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The strong-weak coupling symmetry in 2D Φ^4 field models

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It is found that the exact beta-function $\beta(g)$ of the continuous 2D $g\Phi^4$ model possesses two types of dual symmetries, these being the Kramers-Wannier (KW) duality symmetry and the strong-weak (SW) coupling symmetry $f(g)$, or S-duality. All these transformations are explicitly constructed. The S-duality transformation $f(g)$ is shown to connect domains of weak and strong couplings, i.e. above and below g^*. Basically it means that there is a tempting possibility to compute multiloop Feynman diagrams for the β-function using high-temperature lattice expansions. The regular scheme developed is found to be strongly unstable.

Approximate values of the renormalized coupling constant g^* found from duality symmetry equations are in an agreement with available numerical results.

Key words: *Kramers-Wannier duality, S-duality, renormalization group, beta-function*

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1. Introduction

The 2D Ising model and some other lattice spin models are known to possess the remarkable Kramers-Wannier(KW) duality symmetry, playing an important role both in statistical mechanics, quantum field theory [1,2] and in superstring models [3]. The self-duality of the isotropic 2D Ising model means that there exists an exact mapping between the high-T and low-T expansions of the partition function [2]. In the transfer-matrix language this implies that the transfer-matrix of the model under discussion is covariant under the duality transformation. If we assume that the critical point is unique, the KW self-duality would yield the exact Curie temperature of the model. This holds for a large set of lattice spin models including systems with quenched disorder (for a review see [4]). Recently, the Kramers-Wannier duality

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symmetry was extended to the continuous 2D $g\Phi^4$ model [5] in the strong-coupling regime, i.e. for $g > g^*$.

This beta-function $\beta(g)$ is to date known only in the five-loop approximation within the framework of the conventional perturbation theory at the fixed dimension $d = 2$ [6].

The strong coupling expansion for the calculation of the beta-function of the 2D scalar $g\Phi^4$ theory as an alternative approach to the convemtional perturbation theory (described in the excellent textbook, known among experts in the field as Bible [7]) and was developed in [8].

It is well known from quantum field theory and statistical mechanics that any strong coupling expansions are closely connected with the high-temperature (HT) series expansions for lattice models. From the field-theoretical point of view the HT series are nothing but strong coupling expansions for field models, the lattices being considered as a technical device to define cut-off field theories (see [8–11] and references therein).

Calculations of beta-functions are of great interest in statistical mechanics and in quantum field theory. The beta-function contains the essential information on the renormalized coupling constant g^*, this being important in constructing the equation of state of the 2D Ising model, [7].

In this paper we study other duality symmetries of the beta-function $\beta(g)$ for the 2D $g\Phi^4$ theory regarded as a non-integrable continuum limit of the exactly solvable 2D Ising model. The main purpose is to construct exlicitly the strong-weak (SW) coupling duality transformation $f(g)$ connecting domains of weak and strong couplings, i.e. above and below $g*$. The last transformation allows one to compute the yet unknown multiloop orders $(6, 7, \ldots)$ of the β-functions based on the lattice expansions [11].

The paper is organized as follows. In section 2 we set up basic notations and define both the correlation length and beta-function $\beta(g)$. In section 3 the duality symmetry transformation $\tilde{g} = d(g)$ is derived. Then it is proved that $\beta(d(g)) = d'(g)\beta(g)$. An approximate expression for $d(g)$ is also found. Section 4 contains an explicit derivation of the strong-weak coupling transformation whilst in section 5 in order to illustrate our approach the sixth-order term of $\beta(g)$ is approximately computed. Section 4 contains disussion and some concluding remarks.

2. The correlation length and the Coupling constant

We begin by considering the classical Hamiltonian of the 2D Ising model (in the absence of an external magnetic field), defined on a square lattice with periodic boundary conditions; as usual:

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \,, \tag{1}$$

where $\langle i, j \rangle$ indicates that the summation is over all nearest-neighboring sites; $\sigma_i = \pm 1$ are spin variables and J is a spin coupling. The standard definition of the

spin-pair correlation function reads:

$$G(R) = \langle \sigma_{\mathbf{R}} \sigma_{\mathbf{0}} \rangle, \tag{2}$$

where $\langle \cdots \rangle$ stands for the conventional Gibbs average.

