x, u^a, y_i^a, \hat{u}^a, \hat{y}_i^a)$. Equip $\hat{M}$ with the contact 1-forms $w^a = \mathrm{d}u^a - y_i^a\mathrm{d}x^i$ and $\hat{w}^a = \mathrm{d}\hat{u}^a - \hat{y}_i^a\mathrm{d}x^i$. In addition to the n-forms Q_a, we introduce N auxiliary n-forms $\hat{Q}_a = \hat{V}_a\mu - \mathrm{d}\hat{V}_a^i \wedge \mu_i$, where $\hat{V}_a$ and $\hat{V}_a^i$ are given by

$$\hat{V}_a = \partial_a \left(V_b\hat{u}^b + V_b^j\hat{y}_j^b\right), \tag{6a}$$
$$\hat{V}_a^i = \partial_a^i \left(V_b\hat{u}^b + V_b^j\hat{y}_j^b\right) \tag{6b}$$

with $\partial_a = \partial/\partial u^a$ and $\partial_a^i = \partial/\partial y_i^a$.

Let $L = V_b\hat{u}^b + V_b^j\hat{y}_j^b$. Define the 1-form Q for the extended system $\{Q_a, \hat{Q}_a\}$ by

$$Q = V_a\mathrm{d}\hat{u}^a + V_a^i\mathrm{d}\hat{y}_i^a + \hat{V}_a\mathrm{d}u^a + \hat{V}_a^i\mathrm{d}y_i^a\,. \tag{7}$$

Then $Q \wedge \mu = \mathrm{d}(L\mu)$. Hence the extended system $\{Q_a, \hat{Q}_a\}$ admits a variational principle with the Lagrangian function $L = V_a\hat{u}^a + V_a^i\hat{y}_i^a$. The Euler-Lagrange equations for the extended system can be written as

$$\Phi^* Q_a = \Phi^* \left\{ L_{u^a} - \left(L_{y_i^a\hat{u}_b}\right)\hat{y}_i^b - \left(L_{y_i^a\hat{y}_j^b}\right)\partial_i\hat{y}_j^b \right\}\mu = 0, \tag{8a}$$
$$\Phi^* \hat{Q}_a = \Phi^* \left\{ L_{\hat{u}^a} - \left(L_{\hat{y}_i^a u^b}\right) y_i^b - \left(L_{\hat{y}_i^a y_j^b}\right)\partial_i y_j^b \right\}\mu = 0. \tag{8b}$$

3. Construction of the Lagrangian vector field

In the following discussions we assume that L is independent of x explicitly. In Lagrangian Mechanics, the Lagrangian 1-form θ, the Lagrangian 2-form Ω, and the total energy of the system E are defined as follows [6]:

$$\theta = \sum_i L_{\dot{q}^i} \mathrm{d}q^i, \tag{9a}$$

$$\Omega = L_{\dot{q}^i q^j} \mathrm{d}q^i \wedge \mathrm{d}q^j + L_{\dot{q}^i \dot{q}^j} \mathrm{d}\dot{q}^i \wedge \mathrm{d}q^j, \tag{9b}$$

$$E = \dot{q}^i L_{\dot{q}^i} - L. \tag{9c}$$

In a similar manner, we define the Lagrangian 1-form θ_{L}, the Lagrangian 2-form Ω_{L}, and the energy density E_{L} for the extended system $\{Q_a, \hat{Q}_a\}$ by

$$\theta_{\mathrm{L}} = \sum_i \left[L_{y_i^a} \mathrm{d}u^a + L_{\hat{y}_i^a} \mathrm{d}\hat{u}^a \right], \tag{10a}$$

$$\begin{aligned} \Omega_{\mathrm{L}} = \mathrm{d}\theta_{\mathrm{L}} = \sum_i \Big\{ & \left(L_{y_i^a \hat{y}_j^b} \right) \mathrm{d}\hat{y}_j^b \wedge \mathrm{d}u^a + \left(L_{y_i^a \hat{u}^b} \right) \mathrm{d}\hat{u}^b \wedge \mathrm{d}u^a \\ & + \left(L_{\hat{y}_i^a y_j^b} \right) \mathrm{d}y_j^b \wedge \mathrm{d}\hat{u}^a + \left(L_{\hat{y}_i^a u^b} \right) \mathrm{d}u^b \wedge \mathrm{d}\hat{u}^a \Big\}, \end{aligned} \tag{10b}$$

$$E_{\mathrm{L}} = y_i^a L_{y_i^a} + \hat{y}_j^b L_{\hat{y}_j^b} - L. \tag{10c}$$

