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Interacting N-vector order parameters with $O(N)$ symmetry

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We consider the critical behavior of the most general system of two N-vector order parameters that is $O(N)$ invariant. We show that it may have a multicritical transition with enlarged symmetry controlled by the chiral $O(2) \otimes O(N)$ fixed point. For $N = 2, 3, 4$, if the system is also invariant under the exchange of the two order parameters and under independent parity transformations, one may observe a critical transition controlled by a fixed point belonging to the mn model. Also in this case there is a symmetry enlargement at the transition, the symmetry being $[SO(N) \oplus SO(N)] \otimes C_2$, where C_2 is the symmetry group of the square.

Key words: *N-vector model, $O(N)$ symmetry, multicritical transitions*

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1. Introduction

The critical behavior of a system with a single N-vector order parameter is well known [1,2]. In this paper we investigate the critical behavior of a system with two N-vector parameters that is invariant under $O(N)$ transformations and independent parity transformations.

If the two N-vector order parameters are identical, i.e. the model is symmetric under their exchange, the most general Landau-Ginzburg-Wilson (LGW) Φ^4 Hamiltonian is given by

$$\mathcal{H}_{\rm cr} = \int {\rm d}^3x \left[\frac{1}{2} \sum_\mu (\partial_\mu \phi \cdot \partial_\mu \phi + \partial_\mu \psi \cdot \partial_\mu \psi) + \frac{r}{2} \left(\phi^2 + \psi^2\right) + \frac{u_0}{4!} \left(\phi^4 + \psi^4\right) + \frac{w_0}{4!} \phi^2 \psi^2 + \frac{z_0}{4!} (\phi \cdot \psi)^2 \right], \quad (1)$$

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where ψ_a and ϕ_a are two N-dimensional vectors. This Hamiltonian is well defined for $u_0 > 0$, $2u_0 + w_0 > 0$, and $2u_0 + w_0 + z_0 > 0$. Hamiltonian (1) is invariant under the transformations

$$(\mathbb{Z}_2)_{\rm exch} \otimes (\mathbb{Z}_2)_{\rm par} \otimes O(N). \tag{2}$$

The first $\mathbb{Z}_2$ group is related to the exchange transformations $\phi \leftrightarrow \psi$, while the second group is related to the parity transformations $\phi \to -\phi$, $\psi \to \psi$, or, equivalently, $\phi \to \phi$, $\psi \to -\psi$ (note that the transformation $\phi \to -\phi$, $\psi \to -\psi$ is already accounted for by the $O(N)$ group).

If the two order parameters are not identical and therefore the symmetry is only

$$(\mathbb{Z}_2)_{\rm par} \otimes O(N), \tag{3}$$

the corresponding LGW Φ^4 Hamiltonian is

$$\begin{aligned} \mathcal{H}_{\rm mcr} &= \int {\rm d}^3x \left[\frac{1}{2} \sum_\mu (\partial_\mu \phi \cdot \partial_\mu \phi + \partial_\mu \psi \cdot \partial_\mu \psi) + \frac{r_1}{2} \phi^2 + \frac{r_2}{2} \psi^2 \right. \\ &\left. + \frac{u_0}{4!} \phi^4 + \frac{v_0}{4!} \psi^4 + \frac{w_0}{4!} \phi^2 \psi^2 + \frac{z_0}{4!} (\phi \cdot \psi)^2 \right], \end{aligned} \tag{4}$$

that is well defined for $u_0 > 0$, $v_0 > 0$, $w_0 + 2\sqrt{u_0 v_0} > 0$, and $w_0 + z_0 + 2\sqrt{u_0 v_0} > 0$. Hamiltonian (4) has two different mass terms and thus it gives rise to a variety of critical and multicritical behaviors. It generalizes the multicritical Hamiltonian considered in [3] that has $w_0 = 0$ and is symmetric under the larger symmetry group $O(N) \oplus O(N)$.

For $N = 2$ there is a transformation of the fields and couplings that leaves Hamiltonians (1) or (4) invariant. If we transform the fields as $\phi'_a = \sum_b \epsilon_{ab} \phi_b$, $\psi'_a = \psi_a$ and the couplings as

$$u'_0 = u_0, \qquad v'_0 = v_0, \qquad w'_0 = w_0 + z_0, \qquad z'_0 = -z_0, \tag{5}$$

we reobtain Hamiltonians (1) and (4) expressed in terms of the primed fields and couplings [4]. This implies that, for any FP with $z > 0$, there exist an equivalent one with the same stability properties and $z < 0$.

Finally, if we do not require the invariance of the model under independent parity transformations, i.e., the model is only $O(N)$ symmetric, we must add an additional quadratic term $\phi \cdot \psi$ and two additional quartic terms, $(\phi \cdot \psi)\phi^2$ and $(\phi \cdot \psi)\psi^2$. In this case, the general analysis becomes more complex since we have to deal with a multicritical theory with three quadratic terms.

In this paper we investigate whether theories (1) and (4) have stable fixed points (FPs) in three dimensions. We do not determine the renormalization-group (RG) flow in the full theory, but rather we show that stable FPs can be identified by an analysis of the submodels whose RG flow is already known. We consider the stable FPs of the submodels and determine their stability properties with respect to the perturbations that are present in the complete theory. In this way we are able to

identify three stable FPs. For any N, there is an $O(2) \otimes O(N)$ symmetric FP. This FP may be the relevant one for models with $z_0 > 0$, which may therefore show a symmetry enlargement at the (multi)critical transition. For $N = 2$ there is an equivalent $O(2) \otimes O(2)/\mathbb{Z}_2$ symmetric FP with $z < 0$, a consequence of symmetry (5), which may be the relevant one for models with $z_0 < 0$. For $N = 2, 3, 4$ we find that $\mathcal{H}_{\rm cr}$ – but not the multicritical theory $\mathcal{H}_{\rm mcr}$ – has another stable FP that belongs to the so-called mn model [5] with $n = 2$ and $m = N$. Also in this case there is a symmetry enlargement at the transition: the FP is symmetric under the group $[SO(N) \oplus SO(N)] \otimes C_2$ where C_2 is the symmetry group of the square. It is interesting to note that the chiral $O(2) \otimes O(N)$ FP is also stable if we do not require the model to be invariant under independent parity transformations. Indeed, the additional terms $(\phi \cdot \psi)\phi^2$ and $(\phi \cdot \psi)\psi^2$ are irrelevant perturbations at the chiral FP.

