Condensed Matter Physics, 2000, Vol. 3, No. 3(23), pp. 559–575

CONDENSED MATTER PHYSICS

Renormalization group domains of the scalar Hamiltonian

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Received March 9, 2000

Using the local potential approximation of the exact renormalization group (RG) equation, we show various domains of values of the parameters of the $O(1)$ -symmetric scalar Hamiltonian. In three dimensions, in addition to the usual critical surface $S_{\rm c}$ (attraction domain of the Wilson-Fisher fixed point), we explicitly show the existence of a first-order phase transition domain $S_{\rm f}$ separated from $S_{\rm c}$ by the tricritical surface $S_{\rm t}$ (attraction domain of the Gaussian fixed point). $S_{\rm f}$ and $S_{\rm c}$ are two distinct domains of repulsion for the Gaussian fixed point, but $S_{\rm f}$ is not the basin of attraction of a fixed point. $S_{\rm f}$ is characterized by an endless renormalized trajectory lying entirely in the domain of negative values of the φ^4 -coupling. This renormalized trajectory also exists in four dimensions making the Gaussian fixed point ultra-violet stable (and the φ_4^4 renormalized field theory asymptotically free but with a wrong sign of the perfect action). We also show that a very retarded classical-to-Ising crossover may exist in three dimensions (in fact below four dimensions). This could be an explanation of the unexpected classical critical behaviour observed in some ionic systems.

Key words: *renormalization group, critical, tricritical, first-order phase transition, crossover*

PACS: *05.10.Cc, 05.70.Jk, 11.10.Hi, 61.20.Qg*

1. Introduction

The object of this paper[1] is to carry on studying the local potential approximation of the exact renormalization group (RG) equation for the scalar theory [1]. In the previous publication [2] (to be considered as part I of the present work), we had already considered this approximation with a view to qualitatively discuss the connection between the standard perturbative renormalization of field theory (as it

[1] Based on a talk given at "RG 2000", Taxco, Mexico, January 1999

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can be found in most textbooks on field theory, see for example [3]) and the modern view [4] in which the renormalized parameters of a field theory are introduced as the "relevant" directions of a fixed point (FP) of a RG transform. Actually the local potential approximation, which allows us to consider all the powers of the field φ on the same footing, is an excellent textbook example of the way in which infinitely many degrees of freedom are accounted for in (nonperturbative) RG theory. Almost all the characteristics of the RG theory are involved in this approximation. The only lacking features are related to phenomena highly correlated to the non-local parts neglected in the approximation and, when the critical exponent η is small (especially for $d = 4$ and $d = 3$), one expects the approximation to be qualitatively correct regarding all aspects of the RG theory [1].

In the following we look at the domains of attraction or of repulsion of fixed points in the $O(1)$ scalar theory in three and four dimensions ($d = 3$ and $d = 4$).

At first sight, one could think that the issue considered is very simple since, with regard to criticality, the $O(1)$-symmetric systems in three dimensions are known to belong to the same class of universality (the Ising class). Now, because the Ising class is associated to the domain of attraction of the unique (non-trivial) Wilson-Fisher fixed point [5], then by adjusting one parameter (in order to reach the critical temperature[2]) any $O(1)$ scalar Hamiltonian should be driven to the Wilson-Fisher fixed point under the action of renormalization. Consequently there would be only two domains for the $O(1)$ scalar theory: the critical subspace $S_{\rm c}$ (of codimension 1) in the Wilson space (S) of infinite dimensions of the Hamiltonian parameters (in which the RG transforms generate flows) and the complement to S of $S_{\rm c}$ (corresponding to noncritical Hamiltonians).

In fact, this is not correct because there is another fixed point in S: the Gaussian fixed point which, although trivial, controls tricritical behaviours in three dimensions. Now each FP has its own basin of attraction in S [5]. The attraction domain of the Gaussian FP is the tricritical subspace $S_{\rm t}$ of codimension 2 (with no intersection with $S_{\rm c}$). In addition, we show that there is a second subspace of codimension 1 in S, called $S_{\rm f}$, which is different from $S_{\rm c}$, and thus which is not a domain of attraction to the Wilson-Fisher fixed point. There is no FP to which a point of $S_{\rm f}$ is attracted. $S_{\rm f}$ is characterized by a negative sign of the φ^4-Hamiltonian parameter u_4 and is associated with systems undergoing a first-order phase transition [6]. We show that an endless attractive RG trajectory is associated to this domain of first-order transitions. It is a renormalized trajectory (denoted below by T_1'') that emanates from the Gaussian fixed point. The frontier between $S_{\rm f}$ and $S_{\rm c}$ corresponds to the tricritical subspace $S_{\rm t}$ which is the domain of attraction of the Gaussian fixed point while $S_{\rm f}$ and $S_{\rm c}$ are two distinct domains of repulsion for the Gaussian fixed point.