We assume that the Lagrangian function L is nondegenerate, i.e., $L_{y_i^a \hat{y}_j^b}, L_{\hat{y}_i^a y_j^b}$ are nowhere vanishing on $\hat{M}$. Since L, Ω_{L} and E_{L} are independent of x, we define the Lagrangian vector field $X_{\mathrm{L}} \in T\hat{M}$[1] by

$$X_{\mathrm{L}} = Y_k^c D_{ku^c} + \hat{Y}_k^b D_{k\hat{u}^b} + Z_{kl}^c D_{ky_l^c} + \hat{Z}_{kl}^c D_{k\hat{y}_l^c},$$

where $Y_k^c, \hat{Y}_k^b, Z_{kl}^c, \hat{Z}_{kl}^c$ are functions defined on $\hat{M}$, while $D_{ku^c}, D_{k\hat{u}^b}, D_{ky_l^c}, D_{k\hat{y}_l^c}$ are partial differentiation operators with respect to $u^c, \hat{u}^b, y_l^c, \hat{y}_l^c$ respectively. Notice that we have adopted double subscripts for the D operators to emphasize that the flows of X_{L} are given by

$$Y_k^c = \frac{\partial u^c}{\partial t^k}, \qquad \hat{Y}_k^b = \frac{\partial \hat{u}^b}{\partial t^k}, \qquad Z_{kl}^c = \frac{\partial y_l^c}{\partial t^k}, \qquad \hat{Z}_{kl}^c = \frac{\partial \hat{y}_l^c}{\partial t^k}, \tag{11}$$

where $(t^1, t^2, \ldots, t^n)$ are n parameters.

In Lagrangian mechanics the Lagrangian vector field X is constructed by the condition $X \lrcorner \Omega = -\mathrm{d}E$. It is well known that the flows of X satisfy the Euler-Lagrange equations [6]. In a similar manner, we determine the Lagrangian vector field X_{L} by the condition

$$X_{\mathrm{L}} \lrcorner \Omega_{\mathrm{L}} = -\mathrm{d}E_{\mathrm{L}}. \tag{12}$$

By comparing the coefficients of $\mathrm{d}y_i^a, \mathrm{d}\hat{y}_j^b, \mathrm{d}u^a$ and $\mathrm{d}\hat{u}^b$ on both sides of (12) we can obtain the following results:

$$-Y_i^a (L_{y_i^a \hat{y}_j^b}) = -y_i^a (L_{y_i^a \hat{y}_j^b}), \tag{13a}$$

$$-\hat{Y}_j^b (L_{\hat{y}_j^b y_i^a}) = -\hat{y}_j^b (L_{\hat{y}_j^b y_i^a}), \tag{13b}$$

$$(L_{y_i^a \hat{u}^b}) \hat{Y}_i^b + (L_{y_i^a \hat{y}_j^b}) \hat{Z}_{ij}^b = L_{u^a}, \tag{13c}$$

$$(L_{\hat{y}_j^b u^a}) Y_j^a + (L_{\hat{y}_j^b y_i^a}) Z_{ji}^a = L_{\hat{u}^b}. \tag{13d}$$

[1] Alternatively, we can write

$$X_{\mathrm{L}} = W_i \partial_i + Y_k^c D_{ku^c} + \hat{Y}_k^b D_{k\hat{u}^b} + Z_{kl} D_{ky_l^c} + \hat{Z}_{kl}^c D_{k\hat{y}_l^c}.$$

However, W_i turns out to be zero identically.

Since the Lagrangian function is nondegenerate, by (13a) and (13b) we obtain $Y_i^a = y_i^a, \hat{Y}_j^b = \hat{y}_j^b$. On the other hand, $\hat{Z}_{ij}^b$ and Z_{ji}^a can be solved from (13c) and (13d) respectively. In view of (11) we can identify $t^k = x^k$. Thus, $Z_{ji}^a = \partial_j y_i^a, \hat{Z}_{ij}^b = \partial_i \hat{y}_j^b$, and (13c), (13d) become

$$(L_{y_i^a \hat{u}^b})\hat{y}_i^b + (L_{y_i^a \hat{y}_j^b})(\partial_i \hat{y}_j^b) = L_u^a \,, \tag{14a}$$

$$(L_{\hat{y}_j^b u^a})y_j^a + (L_{\hat{y}_j^b y_i^a})(\partial_j y_i^a) = L_{\hat{u}^b} \,. \tag{14b}$$

Hence (14a) and (14b) in conjunction with (11) give rise to (8a) and (8b). Therefore the flows of X_{L} satisfy the Euler-Lagrange equations.