In the analysis we mainly use the minimal-subtraction ($\overline{\rm MS}$) scheme without ϵ expansion (henceforth indicated as $3d$-$\overline{\rm MS}$ scheme) in which no ϵ expansion is performed and ϵ is set to the physical value $\epsilon = 1$ [6]. In order to generate the relevant perturbative series we use a symbolic manipulation program that generates the diagrams and computes symmetry and group factors. For the Feynman integrals we use the results reported in [7]. In this way we obtained five-loop $3d$-$\overline{\rm MS}$ expansions.

2. Mean-field analysis

The mean-field analysis of the critical behavior of Hamiltonian $\mathcal{H}_{\rm cr}$ is quite straightforward. If $r > 0$ the system is paramagnetic, with $\phi = \psi = 0$. For $r < 0$ there are three possible low-temperature phases:

(a) For $w_0 > 2u_0$ and $z_0 > 2u_0 - w_0$, we have $\phi \neq 0$ and $\psi = 0$ (or vice versa). The corresponding symmetry-breaking pattern is $(\mathbb{Z}_2)_{\rm exch} \otimes (\mathbb{Z}_2)_{\rm par} \otimes O(N) \to (\mathbb{Z}_2)_{\rm par} \otimes O(N-1)$.

(b) For $z_0 < 0$ and $-2u_0 < w_0 + z_0 < 2u_0$, we have $\phi = \psi \neq 0$. The corresponding symmetry-breaking pattern is $(\mathbb{Z}_2)_{\rm exch} \otimes (\mathbb{Z}_2)_{\rm par} \otimes O(N) \to (\mathbb{Z}_2)_{\rm exch} \otimes O(N-1)$.

(c) For $z_0 > 0$ and $-2u_0 < w_0 < 2u_0$, we have $|\phi| = |\psi| \neq 0$, $\phi \cdot \psi = 0$. The corresponding symmetry-breaking pattern is $(\mathbb{Z}_2)_{\rm exch} \otimes (\mathbb{Z}_2)_{\rm par} \otimes O(N) \to (\mathbb{Z}_2)_{\rm exch} \otimes O(N-1)$.

The analysis of the mean-field behavior of $\mathcal{H}_{\rm mcr}$ is presented for $N = 4$ in [8] and it is easily extended to the present case. There are three possible phase diagrams:

(a1) For $z_0 < 0$ and $-2\sqrt{u_0 v_0} < w_0 + z_0 < 2\sqrt{u_0 v_0}$, the multicritical point is tetracritical, see figure 1. Phase 1 is paramagnetic with $\phi = \psi = 0$, in phase 2 $\phi \neq 0$ and $\psi = 0$, while in phase 3 the opposite holds, $\phi = 0$ and $\psi \neq 0$; in phase 4 $\phi \neq 0$, $\psi \neq 0$ with $\phi \| \psi$. All transitions are of second order. Transitions 1–2 and 1–3 are associated with the symmetry breaking $\mathbb{Z}_2 \otimes O(N) \to \mathbb{Z}_2 \otimes O(N-1)$. In the presence of fluctuations these transitions belong to the $O(N)$

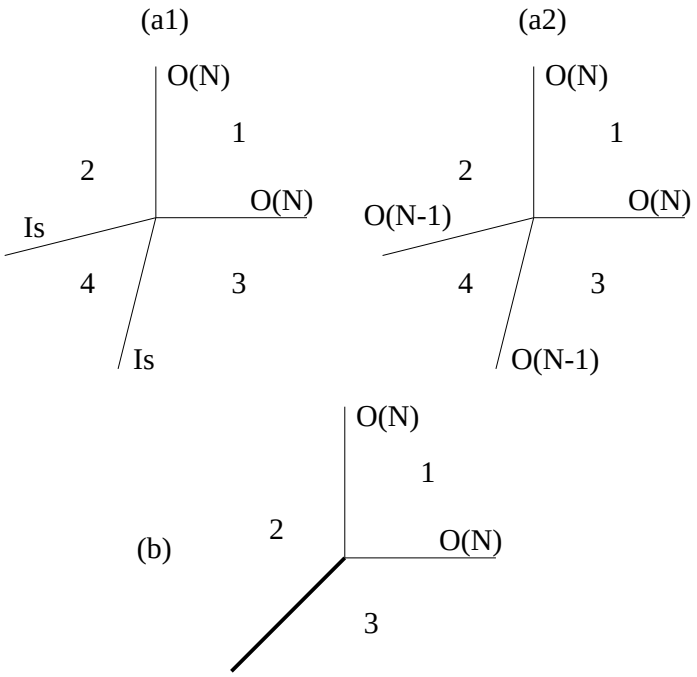

Figure 1. Possible multicritical phase diagrams. Thin lines indicate second-order transitions, while the thick line in case (b) corresponds to a first-order transition. "Is" indicates an Ising transition.

universality class. Transitions 2–4 and 3–4 are associated with the symmetry-breaking pattern $\mathbb{Z}_2 \otimes O(N-1) \to O(N-1)$. In the presence of fluctuations they should belong to the Ising universality class.

(a2) For $z_0 > 0$ and $-2\sqrt{u_0 v_0} < w_0 < 2\sqrt{u_0 v_0}$ the multicritical point is tetracritical, see figure 1. Phases 1, 2, and 3 as well as transitions 1–2 and 1–3 are identical to those discussed in case (a1). In phase 4 $\phi \neq 0$, $\psi \neq 0$ with $\phi \cdot \psi = 0$. All transitions are second-order ones. Transitions 2–4 and 3–4 are associated with the symmetry-breaking pattern $\mathbb{Z}_2 \otimes O(N-1) \to \mathbb{Z}_2 \otimes O(N-2)$ and, in the presence of fluctuations, they should belong to the $O(N-1)$ universality class.