Actually, the situation is in conformity with the usual view. Considering the famous φ^4-model [Landau-Ginzburg-Wilson (LGW) Hamiltonian] in which the associated coupling u_4 is positive, the Hamiltonian at criticality is attracted exclusively to the Wilson-Fisher fixed point, but if u_4 is negative, a φ^6-term is required for sta-

[2]We assume that the second relevant field, corresponding to the magnetic field for magnetic systems, is set equal to zero.

bility, but then one may get either a tricritical phase transition or a second- or a first-order transition [7]. In the present study we do not truncate the Hamiltonian which involves all the powers of the field φ.

We explicitly show that a system which would correspond to an initial point lying very close to the frontier S_t in the critical side (in S_c) would display a retarded classical-to-Ising crossover [8]. This result is interesting with regard to ionic systems (for example) in which a classical behaviour has been observed while an Ising-like critical behaviour was expected. The eventuality of a retarded crossover from the classical to the Ising behaviour has previously been mentioned but without explaining theoretically how this kind of crossover could develop [9]. In [8] a calculation suggests that the RPM model for ionic systems would specifically correspond to a scalar Hamiltonian with a negative sign for the φ^4-Hamiltonian parameter (but the order parameter chosen is not the bulk density [10]). This calculation has motivated the present study.

We also indicate that the renormalized trajectory T_1'' still exists in four dimensions. This makes the Gaussian fixed point ultraviolet stable and the scalar field theory *formally* asymptotically free. However the associated "perfect" action [11] would have the wrong sign to provide us with an acceptable (well defined) field theory.

The paper is arranged as follows. In section 2 we briefly present the local potential approximation of the exact RG equation to be studied. We introduce the strategy we have chosen to solve the resulting nonlinear differential equation with a view to show the trajectories of interest in the space S of infinite dimension. Because the practical approach to the Gaussian fixed point is made difficult due to the logarithmic slowness characteristic of a marginally irrelevant direction (for $d = 3$), we found it useful to first test our numerical method with a close approach to the Wilson-Fisher fixed point. We present the characteristic results of this approach and various kinds of domains corresponding to $u_4 > 0$ (a kind of a summary of [2])

In section 3 we describe various kinds of attraction or repulsion domains of the Gaussian fixed point (for a negative value of the φ^4-Hamiltonian parameter) corresponding to tricritical, critical and first-order subspaces. Then we discuss the consequences and especially explicitly show how a retarded crossover from the classical to the Ising behaviour can be obtained.

We then shortly discuss the case $d = 4$ when $u_4 < 0$.

In two appendices we report on some technical aspects of the numerical treatment of the RG equation studied, in particular on the appearing of spurious nontrivial tricritical fixed points (appendix A.1).

2. The RG equation studied

The local potential approximation was first considered by Nicoll et al. [12] from the sharp cutoff version of the exact RG equation of Wegner and Houghton [13]. It was rederived by Tokar [14] by using approximate functional integrations and rediscovered by Hasenfratz and Hasenfratz [15]. As in [2] we adopt the notation of

the latter authors and consider the following nonlinear differential equation for the simple function $f(l,\varphi)$:

$$\dot{f} = \frac{K_d}{2}\frac{f''}{1+f'} + \left(1-\frac{d}{2}\right)\varphi f' + \left(1+\frac{d}{2}\right) f \tag{2.1}$$

in which a prime refers to a derivative with respect to the constant dimensionless field φ (at constant l) and $f(l,\varphi) = V'(l,\varphi)$ is the derivative of the dimensionless potential $V(l,\varphi)$; $\dot{f}$ stands for $\partial f/\partial l|_\varphi$ in which l is the scale parameter defined by $\Lambda/\Lambda_0 = \mathrm{e}^{-l}$ and corresponding to the reduction to Λ of an arbitrary initial momentum scale of reference Λ_0 (the initial sharp momentum cutoff). Finally, K_d is the surface of the d-dimensional unit sphere divided by $(2\pi)^d$.

A fixed point is a solution of the equation $\dot{f} = 0$. The study of the resulting second order differential equation provides the following results:

- $d \geqslant 4$, no FP is found except the Gaussian fixed point.
- $3 \leqslant d < 4$, one nontrivial FP (the Wilson-Fisher fixed point [5]) is found [15–17].
- A new nontrivial FP emanates from the origin (the Gaussian fixed point) below each dimensional threshold $d_k = 2k/(k-1)$, $k = 2, 3, \ldots, \infty$ [18].

If one represents the function $f(l,\varphi)$ as a sum of monomials of the form:

$$f(l,\varphi) = \sum_n u_{2n}(l)\,\varphi^{2n-1}$$

then, for $d = 3$, the Wilson-Fisher fixed point f^* is located in S at [2]: $u_2^* = -0.461533\cdots$, $u_4^* = 3.27039\cdots$, $u_6^* = 14.4005\cdots$, $u_8^* = 32.31289\cdots$, etc.