To conclude this section we consider an example in extended irreversible thermodynamics:

$$\partial_x^2 T - \frac{\tau}{\chi}\partial_t^2 T - \frac{1}{\chi}\partial_t T = 0, \tag{15}$$

where T is the temperature, τ and χ are constants.

Let $x^1 = x, x^2 = t$. Then $y_1 = \partial_1 T, y_2 = \partial_2 T, \mu = \mathrm{d}x \wedge \mathrm{d}t = \mathrm{d}x^1 \wedge \mathrm{d}x^2, \mu_1 = \partial_1 \lrcorner \mu = \mathrm{d}x^2, \mu_2 = \partial_2 \lrcorner \mu = -\mathrm{d}x^1$. The contact manifold M has the global coordinate cover (x^1, x^2, u, y_1, y_2), and is equipped with the contact 1-form $\omega = \mathrm{d}u - y_i \mathrm{d}x^i$, $i = 1, 2$.

(i) By (15) the 2-form E is given by

$$E = -\mathrm{d}\left(\frac{1}{\chi}u + \frac{\tau}{\chi}y_2\right) \wedge \mu_2 + \mathrm{d}y_1 \wedge \mu_1 \,.$$

The regular map $\Phi : B_2 \to M$ yields $\Phi^* E = 0$, which is identical to (15). Let $Q_1 = E = v_1 \mu - \mathrm{d}v_1^i \wedge \mu_i$. We have $v_1 = 0, v_1^1 = -y_1, v_1^2 = \frac{1}{\chi}(u + \tau y_2)$. If we set $Q = v_1 \mathrm{d}u + v_1^1 \mathrm{d}y_1 + v_1^2 \mathrm{d}y_2$, then $Q \wedge \mu$ is not a closed 3-form. Consequently (15) does not admit a variational principle[2]. For this reason we enlarge M to $\hat{M}$, where $\hat{M}$ has the coordinate cover $(x, u, y_1, y_2, \hat{u}, \hat{y}_1, \hat{y}_2)$, and is equipped with the contact 1-forms $\omega = \mathrm{d}u - y_i \mathrm{d}x^i, \hat{\omega} = \mathrm{d}\hat{u} - \hat{y}_i \mathrm{d}x^i$.

(ii) Let

$$L = v_1 \hat{u} + v_1^1 \hat{y}_1 + v_1^2 \hat{y}_2 = -y_1 \hat{y}_1 + \left(\frac{1}{\chi}u + \frac{\tau}{\chi}y_2\right)\hat{y}_2 \,.$$

[2]It should be noted that there are many different formulations for a system of second order quasi-linear PDEs to be transcribed into a system on n-forms. For the specific PDE in (15), we could have constructed the 2-form

$$E' = \mathrm{d}y_1 \wedge \mu_1 - \frac{\tau}{\chi}\mathrm{d}y_2 \wedge \mu_2 - \frac{1}{\chi}y_2 \mu.$$

In view of the of the contact 1-form $\omega = \mathrm{d}u - y_i \mathrm{d}x^i$, we can easily show that

$$E = E' - \frac{1}{\chi}\omega \wedge \mu_2,$$

i.e., $E' = E \bmod \omega$. Hence both $\Phi^* E = 0$ and $\Phi^* E' = 0$ give rise to (15). Similarly, if we set $v_1 = -\chi^{-1} y_2$, $\mathrm{d}v_1^1 = -\mathrm{d}y_1$, $\mathrm{d}v_1^2 = \tau\chi^{-1}\mathrm{d}y_2$, then $Q_1' = v_1 \mu - \mathrm{d}v_1^i \wedge \mu_i = Q_1 + \chi^{-1}\omega \wedge \mu_2$. Hence $Q_1' = E'$ and $Q_1 \wedge \mu = Q_1' \wedge \mu$.