The statistical mechanics definition of the correlation length is given by [9]

$$\xi^2 = \frac{\mathrm{d} \ln G(p)}{\mathrm{d}p^2}|_{p=0}\,. \tag{3}$$

The quantity ξ^2 is known to be conveniently expressed in terms of the spherical moments of the spin correlation function itself, namely

$$\mu_l = \sum_{\mathbf{R}} \left(\frac{R}{a}\right)^l G(\mathbf{R}) \tag{4}$$

with a being some lattice spacing. It is easy to see that

$$\xi^2 = \frac{\mu_2}{2d\mu_0}, \tag{5}$$

where d is the spatial dimension (in our case $d = 2$).

In order to extend the KW duality symmetry to the continuous field theory we have a need for a "lattice" model definition of the coupling constant g, equivalent to the conventional one exploited in the RG approach. The renormalization coupling constant g of the $g\Phi^4$ theory is closely related to the fourth derivative of the "Helmholtz free energy", namely $\partial^4 F(T,m)/\partial m^4$, with respect to the order parameter $m = \langle \Phi \rangle$. It may be defined as follows (see [7])

$$g(T,h) = -\frac{(\partial^2 \chi/\partial h^2)}{\chi^2 \xi^d} + 3\frac{(\partial \chi/\partial h)^2}{\chi^3 \xi^d}\,, \tag{6}$$

where χ is the homogeneous magnetic susceptibility

$$\chi = \int \mathrm{d}^2 x G(x). \tag{7}$$

It is in fact easy to show that $g(T,h)$ in equation (6) is merely the standard four-spin correlation function taken at zero external momenta. The renormalized coupling constant of the critical theory is defined by the double limit

$$g^* = \lim_{h\to 0} \lim_{T\to T_\mathrm{c}} g(T,h) \tag{8}$$

and it is well known that these limits do not commute with each other. As a result, g^* is a path-dependent quantity in the thermodynamic (T,h) plane [8].

Here we are mainly concerned with the coupling constant on the isochore line $g(T > T*, h = 0)$ in the disordered phase and with its critical value

$$g_+^* = \lim_{T\to T_\mathrm{c}^+} g(T, h=0) = -\frac{\partial^2 \chi/\partial h^2}{\chi^2 \xi^d}|_{h=0}\,. \tag{9}$$

The "lattice" coupling constant g_+^* defined in equation (9) is a given function of the temperature T_c.

3. The Kramers-Wannier symmetry

The standard KW duality tranformation is known to be as follows [1–3]

$$\sinh(2\tilde{K}) = \frac{1}{\sinh(2K)}. \tag{10}$$

We shall see that it will be more convenient to deal with a new variable $s = \exp(2K)\tanh(K)$, where $K = J/T$.

It follows from the definition that s transforms as $\tilde{s} = 1/s$; this implies that the correlation length of the 2D Ising model given by $\xi^2 = s/(1-s)^2$ is a self-dual quantity [5]. Now, on the one hand, we have the formal relation

$$\xi \frac{\mathrm{d}s(g)}{\mathrm{d}\xi} = \frac{\mathrm{d}s(g)}{\mathrm{d}g}\beta(g), \tag{11}$$

where $s(g)$ is defined as the inverse function of $g(s)$, i.e. $g(s(g)) = g$ and the beta-function is given, as usual, by

$$\xi \frac{\mathrm{d}g}{\mathrm{d}\xi} = \beta(g). \tag{12}$$

On the other hand, it is known from [5]

$$\xi \frac{\mathrm{d}s}{\mathrm{d}\xi} = \frac{2s(1-s)}{(1+s)}. \tag{13}$$

From equations (11)–(13), a useful representation of the beta-function in terms of the $s(g)$ function thus follows

$$\beta(g) = \frac{2s(g)(1-s(g))}{(1+s(g))\,(\mathrm{d}s(g)/\mathrm{d}g)}. \tag{14}$$

It follows from the representation (14) that the beta-function vanishes at the point $g = g^*$ where $s(g*) = 1$. If one assumes that the fixed point is not singular (although this is not neccesary and not obvious),then from this equation it would follow that $\omega = \beta'(g)|_{g=g_c} = 1$ in agreement with the classical paper [12].

The correct solution of this non-trivial problem was found in remarkable papers belonging to the Italian group of researchers [13,15,16]. The main result is that $\omega = \beta'(g)|_{g=g_c} = 1.75$.