Once the FP is known, one may study its vicinity which is characterized by orthogonal directions corresponding to the infinite set of eigenvectors, solutions of the differential equation (2.1) linearized at f^*. The eigenvectors associated to positive eigenvalues are said to be relevant; when the eigenvalues are negative they are said to be irrelevant and marginal otherwise [19].

The relevant eigenvectors correspond to directions along which the RG trajectories go away from the FP and the irrelevant eigenvectors correspond to directions along which the trajectories go into the FP. A marginal eigenvector may be relevant or irrelevant.

Our present FP f^* has only one relevant direction and infinitely many irrelevant directions (no marginal direction, still see [1]). As already explained and shown in [2], in order to approach f^* starting from an initial point in S, one must adjust one parameter of the initial function $f(0,\varphi)$. This amounts to fixing the temperature of a system to its critical temperature.

Starting with a known initial function (at "time" $l = 0$) say:

$$f(0,\varphi) = u_2(0)\,\varphi + u_4(0)\,\varphi^3,$$

we adjust $u_2\,(0)$ to the critical value $u_2^{\rm c} = -0.29958691\cdots$ corresponding to $u_4\,(0) = 3$ so that $f(l,\varphi)$ [solution at time l of the differential equation (2.1)] approaches f^* when $l \to \infty$. The approach to f^* is characterized by the least negative eigenvalue $\lambda_2 = -1/\omega_1$ (ω_1 was noted ω in [2]). This means that, in the vicinity of f^* any parameter $u_n\,(l)$ evolves as follows ($l \to \infty$):

$$u_n\,(l) \simeq u_n^* + a_n \exp\,(-\omega_1 l)\,.$$

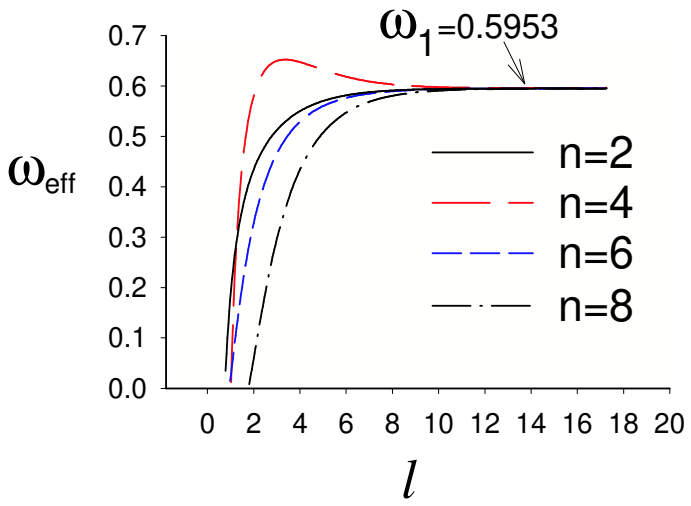

Figure 1. Evolutions for $d = 3$ of the first four Hamiltonian parameters $u_2(l)$, $u_4(l)$, $u_6(l)$, $u_8(l)$ in a close approach to the Wilson-Fisher fixed point f^* along T_1 or $\mathrm{T}_1^{'}$. The effective inverse eigenvalue $\omega_{\rm eff}\,(l)$ is given by equation (2.2) for $n = 2, 4, 6, 8$. All these quantities reach the same universal value ω_1 characteristic of the least irrelevant eigendirection of f^*. To get this close approach to f^* from equation (2.1), the initial critical value $u_2^{\rm c}$ corresponding to $u_4(0) = 3$, has been determined with more than twenty digits.

Figure 1 illustrates this feature for the first four $u_n\,(l)$'s in the approach to f^*. In [2] the two associated attractive trajectories (locally tangent to the least irrelevant eigenvector in the vicinity of f^*) were noted T_1 and T_1'.

One may also constrain the trajectory to approach f^* along the second irrelevant direction (with the associated attractive trajectories noted T_2 or T_2' in [2] and associated with the second least negative eigenvalue $\lambda_3 = -1/\omega_2$). In this case a second parameter of the initial f must be adjusted, e.g., $u_4\,(0)$ must be adjusted to $u_4^{\rm c}$ *and simultaneously* $u_2\,(0)$ to the corresponding $u_2^{\rm c}$, see [16,2]. Then, in the vicinity of f^*, any parameter $u_n\,(l)$ will evolve as follows:

$$u_n\,(l) \simeq u_n^* + a_n'\exp\,(-\omega_2 l)\,.$$

Looking for this kind of approach to f^*, we have found that $6.66151663 < u_4^{\rm c} < 6.66151669$ and $u_2^{\rm c} = -0.58328898880579\cdots$ This has allowed us to estimate $\omega_2 \cong 2.84$. Although the shooting method is certainly not well adapted to determining