Then
$$\hat{v}_1 = \partial_u L = \frac{1}{\chi}\hat{y}_2, \hat{v}_1^1 = L_{y_1} = -\hat{y}_1, \hat{v}_1^2 = L_{y_2} = \frac{\tau}{\chi}\hat{y}_2.$$
Thus
$$\hat{Q}_1 = \hat{v}_1\mu - \mathrm{d}\hat{v}_1^i \wedge \mu_i = \frac{1}{\chi}\hat{y}_2\mu + \mathrm{d}\hat{y}_1 \wedge \mu_1 - \frac{\tau}{\chi}\mathrm{d}\hat{y}_2 \wedge \mu_2. \tag{16}$$
Set
$$Q = v_1\mathrm{d}\hat{u} + v_1^1\mathrm{d}\hat{y}_1 + v_1^2\mathrm{d}\hat{y}_2 + \hat{v}_1\mathrm{d}u + \hat{v}_1^1\mathrm{d}y_1 + \hat{v}_1^2\mathrm{d}y_2\,.$$
Then $Q = \mathrm{d}L$. Hence $Q \wedge \mu = \mathrm{d}(L\mu)$ and, the extended system $\{Q_1, \hat{Q}_1\}$ admits a variational principle with the Lagrange function L given by
$$L = -y_1\hat{y}_1 + \frac{\tau}{\chi}y_2\hat{y}_2 + \frac{1}{\chi}u\hat{y}_2.$$

(iii) Based on the Lagrange function L, we can construct the Euler-Lagrange 2-forms as follows:
$$\begin{aligned} E(L) &= L_u\mu - \mathrm{d}L_{y_i} \wedge \mu_i = \frac{1}{\chi}\hat{y}_2\mu + \mathrm{d}\hat{y}_1 \wedge \mu_1 - \frac{\tau}{\chi}\mathrm{d}\hat{y}_2 \wedge \mu_2 = \hat{Q}_1, \\ \hat{E}(L) &= L_{\hat{u}}\mu - \mathrm{d}L_{\hat{y}_i} \wedge \mu_i = \mathrm{d}y_1 \wedge \mu_1 - \frac{\tau}{\chi}\mathrm{d}y_2 \wedge \mu_2 - \frac{1}{\chi}\mathrm{d}u \wedge \mu_2 = Q_1\,. \end{aligned}$$

Hence the Euler-Lagrange equations for the extended system $\{Q_1, \hat{Q}_1\}$ become
$$\begin{aligned} \Phi^* E(L) &= \Phi^*\left\{L_u - (L_{y_i\hat{u}})\hat{y}_i - (L_{y_i\hat{y}_j})\partial_i\hat{y}_j\right\}\mu \\ &= \Phi^*\left\{\frac{1}{\chi}\hat{y}_2 + \partial_1\hat{y}_1 - \frac{\tau}{\chi}\partial_2\hat{y}_2\right\}\mu = 0, \qquad (17a) \\ \Phi^*\hat{E}(L) &= \Phi^*\left\{L_{\hat{u}} - (L_{\hat{y}_i u})y_i - (L_{\hat{y}_i y_j})\partial_i y_j\right\}\mu \\ &= \Phi^*\left\{\frac{1}{\chi}y_2 - \partial_1 y_1 + \frac{\tau}{\chi}\partial_2 y_2\right\}\mu = 0. \qquad (17b) \end{aligned}$$

Equations (17a) and (17b) in turn yield
$$\partial_x^2\hat{T} + \frac{1}{\chi}\partial_t\hat{T} - \frac{\tau}{\chi}\partial_t^2\hat{T} = 0, \tag{18a}$$
$$\partial_x^2 T - \frac{1}{\chi}\partial_t T - \frac{\tau}{\chi}\partial_t^2 T = 0. \tag{18b}$$

(iv) By the Lagrangian function
$$L = -y_1\hat{y}_1 + \left(\frac{1}{\chi}u + \frac{\tau}{\chi}y_2\right)\hat{y}_2$$
we construct the Lagrangian 1-form
$$\theta_{\mathrm{L}} = \sum_i (L_{y_i}\mathrm{d}u + L_{\hat{y}_i}\mathrm{d}\hat{u}) = -\hat{y}_1\mathrm{d}u + \frac{\tau}{\chi}\hat{y}_2\mathrm{d}u - y_1\mathrm{d}\hat{u} + (\frac{\tau}{\chi}y_2 + \frac{1}{\chi}u)\mathrm{d}\hat{u}.$$

Then the Lagrangian 2-form $\Omega_{\rm L}$ becomes

$$\Omega_{\rm L} = {\rm d}\theta_{\rm L} = -{\rm d}\hat{y}_1 \wedge {\rm d}u + \frac{\tau}{\chi}{\rm d}\hat{y}_2 \wedge {\rm d}u - {\rm d}y_1 \wedge {\rm d}\hat{u} + \frac{\tau}{\chi}{\rm d}y_2 \wedge {\rm d}\hat{u} + \frac{1}{\chi}{\rm d}u \wedge {\rm d}\hat{u}.$$