Let us define the dual coupling constant $\tilde{g}$ and the duality transformation function $d(g)$ as

$$s(\tilde{g}) = \frac{1}{s(g)}, \qquad \tilde{g} \equiv d(g) = s^{-1}\left(\frac{1}{s(g)}\right), \tag{15}$$

where $s^{-1}(x)$ stands for the inverse function of $x = s(g)$. It is easy to check that a further application of the duality map $d(g)$ gives back the original coupling constant, i.e. $d(d(g)) = g$, as it should be. Notice also that the definition of the duality

transformation given by equation (15) has a form similiar to the standard KW duality equation, equation (10).

Consider now the symmetry properties of $\beta(g)$. We shall see that the KW duality symmetry property, equation (10), results in the beta-function being covariant under the operation $g \to d(g)$:

$$\beta(d(g)) = d'(g)\beta(g). \tag{16}$$

To prove it let us evaluate $\beta(d(g))$. Then equation (14) yields

$$\beta\left(d(g)\right) = \frac{2s(\tilde{g})(1-s(\tilde{g}))}{(1+s(\tilde{g}))\left(\mathrm{d}s(\tilde{g})/\mathrm{d}\tilde{g}\right)}\,. \tag{17}$$

Bearing in mind equation (15) one is led to

$$\beta(d(g)) = \frac{2s(g)-2}{s(g)(1+s(g))\left(\mathrm{d}s(\tilde{g})/\mathrm{d}\tilde{g}\right)}\,. \tag{18}$$

The derivative in the r.h.s. of equation (18) should be rewritten in terms of $s(g)$ and $d(g)$. It may be easily done by applying equation (15):

$$\frac{\mathrm{d}s(\tilde{g})}{\mathrm{d}\tilde{g}} = \frac{\mathrm{d}}{\mathrm{d}\tilde{g}}\frac{1}{s(g)} = -\frac{s'(g)}{s^2(g)}\frac{1}{d'(g)}\,. \tag{19}$$

Substituting the r.h.s. of equation (19) into equation (18) one obtains the desired symmetry relation, equation (16).

Therefore, the self-duality of the model allows us to determine the fixed point value in another way, namely from the duality equation $d(g^*) = g^*$.

Making use of a rough approximation for $s(g)$, one gets [5]

$$s(g) \simeq \frac{2}{g} + \frac{24}{g^2} \simeq \frac{2}{g}\frac{1}{1-12/g} = \frac{2}{g-12}\,. \tag{20}$$

Combining this Padé-approximant with the definition of $d(g)$, equation (15), one is led to

$$d(g) = 4\frac{3g-35}{g-12}\,. \tag{21}$$

The fixed point of this function, $d(g^*) = g^*$, is easily seen to be $g^*_+ = 14$. On the one hand, that is a rough approximation, on the other hand best numerical and analytical estimates obtained by making use of lattice and conformal field theory yield $g^*_+ = 14.697323(20)$ see [13,15,16]). and references therein).

4. The weak-strong coupling dual symmetry

The beta-function of the model under discussion possesses the important algebraic property (14) (KW duality) which permits to develop the weak-strong-duality transformation $f(g)$ connecting both the weak-coupling and strong coupling regimes.

Nowadays both the five-loop approximation results ([6]) and the strong coupling expansion for the beta-function [9] are known rather well. These are given by

$$\begin{aligned}\beta_1(g) &= 2g - 2g^2 + 1.432347241g^3 - 1.861532885g^4 \\ &\quad + 3.164776688g^5 - 6.520837458g^6 + O(g^7), \end{aligned} \tag{22}$$

$$\begin{aligned}\beta_2(g) &= -2g + \frac{12}{\pi} - \frac{9}{\pi^2 g} + \frac{27}{\pi^3 g^2} + \frac{81}{8\pi^4 g^3} - \frac{3645}{16\pi^5 g^4} - \frac{15309}{32\pi^6 g^5} \\ &\quad + \frac{2187}{64\pi^7 g^6} + O(g^{-7}). \end{aligned} \tag{23}$$

Here indices $1, 2$ stand for the weak and strong coupling regimes, respectively. The main goal of this section is to determine a dual transformation $f(g)$ such as $f[f(g)] = g$ relating beta-functions $\beta_1(g)$ and $\beta_2(g)$.