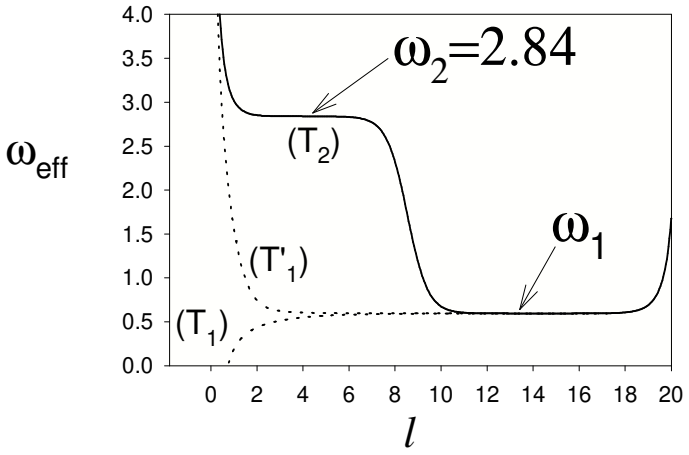

Figure 2. When a second condition is imposed on the initial Hamiltonian parameters, the approach to f^* may be adjusted such as to asymptotically take the second least irrelevant eigendirection. Here $\omega_{\text{eff}}(l)$ is given by equation (2.2) for $n = 2$ it clearly undergoes (full line) a flat inflection point at the value $\omega_2 = 2.84$ corresponding to an approach to f^* along T_2, the greater the critical parameter u_4^{c} is accurately determined the longer is the flat extremum. Because u_4^{c} is not completely determined [within the available accuracy in solving equation (2.1)] the trajectory leaves the direction of T_2 to take one of the two directions of approach associated to the least irrelevant inverse eigenvalue ω_1 (corresponding to T_1 or $\mathrm{T}_1^{'}$ as indicated by dashed curves). Here, the trajectory corresponding to the full line goes along $\mathrm{T}_1^{'}$. Again a flat extremum of $\omega_{\text{eff}}(l)$ indicates the approach along an eigenvector of f^* and requires an accurate determination of the critical value u_2^{c}. Because this determination is not complete, the trajectory ends up going away from f^* as indicated by the sudden departure of $\omega_{\text{eff}}(l)$ from ω_1 for the large values of l.

the eigenvalues (see the huge number of digits required in determining u_2^{c} and u_4^{c}), our estimate is close to $\omega_2 \cong 2.8384$ found by Comellas and Travesset [20].

Because u_4^{c} cannot be perfectly determined, the trajectory leaves the trajectory T_2 before reaching f^* to take one of the two directions T_1 or $\mathrm{T}_1^{'}$ (corresponding to ω_1). Figure 2 illustrates this effect with the evolution, for $n = 2$, of the following effective eigenvalue:

$$\omega_{\text{eff}}^{(n)}(l) = -\frac{\mathrm{d}^2 u_n(l)/\mathrm{d}l^2}{\mathrm{d}u_n(l)/\mathrm{d}l}, \tag{2.2}$$

the definition of which does not refer explicitly to f^*. The evolution of $\omega_{\text{eff}}(l)$ shows a flat extremum (or a flat inflection point) at an RG eigenvalue of f^* each time the RG flow runs along an eigendirection in the vicinity of f^*.

Similar to u_4^{c}, the value u_2^{c} cannot be perfectly determined. Consequently, the trajectory ends up going away from the fixed point. This provides us with the opportunity of determining the only positive (the relevant) eigenvalue corresponding to the critical exponent $\nu = \lambda_1 = -1/\omega_0$ [$\omega_{\text{eff}}(l)$ shows then a flat extremum at ω_0 when the flow still runs in close vicinity of f^*]. Finally, far away from the fixed point,