(b) For $w_0 > 2\sqrt{u_0 v_0}$ and $w_0 + z_0 > 2\sqrt{u_0 v_0}$ the multicritical point is bicritical, see figure 1. Phases 1, 2, and 3 as well as transitions 1–2 and 1–3 are identical to those discussed in case (a1). The transition between phases 2 and 3 is of first order.

3. Analysis of some particular cases

3.1. Particular models and fixed points

The three-dimensional properties of the RG flow are determined by its FPs. Some of them can be identified by considering particular cases in which some of the quartic parameters vanish. For $\mathcal{H}_{\rm cr}$ we can easily recognize two submodels:

(a) The $O(2)\otimes O(N)$ model with Hamiltonian [9]

$$\mathcal{H}_{\rm ch} = \int \mathrm{d}^d x \left\{ \frac{1}{2}\sum_{ai}\left[\sum_{\mu}(\partial_\mu \Phi_{ai})^2 + r\Phi_{ai}^2\right] + \frac{g_{1,0}}{4!}(\sum_{ai}\Phi_{ai}^2)^2 + \frac{g_{2,0}}{4!}\left[\sum_{i,j}\left(\sum_a \Phi_{ai}\Phi_{aj}\right)^2 - \left(\sum_{ai}\Phi_{ai}^2\right)^2\right]\right\}, \quad (6)$$

where Φ_{ai} is an $N\times 2$ matrix, i.e., $a=1,\ldots,N$ and $i=1,2$. Hamiltonian (1) reduces to (6) for $2u_0 - w_0 - z_0 = 0$, if we set $\Phi_{a1}=\phi_a$, $\Phi_{a2}=\psi_a$, $u_0 = g_{1,0}$, $w_0 = 2(g_{1,0}-g_{2,0})$, and $z_0 = 2g_{2,0}$. The properties of $O(2)\otimes O(N)$ models are reviewed in [2,10–12]. In three dimensions perturbative calculations within the three-dimensional massive zero-momentum (MZM) scheme [13,14] and within the $3d$-$\overline{\rm MS}$ scheme [12] indicate the presence of a stable FP with attraction domain in the region $g_{2,0}>0$ for all values of N (only for $N=6$ the evidence is less clear: a FP is identified in the $3d$-$\overline{\rm MS}$ scheme but not in the MZM scheme). For $N=2$, these conclusions have been recently confirmed by a Monte Carlo calculation [12]. On the other hand, near four dimensions, a stable FP is found only for large values of N, i.e., $N > N_c = 21.80 - 23.43\epsilon + 7.09\epsilon^2 + O(\epsilon^3)$ [9,15–17]. A stable FP with attraction domain in the region $g_{2,0}<0$ exists for $N=2$ (it belongs to the XY universality class) [9], for $N=3$ ([18]), and $N=4$ ([4]). Note that nonperturbative approximate RG calculations have so far found no evidence of stable FPs for $N=2$ and 3 [11,19,20]. In the following we call the FP with $g_2>0$ *chiral* FP (we indicate it with $g^*_{1,\rm ch}$, $g^*_{2,\rm ch}>0$), while the FP with $g_2<0$ is named *collinear* FP and indicated with $g^*_{1,\rm cl}$, $g^*_{2,\rm cl}<0$.

(b) The so-called mn model with Hamiltonian [5]

$$\mathcal{H}_{\rm mn} = \int \mathrm{d}^d x \left\{ \frac{1}{2}\sum_{ai}\left[\sum_{\mu}(\partial_\mu \Phi_{ai})^2 + r\Phi_{ai}^2\right] + \frac{g_{1,0}}{4!}(\sum_{ai}\Phi_{ai}^2)^2 + \frac{g_{2,0}}{4!}\sum_{abi}\Phi_{ai}^2\Phi_{bi}^2\right\}, \quad (7)$$

where Φ_{ai} is an $m\times n$ matrix, i.e., $a=1,\ldots,m$ and $i=1,\ldots,n$. Hamiltonian (1) reduces to (6) for $n=2$, $m=N$, and $z_0=0$, if we set $\Phi_{a1}=\phi_a$, $\Phi_{a2}=\psi_a$, $u_0 = g_{1,0}+g_{2,0}$ and $w_0 = 2g_{1,0}$. A stable FP is the $O(m)$ FP with $g_1=0$ and

$g_2 = g_m^*$, where g_m^* is the FP value of the renormalized coupling in the $O(m)$ model. In App. A we show that the model has a second stable FP with $g_2 < 0$ for $n = 2$ and $m = 2$, 3, and 4. We name this FP the mn FP and we label the corresponding coordinates by $g_{1,mn}^*$ and $g_{2,mn}^*$.

The presence of these two submodels that have one parameter less than the original model imply that the quartic parameter space splits into four regions such that the RG flow does not cross the two planes $z = 0$ and $2u - w - z = 0$. Note that, for $N = 2$, because of symmetry (5), we should only consider the region $z \geqslant 0$.

The results for models (a) and (b) allow us to identify four possible FPs that are candidates for being stable FPs of the full theory:

(1) $u = g_{1,\rm ch}^*$, $w = 2(g_{1,\rm ch}^* - g_{2,\rm ch}^*)$, $z = 2g_{2,\rm ch}^*$; this FP may be the stable FP of the trajectories that start in the region $z_0 > 0$;

(2) $u = g_{1,\rm cl}^*$, $w = 2(g_{1,\rm cl}^* - g_{2,\rm cl}^*)$, $z = 2g_{2,\rm cl}^*$; this FP may be the stable FP of the trajectories that start in the region $z_0 < 0$;

(3) $u = g_N^*$, $w = 0$, $z = 0$; this FP may be the stable FP of the trajectories that start in the region $2u_0 - w_0 - z_0 > 0$;

(4) $u = g_{1,mn}^* + g_{2,mn}^*$, $w = 2g_{1,mn}^*$, $z = 0$ for $N = 2, 3, 4$; this FP may be the stable FP of the trajectories that start in the region $2u_0 - w_0 - z_0 < 0$.

Note that, because of symmetry (5), for $N = 2$ there is also a chiral (resp. collinear) FP with $z < 0$ (resp. $z > 0$).