From equation (14) one can easily find the functions $S_1(g), S_2(g)$ and their inverse functions $G_1(s) = S_1^{-1}(g), G_2(s) = S_2^{-1}(g)$ corresponding to two regimes. Simple but cumbersome calculations lead to

$$\begin{aligned}G_1(s) &= s + s^2 + 0.3580868104s^3 - 0.1166327797s^4 \\ &\quad - 0.1968226859s^5 - 0.1299831557s^6 + O(s^7), \end{aligned}$$

$$\begin{aligned}S_1(g) &= g - g^2 + 1.6419131896g^3 - 3.09293317g^4 \\ &\quad + 6.361881481g^5 - 13.78545095g^6 + O(g^7), \\ &\quad s \in [0,1], \qquad g \in [0, g^*], \end{aligned} \tag{24}$$

$$\begin{aligned}G_2(s) &= \frac{3}{4\pi s} + \frac{9}{2\pi} - \frac{9s}{4\pi} + \frac{18s^2}{\pi} - \frac{108s^3}{\pi} \\ &\quad + \frac{618s^4}{\pi} - \frac{3474s^5}{\pi} + \frac{19494s^6}{\pi} + O(s^7), \end{aligned}$$

$$\begin{aligned}S_2(g) &= \frac{2*3}{8\pi g} + \frac{24*3^2}{(8\pi g)^2} + \frac{264*3^3}{(8\pi g)^3} + \frac{2976*3^4}{(8\pi g)^4} + \frac{35136*3^5}{(8\pi g)^5} \\ &\quad + \frac{423680*3^6}{(8\pi g)^6} + \frac{5149824*3^7}{(8\pi g)^7} + \frac{63275520*3^8}{(8\pi g)^8} + O(g^{-9}), \\ &\quad s \in [0,1], \qquad g \in [g^*, \infty). \end{aligned} \tag{25}$$

Being equipped with these formulas one may easily construct two branches of the same duality transformation function $f_{12}(g)$ and $f_{21}(g)$ defined in different domains of g. The functions are

$$\begin{aligned}\frac{1}{f_{21}(g)} &\equiv \frac{1}{G_2(S_1(g))} = \frac{4\pi g}{3} - \frac{28\pi g^2}{3} + 220.5059303g^3 \\ &\quad - 1766.8145g^4 + 14816.94007g^5 - 127842.5955g^6\,, \\ &\quad g \in [0, g^*], \qquad f_{21}(g) \in [g^*, \infty], \end{aligned} \tag{26}$$

$$\begin{aligned} f_{12}(g) &\equiv G_1(S_2(g)) = \frac{3}{4\pi g} + \frac{63}{16\pi^2 g^2} + \frac{0.61714739472}{g^3} \\ &\quad + \frac{0.9560453953}{g^4} + \frac{1.502156783}{g^5} + \frac{2.368311503}{g^6} + O(g^7), \\ &\quad g \in [g^*, \infty), \qquad f_{12}(g) \in [0, g^*]. \end{aligned} \tag{27}$$

Functions found above look like inversion, but they are not so simple. A nontrivial example of the 2D model disordered Dirac fermions was discovered in [14]. It was shown that the beta-function of the (nonintegrable) model under consideration also exhibits the strong-weak coupling duality such as $g^* \to 1/g$ [14].

It is worth noting that the transformation found is dual indeed

$$f_{12}(f_{21}(g)) = f_{21}(f_{12}(g)) \equiv g. \tag{28}$$

Moreover, by definition weak-strong coupling expansions of $\beta(g)$ are related to each other in the following way:

$$\beta_2(g) = \frac{\beta_1(f_{12}(g))}{f'_{12}(g)}\,, \tag{29}$$

$$\beta_1(g) = \frac{\beta_2(f_{21}(g))}{f'_{21}(g)}\,. \tag{30}$$

It is rather amusing that equation (27) looks like a geometric series. Making use of the Pade method we arrive at

$$\begin{aligned} f_{12}(g) &\approx \frac{0.2387324146g^2 - 0.0745907136g + 0.0850867165}{g^3 - 1.983571753g^2 + 1.086109562g - 0.6919672492} \\ &\quad g \in [g^*, \infty), \qquad f_{12}(g) \in [0, g^*]. \end{aligned} \tag{31}$$

The weak-strong duality equation and strong-coupling expansion yield the following numerical values (the "exact" estimates one may find in the previous section)

$$\begin{aligned} f_{12}(g) - g = 0, &\qquad g^* = 14.38, \\ \beta_2(g^*) = 0, &\qquad g^* = 14.63. \end{aligned} \tag{32}$$

5. Higher-order terms of the beta-function

Finally, let us consider how one can compute the $\beta(g)$ in the multiloop approximation via the strong-coupling expansion and the S-duality function. This is in order to find that one should exploit equation (29), equation (23) and the approximate expression for $f_{12}(g)$ given by equation (31).