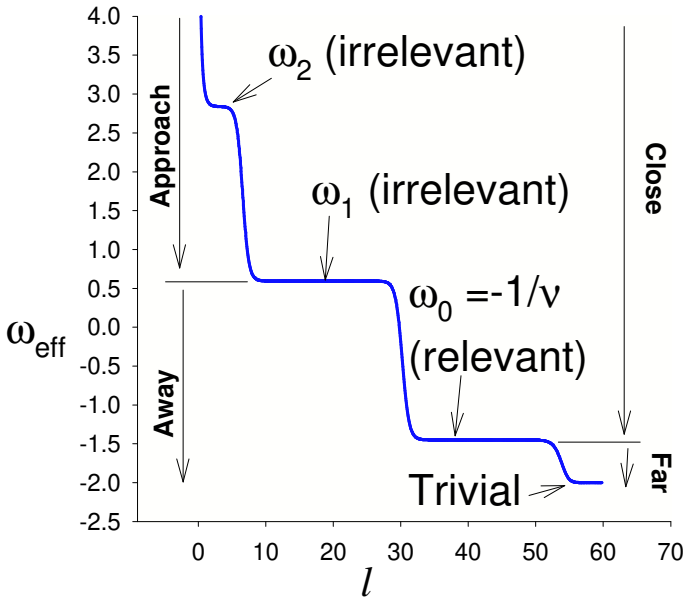

Figure 3. This figure is a continuation of figure 2. It shows the various plateaux that $\omega_{\text{eff}}(l)$ undergoes along a RG trajectory first adjusted to approach f^* along the second irrelevant direction (plateau at $\omega_2 = 2.84$). Because it is not possible to exactly determine the initial conditions, the trajectory always ends up going away from the fixed point towards the trivial high temperature fixed point characterized by the classical value $\frac{1}{2}$ (for minus the inverse of $\omega_{\text{eff}}(l)$, thus the final plateau at -2). In-between, the RG flow has been influenced by the close vicinity of the least irrelevant eigenvector (plateau at ω_1) and that of the relevant eigenvector (plateau at $\omega_0 = -\frac{1}{\nu}$). The various regimes of the RG flows are indicated by the vertical arrows on the left (direction of the flow with respect to the fixed point) and on the right of the figure (distance to the fixed point).

the RG trajectory approaches the trivial high temperature fixed point characterized by a classical eigenvalue (equal to $\frac{1}{2}$). The global picture summarizing the evolution of $\omega_{\text{eff}}(l)$ along the RG trajectory initialized in such a way as to approach f^* first along T_2, is drawn in figure 3.

The values we have determined by this shooting method are (for eigenvalues other than the already mentioned ω_2):

$$\begin{aligned} \omega_1 &\cong 0.5953, \\ \nu &\cong 0.68966, \end{aligned}$$

which are close to the values found, for example, in [15,20]: $\omega_1 \cong 0.5952$ and $\nu \cong 0.6895$.

3. Trajectories for $u_4 < 0$

In the preceding section, we have obtained a RG trajectory approaching the Wilson-Fisher fixed point f^* along T_2 by adjusting two parameters of the initial

Hamiltonian ($u_4^{\rm c}$ and $u_2^{\rm c}$). This is exactly the procedure one must follow to determine a tricritical RG trajectory approaching the Gaussian fixed point in three dimensions (because of its two relevant directions). The only difficulty is to discover initial points in S which are attracted to the Gaussian fixed point. To this end, we again use the shooting method.

Based on the usual arguments regarding the LGW Hamiltonian as well as taking into account the work done by Aharony on compressible ferromagnets [6], one expects to find the tricritical surface in the sector $u_4 < 0$ (and with $u_2 > 0$). Thus we have tried to approach the Gaussian fixed point starting with initial function $f(0)$ of the form:

$$f(0,\varphi) = u_2\,(0)\,\varphi + u_4\,(0)\,\varphi^3 + u_6(0)\varphi^5 \tag{3.1}$$

with (not large) negative values of $u_4\,(0)$, for example $u_4\,(0) = -1$.

Because the Gaussian fixed point is twice unstable, we must adjust two parameters to approach it starting with (3.1). We do that by successive tries (shooting method). For example, if we choose $u_4\,(0) = -1$ and $u_6(0) = 3$ and determine a value of $u_2\,(0)$ such as to get a trajectory which does not go immediately towards the trivial high temperature fixed point, the best we obtain is a trajectory which approaches the Wilson-Fisher fixed point (thus the corresponding initial point belongs to the attraction domain of f^* although $u_4\,(0) < 0$ [6]). But if $u_6(0) = 2$, the adjustment of $u_2\,(0)$ with a view to counterbalance the effect of the most relevant direction of the Gaussian fixed point (which would drive the trajectory toward the high temperature FP) yields a runaway RG flow towards larger and larger negative values of $u_4\,(l)$. From now on, the target is bracketed: the tricritical trajectory corresponding to $u_4\,(0) = -1$ can be obtained with a value of $u_6(0)$ in the range $]2,3[$ (we actually find a rather close approach to the Gaussian fixed point for $2.462280 > u_6(0) > 2.4622788$ and $6.4618440\cdots > u_2(0) > 6.4618407\cdots$).