The analysis of the particular cases of Hamiltonian (4) is very similar. There are two relevant submodels:

(a) The forementioned $O(2) \otimes O(N)$ model for $u_0 = v_0$ and $w_0 + z_0 = 2u_0$. The identication is $u_0 = v_0 = g_{1,0}$, $w_0 = 2g_{1,0} - 2g_{2,0}$, and $z_0 = 2g_{2,0}$.

(b) The $O(N) \oplus O(N)$ model [3]:

$$\mathcal{H}_{\rm mcr,2} = \int {\rm d}^3x \left[\frac{1}{2} \sum_\mu (\partial_\mu \phi \cdot \partial_\mu \phi + \partial_\mu \psi \cdot \partial_\mu \psi) + \frac{r_1}{2}\phi^2 + \frac{r_2}{2}\psi^2 + \frac{f_{1,0}}{4!}\phi^4 + \frac{f_{2,0}}{4!}\psi^4 + \frac{f_{3,0}}{4!}\phi^2\psi^2 \right]. \tag{8}$$

Hamiltonian $\mathcal{H}_{\rm mcr}$ reduces to this model for $z_0 = 0$, with the obvious identification of the parameters. Hamiltonian (8) describes the multicritical behavior of a model with two N-vector order parameters that is symmetric under independent $O(N)$ transformations of the two order parameters, i.e. that is invariant under the symmetry group $O(N) \oplus O(N)$ [3]. In the case we are interested in, i.e. for $N \geqslant 2$, the stable FP is the decoupled FP [21,22], i.e., $f_3 = 0$, $f_1 = f_2 = g_N^*$ (see also App. A).

Note that in this case the $O(2) \otimes O(N)$ model has two parameters less than the original one and thus its presence does not imply any separation of the RG flow. Instead, the second model implies that the quartic parameter space splits into two regions such that the RG flow does not cross the plane $z = 0$. The analysis of the possible FPs is identical to that presented above, since the FPs we have identified are exactly those we have already described.

Table 1. Estimates of the RG dimensions $y_{2,2}$, $y_{4,2}$, and $y_{4,4}$ of the operators $V^{(2,2)}$, $V^{(4,2)}$, and $V^{(4,4)}$ at the chiral (ch) FP and at the collinear (cl) FP. They have been obtained from a conformal-mapping analysis of the corresponding $3d$-$\overline{\rm MS}$ 5-loop perturbative expansions. The results for $y_{2,2}$ are taken from [4]. The results at the collinear FP for $N = 2$ have been computed by using the mapping with the XY model and the results of [21].

	$y_{2,2}$	$y_{4,2}$	$y_{4,4}$
ch,2	1.34(15)	−1.6(1.1)	−0.9(4)
ch,3	1.21(9)	−1.4(8)	−1.0(3)
ch,4	1.17(8)	−1.1(5)	−1.0(3)
ch,6	1.13(9)	−0.9(4)	−0.9(2)
ch,8	1.13(8)	−0.9(5)	−0.9(2)
ch,16	1.08(2)	−1.0(3)	−0.94(11)
ch,∞	1	−1	−1
cl,2	1.9620(8)	0	0.532(12)
cl,3	2.05(15)		0.9(3)
cl,4	2.05(15)		0.9(7)

3.2. Stability of the O(2) $\otimes$ O(N) fixed points

In this section we study the stability properties of the two FPs that appear in the $O(2) \otimes O(N)$ model. For this purpose we need to classify the perturbations of the $O(2) \otimes O(N)$ model that do not break the $O(N)$ invariance. The multicritical Hamiltonian (4) can be rewritten as

$$\mathcal{H}_{\rm mcr} = \mathcal{H}_{\rm ch} + \frac{1}{2} r_{2,2} V^{(2,2)} + \frac{1}{4!} f_{4,4} V^{(4,4)} + \frac{1}{4!} f_{4,2} V^{(4,2)}, \tag{9}$$

where $\mathcal{H}_{\rm ch}$ is the $O(2)\otimes O(N)$ Hamiltonian (6) with $r = (r_1+r_2)/2$, $g_{1,0} = (2u_0+2v_0+w_0+z_0)/6$, $g_{2,0} = (2u_0+2v_0-2w_0+z_0)/6$, and $r_{2,2} = (r_1-r_2)/2$, $f_{4,2} = (u_0-v_0)/2$, and $f_{4,4} = (u_0+v_0-w_0-z_0)/6$. Here, $V^{(2,2)}$, $V^{(4,4)}$, and $V^{(4,2)}$, are respectively a quadratic term that transforms as a spin-2 operator under the $O(2)$ group and two quartic terms that transform as a spin-4 and as a spin-2 operator respectively. Their explicit expressions are:

$$V^{(2,2)} \equiv \phi^2 - \psi^2,$$

$$
\begin{aligned}
V^{(4,2)} &\equiv (\phi^2+\psi^2)V^{(2,2)}, \\
V^{(4,4)} &\equiv (\phi^2)^2+(\psi^2)^2-2\phi^2\psi^2-4(\phi\cdot\psi)^2.
\end{aligned} \tag{10}
$$

A detailed description of all possible perturbations of the $O(2)\otimes O(N)$ FP that leave the $O(N)$ group invariant can be found in App. B of [4]. Note that for $\mathcal{H}_{\rm cr}$ we have $r_{2,2}=f_{4,2}=0$, so that one must only consider the spin-4 quartic perturbation.

Let us first discuss the chiral FP (a) that has $g^*_{2,\rm ch}>0$. In order to estimate the RG dimensions $y_{4,2}$ and $y_{4,4}$ of the above-reported perturbations, we computed the corresponding five-loop $\overline{\rm MS}$ series and we analyzed them within the $3d$-$\overline{\rm MS}$ scheme.

The perturbative series, that are not reported here but are available on request, were analyzed using the conformal-mapping method and the Padé-Borel method, following closely [23], to which we refer for details. The error on the conformal-method results takes into account the spread of the results as the parameters α and b are varied (cf. [23] for definitions) and the error due to the uncertainty of the FP location (we use the estimates reported in [4,12–14]). The results of the analyses using the conformal-mapping method are reported in table 1. Completely consistent results are obtained by using Padé-Borel approximants. As it can be seen, $y_{4,2}$ and $y_{4,4}$ are always negative, indicating that the chiral FP is stable for any N. This FP is therefore expected to be the relevant FP whenever the RG flow starts in the region $z_0>0$. For $N=2$, symmetry (5) implies that a chiral FP [the equivalent one that is obtained by using (5)] may also be reached from the region $z_0<0$.