Define the Lagrangian vector field $X_{\rm L}$ by

$$X_{\rm L} = Y_i D_{iu} + \hat{Y}_i D_{i\hat{u}} + Z_{ij} D_{iy_j} + \hat{Z}_{ij} D_{i\hat{y}_j}\,.$$

The flows of $X_{\rm L}$ can be written as

$$\frac{\partial u}{\partial t^i} = Y_i\,, \qquad \frac{\partial \hat{u}}{\partial t^i} = \hat{Y}_i\,, \qquad \frac{\partial y_j}{\partial t^i} = Z_{ij}\,, \qquad \frac{\partial \hat{y}_j}{\partial t^i} = \hat{Z}_{ij}\,.$$

On the other hand, the energy density of the extended system $\{Q_1, \hat{Q}_1\}$ is given by[3]

$$E_{\rm L} = y_i L_{y_i} + \hat{y}_i L_{\hat{y}_i} - L = -y_1\hat{y}_1 + \frac{\tau}{\chi} y_2\hat{y}_2\,.$$

Now $X_{\rm L}$ can be determined by the equation $X_{\rm L} \lrcorner \Omega_{\rm L} = -{\rm d}E_{\rm L}$. Hence

$$Y_i = y_i\,, \qquad \hat{Y}_i = \hat{y}_i\,,$$

$$\frac{\tau}{\chi} Z_{22} + \frac{1}{\chi} Y_2 - Z_{11} = \frac{\tau}{\chi}\partial_2 y_2 + \frac{1}{\chi} y_2 - \partial_1 y_1 = 0, \tag{19a}$$

$$\frac{\tau}{\chi} \hat{Z}_{22} - \frac{1}{\chi} \hat{Y}_2 - \hat{Z}_{11} = \frac{\tau}{\chi}\partial_2 \hat{y}_2 - \frac{1}{\chi} \hat{y}_2 - \partial_1 \hat{y}_1 = 0. \tag{19b}$$

Applying Φ^* on (19a) and (19b) we can recover (18a) and (18b) respectively. Hence the flows of $X_{\rm L}$ give rise to the Euler-Lagrange equations.

4. Conclusions

It is well known that many important theories in physical sciences can be described by ordinary or partial differential equations. In general these equations can be formulated via the variational principle in terms of a Lagrangian function. In this paper we consider the Lagrangian formulation of a system of quasilinear PDEs, where the underlying system of equations can be recast in terms of the Euler-Lagrange equations. Then we construct a vector field such that the flows of the vector field satisfy the Euler-Lagrange equations.

A similar approach to the problem considered in this paper has also been developed by Guo, Shang and Mei [7]. Based on the idea of adjoint symmetries, they modified a nonconservative system by adding a set of adjoint equations to the system. Thus the dimension of the configurational manifold is double of the dimension of the original system. Consequently the modified system admits a regular Lagrangian. From this point of view, there appears to be some similarity between our work and the work of Guo, *et al*. However, the methodology is very different, especially in the construction of the Lagrangian vector field. It would be interesting to extend this work to a wider class of PDEs, or to consider the Hamiltonian formulation of PDEs and the construction of the Hamiltonian vector field.

[3]We have abused the definition of the energy density $E_{\rm L}$ for the extended system $\{Q_1, \hat{Q}_1\}$. If $E_{\rm L}$ is given by $E_{\rm L} = y_1\hat{y}_1 + \tau\chi^{-1}y_2\hat{y}_2$, then $\int_{B_n} E_{\rm L}\mu$ is a constant. But the determination of $X_{\rm L}$ by the condition $X_{\rm L} \lrcorner \Omega_{\rm L} = -{\rm d}E_{\rm L}$ leads to contradiction.

Acknowledgements

The author wishes to thank the reviewer for many valuable comments, in particular, for bringing reference [7] to the attention of the author.

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Векторне поле Лагранжа та лагранжове формулювання парціальних диференціальних рівнянь

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Отримано 4 листопада 2004 р.

В роботі розглядається лагранжове формулювання системи квазілінійних парціальних диференціальних рівнянь другого порядку. Зокрема, представлена конструкція векторного поля Лагранжа у формі, коли потік векторного поля задовільняє вихідні системи парціальних диференціальних рівнянь.

Ключові слова: *рівняння Ейлера-Лагранжа, векторні поля, квазілінійні парціальні диференціальні рівняння*

PACS: *45.20.Jj, 02.30.Jr*