In order to understand the origin of the direction of runaway in the sector of negative values of u_4, it is worth studying the properties of the Gaussian fixed point by linearization of the RG flow equation in the vicinity of the origin. If we request the effective potential to be bounded by polynomials then the linearization of equation (2.1) is identified with the differential equation of Hermite's polynomials of degree $n = 2k-1$ for the set of discrete values of λ satisfying [15]:

$$\frac{2+d-2\lambda_k}{d-2} = 2k-1 \qquad k = 1,2,3,\ldots \tag{3.2}$$

from which it follows that

- for $d = 4$: $\lambda_k = 4-2k$, $k = 1,2,3,\ldots$, there are two non-negative eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = 0$;
- for $d = 3$: $\lambda_k = 3-k$, $k = 1,2,3,\ldots$, there are three non-negative eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 1$ et $\lambda_3 = 0$.

If we denote by $\chi_k(\varphi)$ the eigenfunctions associated to the eigenvalue λ_k, it comes:

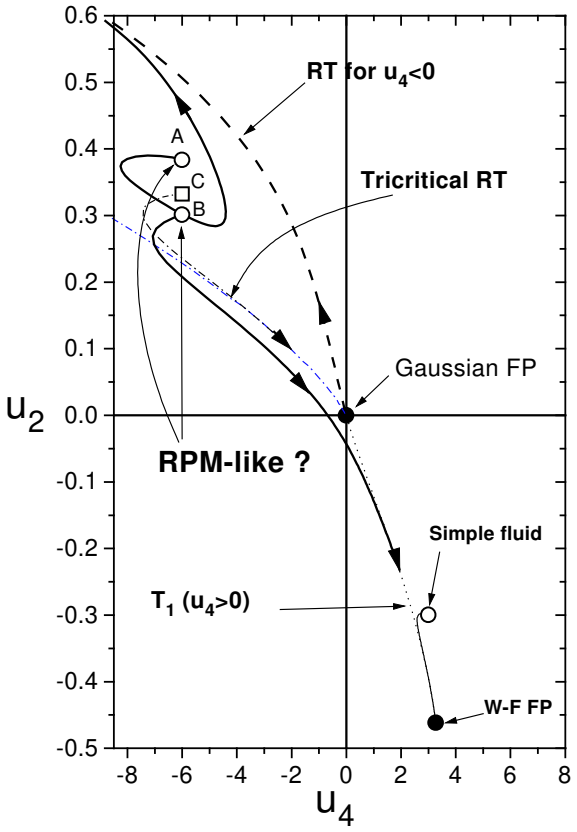

Figure 4. Domains of attraction and repulsion of the Gaussian fixed point. The figure represents projections onto the plane $\{u_2, u_4\}$ of various RG trajectories running in the space S minus one dimension. The flows have been obtained by solving equation (2.1). Black circles represent the Gaussian and the Wilson-Fisher (W-F FP) fixed points. The arrows indicate the directions of the RG flows on the trajectories. The ideal trajectory (dot line) which interpolates between these two fixed points represents the usual renormalized trajectory T_1 corresponding to the so-called φ_3^4 renormalized field theory in three dimensions (usual RT for $u_4 > 0$). White circles represent the projections onto the plane of initial critical Hamiltonians. For $u_4(0) > 0$, the effective Hamiltonians run toward the Ising fixed point asymptotically along T_1 (simple fluid). Instead, for $u_4(0) < 0$ and according to the values of Hamiltonian coefficients of higher order (u_6, u_8, etc.), the RG trajectories either (A) meet an endless RT emerging from the Gaussian FP (dashed curve) and lying entirely in the sector $u_4 < 0$ or (B) meet the usual RT T_1 to reach the Ising fixed point. The frontier which separates these two very different cases (A and B) corresponds to initial Hamiltonians lying on the tricritical subspace S_t (white square C). This is a source of RG trajectories flowing asymptotically toward the Gaussian FP along the tricritical RT. Notice that the coincidence of the initial point B with the RG trajectory starting at point A is not real (it is accidental, due to a projection onto a plane of trajectories lying in a space of infinite dimension). The points A or B could correspond to the restricted primitive model of ionic systems (see [8]).

- $\chi_1^+ = \varphi$, $\chi_2^+ = \varphi^3 - \frac{3}{2}\varphi$, $\chi_3^+ = \varphi^5 - 5\varphi^3 + \frac{15}{4}\varphi$, ..., whatever the spatial dimensionality d.

The superscript "+" is just a reminder of the fact that the eigenfunctions are defined up to a global factor and thus the functions $\chi_k^-(\varphi) = -\chi_k^+(\varphi)$ are also eigenfunctions with the same eigenvalue λ_k.

3.1. Case $d = 3$

Similar to χ_2^+, the direction provided by χ_2^- in S is a direction of instability of the Gaussian fixed point. Now χ_2^+ is associated with the well known renormalized trajectory T_1 on which the usual (massless) φ_3^4-field theory [16,2] is defined, for the same reasons a renormalized trajectory T_1'' locally tangent to χ_2^- in the vicinity of the origin of S exists with the same properties as T_1 (see [2]). The difference is that T_1'' lies entirely in the sector $u_4 < 0$ and is endless (not ended by a fixed point).