It is of interest to compute also the RG dimension $y_{2,2}$ of the quadratic perturbation. For the multicritical Hamiltonian it is related to the crossover exponent ϕ: $\phi=\nu y_{2,2}$, where ν is the correlation-length exponent at the chiral FP (see [12–14] for numerical estimates). The exponent $y_{2,2}$ has already been computed for several values of N in [4]: indeed, $y_{2,2}=y_4$, where y_4 is the RG dimension of the operator $O^{(4)}$ defined in App. C of [4]. Numerical estimates, taken from [4], are reported in table 1.

Now, let us consider the collinear FP (b) that has $g^*_{2,\rm cl}(N)<0$ for $2\leqslant N\leqslant 4$. For $N=2$ the RG dimensions at the collinear FP can be related to the RG dimensions of operators in the XY model. Indeed, the O(2)⊗O(2) collinear FP is equivalent to an XY FP. The mapping is as follows. One defines two fields a_i and b_i, $i=1,2$, and considers [9]

$$
\begin{aligned}
\phi_{11} &= (a_1-b_2)/\sqrt{2}, \\
\phi_{22} &= (a_1+b_2)/\sqrt{2}, \\
\phi_{12} &= (b_1-a_2)/\sqrt{2}, \\
\phi_{21} &= (b_1+a_2)/\sqrt{2}.
\end{aligned} \tag{11}
$$

At the collinear FP, fields a and b represent two independent XY fields. Using this mapping it is easy to show that $V^{(4,2)}\sim \mathcal{O}_i^{(3,1)}(a)b_j$, $\mathcal{O}_i^{(3,1)}(b)a_j$, and $V^{(4,4)}\sim T_{11}(a)T_{11}(b)$, $T_{12}(a)T_{12}(b)$, where

$$
\mathcal{O}_i^{(3,1)}(a)\equiv a_i a^2, \qquad T_{ij}(a)\equiv a_i a_j-\frac{1}{2}\delta_{ij}a^2. \tag{12}
$$

Thus, if $y_{3,1}$ and y_2 are the RG dimensions of $\mathcal{O}_i^{(3,1)}$ and T_{ij} in the XY model, we have

$$y_{4,2} = y_h + y_{3,1} - 3, \qquad y_{4,4} = 2y_2 - 3. \tag{13}$$

By using the equations of motion, one can relate $\mathcal{O}_i^{(3,1)}$ to a_i [24] and obtain $y_{3,1} = 3-y_h$, so that $y_{4,2} = 0$ exactly (this holds in three dimensions; in generic dimension d, $y_{4,2} = 3-d$). For y_2 we can use the result reported in [21], obtaining $y_{4,4} = 0.532(12)$. The analysis of the perturbative series gives results that are fully consistent: $y_{4,2} = 0.0(1)$, $y_{4,4} = 0.57(4)$.

In order to determine $y_{4,2}$ and $y_{4,4}$ for $N = 3$ and 4 we analyzed the corresponding 5-loop $3d$-$\overline{\rm MS}$ expansions. The results for $y_{4,4}$ are reported in table 1. They indicate that $y_{4,4}$ is positive, which implies that the spin-4 quartic perturbation is relevant and therefore the collinear FP is unstable. We do not quote any result for $y_{4,2}$. The perturbative analysis does not allow us to obtain any reliable result: the estimates vary significantly with the parameters b and α and with the perturbative order.

It is interesting to observe that, based on the group-theoretical analysis reported in App. B of [4], any quartic perturbation of the chiral FP that leaves the $O(N)$ symmetry invariant is a combination of spin-2 and spin-4 operators. Thus, the results presented here indicate that the chiral $O(2) \otimes O(N)$ FP is stable under *any* perturbation that preserves the $O(N)$ symmetry. In particular, it is also stable under a perturbation of the form $(\phi \cdot \psi)(a\phi^2 + b\psi^2)$ that may arise if the model is not invariant under independent parity transformations. Indeed, such a term is nothing but a particular combination of spin-2 and spin-4 perturbations. In the notations of App. B of [4] (note that M and N of [4] should be replaced by N and 2 respectively) we have

$$(\phi \cdot \psi)(a\phi^2 + b\psi^2) = \frac{1}{2}(a+b)\mathcal{O}_{12}^{(4,2,1)} + \frac{1}{3}(a-b)\mathcal{O}_{1112}^{(4,4)}\,. \tag{14}$$

Moreover, the additional quadratic term $\phi \cdot \psi$ is nothing but a component of the spin-2 quadratic term that breaks the $O(2)$ group, so that the associated crossover exponent is again $\phi = \nu y_{2,2}$, $y_{2,2}$ being reported in table 1. This is a general result that follows from the analysis of [4]: any quadratic perturbation of the $O(2) \otimes O(N)$ FP that does not break the $O(N)$ invariance is a combination of the components of the spin-2 quadratic operator. Thus, any perturbation is always associated with the same crossover exponent $\phi = \nu y_{2,2}$.

3.3. Stability of the decoupled $O(N) \oplus O(N)$ fixed point

We now consider FP (3) discussed in section 3.1. In order to check its stability, we must determine the RG dimensions at the FP of the perturbations

$$P_E \equiv \phi^2\psi^2, \qquad P_T \equiv \sum_{ij} T_{\phi,ij}T_{\psi,ij}\,, \tag{15}$$

where $T_{\phi,ij} = \phi_i\phi_j - \frac{1}{2}\delta_{ij}\phi^2$. Simple RG arguments show that the RG dimensions are given by

$$y_E = \frac{2}{\nu_N} - 3 = \frac{\alpha_N}{\nu_N}, \qquad y_T = 2y_2 - 3, \tag{16}$$

where α_N and ν_N are the critical exponents of the 3-dimensional $O(N)$ universality class (see [2] for a comprehensive review of results), while y_2 is the exponent associated with the quadratic spin-2 perturbation in the $O(N)$ model [21,25,26].