After some tedious but routine calculations we arrive at some polynomial of 7th degree for $\beta_1(g)$.

$$\begin{aligned}\beta_1(g) &= 2g - 2g^2 + 1.432347241g^3 - 1.861532885g^4 \\ &\quad + 3.164776688g^5 - 6.520837458g^6 - 331.454743g^7\,. \end{aligned} \tag{33}$$

It is easily seen that the first 6 terms except for the 7th one are the exact perturbation expansion for $\beta_1(g)$ [6]. It would be tempting but wrong to regard equation (33) as a $\beta(g)$-function in the 7th loop approximation. In fact, the function in equation (31) is approximate, so that we have to estimate an accuracy of our calculations.

Suppose a difference between the "exact" duality function $f_{12}^{\rm exact}(g)$ and the approximate one given by equation (31) reads

$$f_{12}^{\rm exact}(g) = \frac{0.2387324146g^2 - 0.0745907136g + 0.0850867165}{g^3 - 1.983571753g^2 + 1.086109562g - 0.6919672492} + b/g^7 \tag{34}$$

with b being a fitting parameter. The straightforward calculation shows that a "new" 7th loop contribution computed by making use of the equation (34) depends on the fitting parameter b and differs vastly from the previous one, it being

$$\begin{aligned}\beta_1(g) &= 2g - 2g^2 + 1.432347241g^3 - 1.861532885g^4 \\ &\quad + 3.164776688g^5 - 6.520837458g^6 \\ &\quad + (-331.454743 + 271519.803807b)g^7\,. \end{aligned} \tag{35}$$

Therefore, we see that the approach proposed above provides a regular algorithm for computing higher-order corrections to the $\beta(g)$-function based on the lattice high-order expansions. In other words, one obtaines a tempting possibility to compute (approximately) multiloop Feynman diagrams based on the equation (30) and of high-temperature expansions [9]. Unfortunately, it is very difficult to bring into action this method because of its strong instability, [17].

6. Discussion and concluding remarks

We have shown that the β-function of the $2D$ $g\Phi^4$ theory does have the two types of dual symmetries, these being the Kramers-Wannier symmetry and the weak-strong coupling symmetry (S-duality).

Our proof of the KW symmetry is based on the properties of $g(s), s(g)$ defined only for $0 \leqslant s \leqslant 1$; $g^* \leqslant g < \infty$ and therefore does not cover the weak-coupling

region, $0 \leqslant g \leqslant g^*$. It means that the KW symmetry holds only in the strong-coupling region.

We established the existence of the dual function $f(g)$ or S-duality connecting two domains of both weak coupling and strong coupling. Given both perturbative RG calculations and lattice high-temperature expansions, this function $f(g)$ can be approximately computed. We also explicitly computed high-order terms for $\beta(g)$. A close analysis of the scheme developed shows that this is strongly unstable.

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Симетрія сильно-слабкого зв'язку в 2D Φ^4 польових моделях

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Знайдено, що точні бета-функції $\beta(g)$ неперервної 2D $g\Phi^4$ моделі володіють двома типами дуальної симетрії, а саме дуальною симетрією Крамерса-Ванньє (КВ) і симетрією сильно-слабкого (СС) зв'язку $f(g)$, або S-дуальністю. Усі ці перетворення конструюються явно. Показано, що S-дуальне перетворення $f(g)$ зв'язує домени слабкого і сильного зв'язку, тобто вище і нижче g^*. По суті це означає, що є приваблива можливість обчислювати багатопетлеві діаграми Фейнмана для β-функцій, використовуючи високотемпературні ґраткові розклади. Знайдено, що розвинута регулярна схема є дуже нестійкою.

Знайдені з рівнянь дуальної симетрії наближені значення g^* узгоджуються з доступними чисельними результатами.

Ключові слова: *Крамерса-Ванньє дуальність, S-дуальність, ренормалізаційна група, бета-функція*

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