This endless renormalized trajectory is associated with systems undergoing a first-order phase transition. This is due to the absence of fixed point [21], in which case the correlation length ξ cannot be made infinite although for some systems lying close to T_1'' and attracted to it (i.e. at the transition temperature), ξ may be very large (because T_1'' is endless), in which cases one may say that the transition is almost of the second order [22]. Of course, a domain of the first-order phase transition in S was expected out of the usual arguments [7,6]. We only better specify the conditions of the first-order transition realization in S.

Figure 4 shows the attractive trajectory T_1'' together with the attractive tricritical line approaching the Gaussian fixed point. The tricritical surface S_t separates the first-order surface S_f from the critical surface S_c. Figure 4 also shows that systems lying close to the tricritical surface may still be attracted to the Wilson-Fisher fixed point. In this case the effective exponents may undergo a very retarded crossover to the asymptotic Ising values compared to usual systems corresponding to initial points chosen in the sector $u_4 > 0$ of S. Figure 5 illustrates how minus the inverse of (2.2) provides us with different evolutions [calculated from (2.1)] of the effective exponent $\nu_{eff}(\tau)$ [with $\tau \propto (T - T_\mathrm{c})/T_\mathrm{c}$] according to the initial point chosen in S. It is worth explaining how we have defined $\nu_\mathrm{eff}(\tau)$.

We have seen at the end of section 2 that the quantity (2.2) undergoes a flat extremum (or a flat inflection point) at an RG eigenvalue of f^* each time the RG flow runs along an eigendirection in the vicinity of f^*. Now it happens that this extremum is less and less flat as one chooses larger and larger values of $(u_2(0) - u_2^\mathrm{c})$ (for the eigenvalue ν) but it still exists. This provides us with a way of expressing the evolution of an effective exponent ν_eff when the RG-substitute to τ, namely $(u_2(0) - u_2^\mathrm{c})/u_2^\mathrm{c}$, is varied. Figure 6 shows such an evolution for some initial Hamiltonian (with $u_4(0) = 4$). Notice that for such a Hamiltonian, the extremum disappears before ν_eff reaches the trivial value $\frac{1}{2}$ (associated with the approach to the trivial high temperature fixed point and to a regular – non critical – behaviour) while in the case of a Hamiltonian initialized close to the tricritical surface, the classical-to-Ising crossover is complete (see figure 4). This is because in the latter case the RG

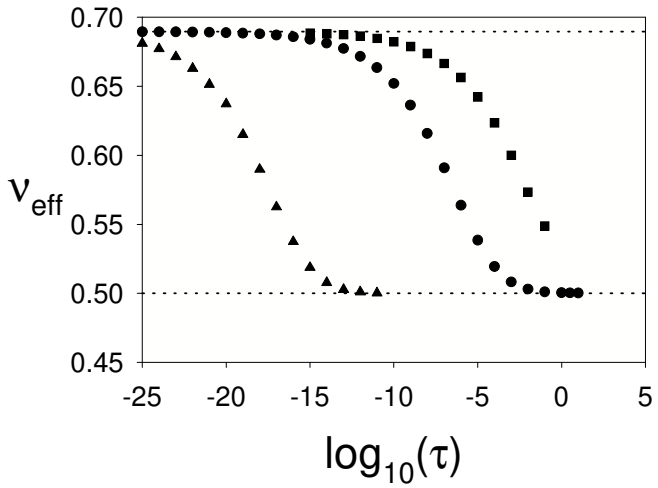

Figure 5. Evolutions of an effective exponent $\nu_{\text{eff}}(\tau)$ [with $\tau \propto (T - T_{\text{c}})/T_{\text{c}}$] along three different families of RG trajectories (see text for additional details). The full squares indicate the evolution of $\nu_{\text{eff}}(\tau)$ for a family of trajectories initialized in the sector $u_4 > 0$ with $u_4(0) = 3$ and for various values of $u_2(0)$ (the same system at criticality corresponds to the white circle "Simple fluid" of figure 4). When $u_2(0) \to u_2^{\text{c}}$ the effective exponent approaches the critical exponent value $\nu \cong 0.69$ compatible with the present study. One observes that the crossover towards the classical value $\frac{1}{2}$ is not complete because $\nu_{\text{eff}}(\tau)$ ceases to make sense before τ becomes large. This is not the case of the evolution represented by the full circles which correspond to trajectories initialized close to the Gaussian fixed point. In this case the complete crossover reproduces the interpolation between the Gaussian and the Wilson-Fisher fixed points and typically corresponds to the usual answer given by field theory [28]. The third evolution (full triangles) corresponds to a family of Hamiltonian initialized close to the tricritical surface but still attracted to the Wilson-Fisher fixed point. One sees that the classical-to-Ising crossover is complete but highly retarded compared to the two other cases. This is because at criticality, the RG flow is first attracted to the Gaussian fixed point (showing then an apparent classical value of ν) before interpolating between the Gaussian and the Wilson-Fisher fixed point.

trajectory comes close to the Gaussian fixed point (and $\nu_{\text{eff}}(\tau)$ has an extremum at $\frac{1}{2}$) before approaching f^*. This reinforces the idea that the so-called classical-to-Ising crossover actually exists only between the Gaussian and Wilson-Fisher fixed points [23].