Since $\alpha_N < 0$ for $N \geqslant 2$ we have $y_E < 0$, i.e. the perturbation P_E is always irrelevant. As for y_T we can use the results reported in [21,25,26]. The spin-2 exponent is equal to $y_2 = 1.766(6)$, $1.790(3)$, $1.813(6)$ for $N = 2$, 3, 4 and increases towards 2 as $N \to \infty$. Correspondingly $y_T = 0.532(12)$, $0.580(6)$, $0.626(12)$, increasing towards 1 as $N \to \infty$. It follows that P_T is always relevant. Thus, the decoupled FP is always irrelevant.

3.4. Stability of the mn fixed point

Here, we wish to consider the stability of the mn FP. For this purpose we must consider the two perturbations

$$P_1 \equiv (\phi^2)^2 - (\psi^2)^2, \qquad P_2 \equiv (\phi \cdot \psi)^2 - \frac{1}{N}\phi^2\psi^2. \tag{17}$$

Note that P_1 is not symmetric under interchange of ϕ and ψ and is therefore not of interest for $\mathcal{H}_{\rm cr}$. The corresponding RG dimensions y_1 and y_2 are computed in App. A: $y_1 = 0.4(3)$, $0.2(2)$, $0.2(2)$ for $N = 2$, 3, 4; $y_2 = -0.9(5)$, $-1.0(8)$, $-0.8(5)$ for the same values of N. They indicate that P_1 is relevant and P_2 is irrelevant at the mn FP. Therefore, the mn FP is a stable FP for $\mathcal{H}_{\rm cr}$ (only P_2 should be considered in this case) and an unstable one for $\mathcal{H}_{\rm mcr}$.

4. Conclusions

In this paper we have investigated the critical behavior of systems described by Hamiltonians (1) and (4). We find that $\mathcal{H}_{\rm cr}$ has three possible stable FPs: for any N, except possibly $N = 6$ (for such a value of N the evidence of this FP is less robust [12]), there is the $O(2) \otimes O(N)$ chiral FP that is relevant for systems with $z_0 > 0$; for $N = 2$ there is a stable chiral FP with $z < 0$ [equivalent to the previous one by symmetry (5)], that is relevant for systems with $z_0 < 0$; for $N = 2$, 3, 4, there is the mn FP that is relevant for systems with $2u_0 - w_0 - z_0 < 0$. In the multicritical theory (4) only the chiral FPs are stable. Thus, systems with $z_0 > 0$ (or, for $N = 2$, with $z_0 \neq 0$) may show a multicritical continuous transition with the larger $O(2) \otimes O(N)$ symmetry.

It is interesting to note that the most general $O(N)$-invariant LGW Hamiltonian for two N-vector parameters includes other couplings beside those present in (1) and (4). One should consider

$$\mathcal{H}_{\rm cr,ext} = \mathcal{H}_{\rm cr} + \frac{r_2}{2}\phi\cdot\psi + \frac{a_0}{4!}(\phi\cdot\psi)(\phi^2+\psi^2), \tag{18}$$

$$\mathcal{H}_{\rm mcr,ext} = \mathcal{H}_{\rm mcr} + \frac{r_3}{2}\phi\cdot\psi + \frac{1}{4!}(\phi\cdot\psi)(a_1\phi^2+a_2\psi^2), \tag{19}$$

depending whether one wants to preserve the symmetry under the exchange of the two fields. As we discussed in section 3.2, the chiral FP is a stable FP also for these two extended models.

Hamiltonians (4) and (18) have two mass parameters and thus symmetry enlargement can be observed only at the multicritical point, where the singular part of the free energy has the form

$$F_{\rm sing} = \mu_t^{2-\alpha} f(\mu_g \mu_t^{-\phi}), \tag{20}$$

where μ_t and μ_g are two linear scaling fields (linear combinations of the temperature and of another relevant parameter), α and ϕ are the specific-heat and the crossover exponents at the $O(2)\otimes O(N)$ model. Note that the same expression, with the same α and ϕ, applies to both models, apart from nonuniversal normalization constants. In $\mathcal{H}_{\rm mcr,ext}$ there are three quadratic parameters and thus the chiral multicritical point can be observed only if three relevant parameters are properly tuned. The singular part of the free energy becomes

$$F_{\rm sing} = \mu_t^{2-\alpha} f(\mu_{g1} \mu_t^{-\phi}, \mu_{g2} \mu_t^{-\phi}), \tag{21}$$

where μ_{g1} and μ_{g2} are two linear scaling fields associated with the same crossover exponent ϕ.

A. The mn model: new fixed points

In this Appendix we consider the mn model defined by Hamiltonian (7), focusing on the case $n = 2$ that is of interest for the present paper. Within the ϵ expansion one finds four FPs, the stable one being the $O(m)$ FP with $g_1 = 0$ and $g_2 = g_m^*$, where g_m^* is the FP value of the renormalized zero-momentum coupling in the $O(m)$ model, see [2,5,27] and references therein. For $m = 2$ (and $n = 2$) the mn model is equivalent [9] to the $O(2)\otimes O(2)$ model defined by Hamiltonian (6). For this model, the results of [12,13] indicate the presence of a new FP that is not predicted by the ϵ-expansion analysis. Because of the mapping, this implies the presence of a new FP in the mn model with $g_2 < 0$. In the MZM scheme the results of [13] imply the presence of a stable FP at $g_1 = 4.4(2)$ and $g_2 = -4.5(2)$, where the renormalized couplings g_1 and g_2 are normalized so that $g_1 = 3g_{1,0}/(16\pi R_{2m} m)$, $g_2 = 3g_{2,0}/(16\pi R_m m)$ at tree level (m is the renormalized zero-momentum mass), where $R_k = 9/(8+k)$. In the $3d$-$\overline{\rm MS}$ scheme, by using the results of [12], we obtain $g_1 = 2.25(13)$ and $g_2 = -2.31(21)$, where $g_i = g_{i,0}\mu^{-\epsilon}/A_d$ with $A_d = 2^{d-1}\pi^{d/2}\Gamma(d/2)$. It is thus of interest to check whether additional FPs are also present for other values of $m > 2$. As we shall show below we find an additional FP for $m = 3$ and $m = 4$. For $m \geqslant 5$ no new FP is found.

In order to check for the presence of additional FPs we considered the six-loop MZM expansions of [28] and we generated 5-loop $3d$-$\overline{\rm MS}$ expansions. For the analysis we used the conformal-mapping method: the position of the Borel singularity in the MZM scheme is reported in [28], while in the $3d$-$\overline{\rm MS}$ we used its trivial generalization. The two β functions were resummed by using several different approximants

Table 2. Results for the mn model for $n = 2$ in two different schemes. The index in column "scheme", $4l$, $5l$, $6l$, refers to the number of loops. We report the coordinate of the FP g_1, g_2, the percentage of approximants that find the zero ($p_{\rm FP}$), and the percentage of approximants that indicate that the FP is stable ($p_{\rm st}$). In the column "info" we report the number of approximants that give real (first number) and complex eigenvalues (second number).

m	scheme	g_1	g_2	$p_{\rm FP}$	$p_{\rm st}$	info
2	$3d$-$\overline{\rm MS}_{5l}$	2.3(2)	−2.3(2)	24/24	19/24	6/24, 18/24
	$3d$-$\overline{\rm MS}_{4l}$	2.4(2)	−2.5(3)	13/24	2/13	13/13, 0/24
	${\rm MZM}_{6l}$	4.60(8)	−4.51(11)	24/24	24/24	0/24, 24/24
	${\rm MZM}_{5l}$	4.7(3)	−4.6(4)	24/24	20/24	4/24, 20/24
3	$3d$-$\overline{\rm MS}_{5l}$	2.5(2)	−2.5(2)	23/24	22/23	18/23, 5/23
	$3d$-$\overline{\rm MS}_{4l}$	2.5(3)	−2.6(5)	13/24	2/13	13/13, 0/13
	${\rm MZM}_{6l}$	5.6(3)	−5.2(3)	23/24	23/23	13/23, 9/23
	${\rm MZM}_{5l}$	5.2(2)	−4.8(2)	24/24	24/24	1/24, 23/24
4	$3d$-$\overline{\rm MS}_{5l}$	2.9(4)	−2.9(3)	19/24	18/19	19/19, 0/19
	$3d$-$\overline{\rm MS}_{4l}$	3.0(3)	−3.0(4)	8/24	0/8	8/8, 0/8
	${\rm MZM}_{6l}$	6.6(6)	−6.0(6)	15/24	15/15	13/15, 2/15
	${\rm MZM}_{5l}$	5.9(3)	−5.2(3)	24/24	24/24	12/24, 12/24

depending on two parameters, b and α (see [23] for definitions). For simplicity, each time we resummed the two β functions by using the same b and α and then determined their common zeroes. In principle, it would have been more natural to consider different values of b and α for the two β functions and all possible combinations. However, as we already tested in the $O(2) \otimes O(N)$ model, the two choices give fully equivalent results. In the analysis we used $\alpha = -1, 0, 1, 2$ and $b = 4, 6, \ldots, 14$, which appeared to be an optimal choice. We report the results in table 2. For comparison, we also performed the analysis for $m = 2$, obtaining results completely consistent with those reported above. In the table we also give the percentage of cases in which a FP was found ($p_{\rm FP}$) and in which this FP was stable ($p_{\rm st}$). Finally, we also indicate the number of cases in which the stability eigenvalues were real or complex.

For $m = 2$ and $m = 3$ the presence of a new FP is unambiguous. Essentially all considered approximants at five and six loops in both schemes give a stable FP. For $m = 4$, the percentages are smaller, although the overall results are still in favor of a new stable FP. For $m \geqslant 5$ there is essentially no evidence. For $m = 2$ the stability eigenvalues are complex, in agreement with the results for the $O(2) \otimes O(2)$ model [12–14]. For $m = 3$ and $m = 4$ the numerical results favor real eigenvalues instead. It is interesting to note that the $3d$-$\overline{\rm MS}$ FPs lie at the boundary of the region in which the expansions are Borel summable, $g_1 + g_2 > 0$. This is not the case for the MZM ones (Borel summability requires $R_{2m} g_1 + R_m g_2 > 0$). Thus, the $3d$-$\overline{\rm MS}$ results should be more reliable in these models.

The mn model for $n = 2$ is invariant under the group $[SO(m) \oplus SO(m)] \otimes$

C_2 where C_2 is the symmetry group of the square. We now consider two quartic operators that break such a symmetry:

$$P_1 \equiv (\Phi_1 \cdot \Phi_1)^2 - (\Phi_2 \cdot \Phi_2)^2, \tag{22}$$

$$P_2 \equiv (\Phi_1 \cdot \Phi_2)^2 - \frac{1}{m}(\Phi_1 \cdot \Phi_1)(\Phi_2 \cdot \Phi_2), \tag{23}$$

where the scalar products are taken in the $O(m)$ space. The first operator is the only quartic one that preserves the continuous symmetry and breaks $C_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$, while the second preserves C_2 but breaks $SO(m) \oplus SO(m) \to SO(m)$. Note that in general P_1 mixes with the lower-dimensional operator $\Phi_1 \cdot \Phi_1 - \Phi_2 \cdot \Phi_2$. Such a mixing should be taken into account in the MZM scheme, but does not occur in the massless $\overline{\rm MS}$ scheme. The operators P_1 and P_2 are relevant for the stability of the FPs of the mn theory in larger models with smaller symmetry group.

We computed the anomalous dimensions of P_1 and P_2 at the new FPs of table 2 by analyzing the corresponding 5-loop $3d$-$\overline{\rm MS}$ series. The exponent y_1 was obtained from the analysis of the inverse series $1/y_1$; the direct analysis of the series of y_1 was very unstable. For y_2 we used instead the corresponding series. The results were not very stable and should be taken with caution. They are:

$$\begin{aligned} m = 2: &\quad y_1 = 0.4(3), \quad y_2 = -0.9(5); \\ m = 3: &\quad y_1 = 0.2(2), \quad y_2 = -1.0(8); \\ m = 4: &\quad y_1 = 0.2(2), \quad y_2 = -0.8(5). \end{aligned}$$

It is interesting to note that the mn model is a submodel of the multicritical Hamiltonian (8) for $r_1 = r_2$ and $f_{1,0} = f_{3,0}$ if we set $\Phi_{a1} = \phi_a$, $\Phi_{a2} = \psi_a$, $f_{1,0} = f_{3,0} = g_{1,0} + g_{2,0}$, and $f_{2,0} = 2g_{1,0}$. This implies that the new FPs may be relevant for the multicritical behavior of $\mathcal{H}_{\rm mcr,2}$. To investigate this possibility we must compute the anomalous dimension of the operator that breaks $[SO(m) \oplus SO(m)] \otimes C_2 \to O(m) \oplus O(m)$, i.e., the operator P_1. As it can be seen, $y_1 > 0$ in all cases, indicating that the mn FP is unstable in the full theory. Thus, the decoupled FP appears to be the only stable FP of the multicritical model (8) [21,22].

References

1. Zinn-Justin J. Quantum Field Theory and Critical Phenomena, fourth edition. Clarendon Press, Oxford, 2001.
2. Pelissetto A., Vicari E., Phys. Rept., 2002, **368**, No. 6, 549–727.
3. Kosterlitz J.M., Nelson D.R., Fisher M.E., Phys. Rev. B, 1976, **13**, No. 1, 412–432.
4. Calabrese P., Pelissetto A., Vicari E., Nucl. Phys., 2005, **709**, No. 3, 550–577.
5. Aharony A. Dependence of universal critical behaviour on symmetry and range of interaction. Phase Transitions and Critical Phenomena, vol. 6, p. 357–424, edited by Domb C. and Green M.S. Academic, New York, 1976.

6. Schloms R., Dohm V., Nucl. Phys. B, 1989, **328**, No. 3, 639–663;
   Phys. Rev. B, 1990, **42**, No. 10, 6142–6152;
   erratum Phys. Rev. B, 1992, **46**, No. 9, 5883.
7. Kleinert H., Schulte-Frohlinde V. Critical Properties of ϕ^4-Theories. World Scientific, Singapore, 2001.
8. Butti A., Pelissetto A., Vicari E., J. High Energy Phys., 2003, **08**, art. 029, 1–27.
9. Kawamura H., Phys. Rev. B, 1988, **38**, No. 7, 4916–4928;
   erratum Phys. Rev. B, 1990, **42**, No. 4, 2610.
10. Kawamura H., J. Phys.: Condens. Matter, 1998, **10**, No. 22, 4707–4754.
11. Delamotte B., Mouhanna D., Tissier M., Phys. Rev. B, 2004, **69**, No. 13, art. 134413, 1–53.
12. Calabrese P., Parruccini P., Pelissetto A., Vicari E., Phys. Rev. B, 2004, **70**, No. 17, art. 174439, 1–23.
13. Pelissetto A., Rossi P., Vicari E., Phys. Rev. B, 2001, **63**, No. 14, art. 140414, 1–4.
14. Calabrese P., Parruccini P., Sokolov A.I., Phys. Rev. B, 2002, **66**, No. 18, art. 180403(R), 1–4;
    Phys. Rev. B, 2003, **68**, No. 09, art. 094415, 1–8.
15. Antonenko S.A., Sokolov A.I., Varnashev K.B., Phys. Lett. A, 1995, **208**, No. 1–2, 161–164.
16. Pelissetto A., Rossi P., Vicari E., Nucl. Phys. B, 2001, **607**, No. 3, 605–634.
17. Calabrese P., Parruccini P., Nucl. Phys. B, 2004, **679**, No. 3, 568–596.
18. De Prato M., Pelissetto A., Vicari E., Phys. Rev. B, 2004, **70**, No. 21, art. 214519, 1–9.
19. Tissier M., Delamotte B., Mouhanna D., Phys. Rev. Lett., 2000, **84**, No. 22, 5208–5211.
20. Kindermann M., Wetterich C., Phys. Rev. Lett., 2001, **86**, No. 6, 1034–1037.
21. Calabrese P., Pelissetto A., Vicari E., Phys. Rev. B, 2003, **67**, No. 05, art. 054505, 1–12.
22. Aharony A., Phys. Rev. Lett., 2002, **88**, No. 05, art. 059703, 1.
23. Carmona J.M., Pelissetto A., Vicari E., Phys. Rev. B, 2000, **61**, No. 22, 15136–15151.
24. De Prato M., Pelissetto A., Vicari E., Phys. Rev. B, 2003, **68**, No. 09, art. 092403, 1–4.
25. Calabrese P., Pelissetto A., Vicari E., Phys. Rev. E, 2002, **65**, No. 4, art. 046115, 1–16.
26. Calabrese P., Pelissetto A., Vicari E. The critical behavior of magnetic systems described by Landau-Ginzburg-Wilson field theories. Frontiers in Superconductivity Research, edited by Barry P. Martins. Nova Science, Hauppauge, NY, 2004; also as: Preprint, cond-mat/0306273, 2003.
27. Dudka M., Holovatch Yu., Yavors'kii T., J. Phys. A, 2004, No. 45, 10727–10734.
28. Pelissetto A., Vicari E., Phys. Rev. B, 2000, **62**, No. 10, 6393–6409.

Взаємодіючі N-векторні параметри порядку з $O(N)$ симетрією

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Ми розглядаємо критичну поведінку найбільш загальної системи двох N-векторних параметрів порядку, яка є $O(N)$ інваріантною. Ми показуємо, що вона може мати мультикритичний перехід з розширеною симетрією контрольованою чіральною $O(2) \otimes O(N)$ нерухомою точкою. Для $N = 2, 3, 4$, якщо система є також інварінтною відносно обміну двох параметрів порядку і відносно незалежного перетворення парності, можна спостерегти критичний перехід контрольований нерухомою точкою, що належить mn моделі. Також у цьому випадку відбувається розширення симетрії при переході, симетрія стає $[SO(N) \oplus SO(N)] \otimes C_2$, де C_2 є групою симетрії на квадраті.

Ключові слова: *N-векторна модель, $O(N)$ симетрія, мультикритичні переходи*

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