The same configuration displayed in figure 4 has been also obtained by Tetradis and Litim [24] while studying analytical solutions of an exact RG equation in the local potential approximation for the $O(N)$-symmetric scalar theory in the large N limit. But they were not able to determine "*the region in parameter space which results in first-order transitions*" [24].

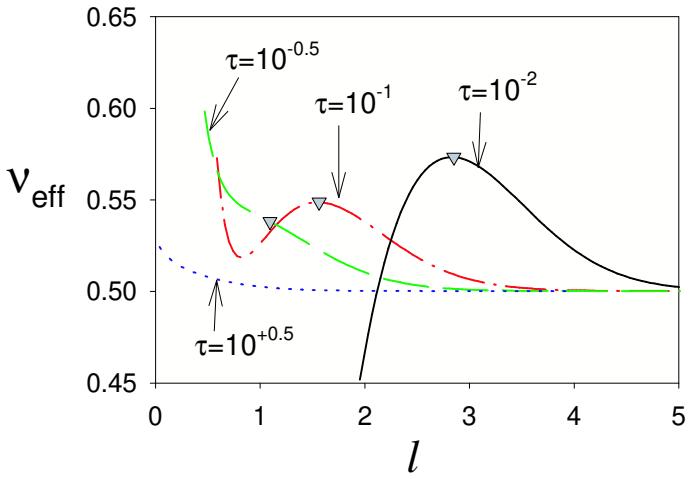

Figure 6. Illustration of the evolution of the extrema $\nu_{\rm eff}(l)$ [minus the inverse of equation (2.2)] for various values of $\tau = (u_2(0) - u_2^{\rm c})/u_2^{\rm c}$ and for the family of RG flows initialized at $u_4(0) = 3$. The extremum (grey triangle) disappears at some not very large value of τ (about $10^{-0.5}$) and does not reach the classical value $\frac{1}{2}$. This induces the partial Ising-to-classical crossover drawn in figure 5 (squares).

3.2. Case $d = 4$

To decide whether the marginal operator (associated with the eigenvalue equal to zero, i.e. λ_2 in four dimensions, or λ_3 in three dimensions) is relevant or irrelevant, one must go beyond the linear approximation. The analysis is presented in [15] for $d = 4$. If one considers a RG flow along χ_2^+ such that $g_2(\varphi, l) = c(l)\chi_2^+(\varphi)$, then one obtains, for small c: $c(l) = c(0)\,[1 - Ac(0)l]$ with $A > 0$. Hence the marginal parameter decreases as l grows. As it is well known, in four dimensions the marginal parameter is irrelevant. However, if one considers the direction opposite to χ_2^+ (i.e. χ_2^-) then the evolution corresponds to changing $c \to -c$. This gives, for small values of c: $c(l) = c(0)\,[1 + A\,|c(0)|\,l]$ and the parameter becomes relevant. The parameter c is the renormalized ϕ^4 coupling constant u_R and it is known that in four dimensions the Gaussian fixed point is IR stable for $u_R > 0$ but IR is unstable for $u_R < 0$ [25].

We have verified that the trajectory T_1'' survives when $d = 4$ (contrary to T_1, see [2]). That trajectory T_1'' is a renormalized trajectory on which we could define a continuum limit for the φ_4^4-field theory and if the corresponding (perfect) action was positive for all φ, one could say that the ϕ_4^4-field theory with a negative coupling is asymptotically free. Unfortunately, because the φ^4-term is dominant for large φ in the vicinity of the origin of S (due to the relevant direction provided by χ_2^-), the negative sign of the renormalized coupling prevents the path integral to be well defined.

However, because the action to which one refers in the continuum limit (the perfect action) is formal (because it involves an infinite number of parameters and cannot be written down, see [2]) we wonder whether the wrong sign of the action is actually a valid argument to reject the φ_4^4-field theory with a negative renormalized coupling. It is worth mentioning that the asymptotically free scalar field theory

which has recently been considered on a lattice [26] could actually be the ϕ_4^4-field theory with a negative coupling to which we refer here.

4. Acknowledgements

We dedicate this article to Professor Yukhnovskii in grateful recognition of his efficient and generous help in fostering the Ukrainian-French Symposium held in Lviv in february 1993, with the hope that in the future the contacts between our two communities will further develop.

A. The finite difference method used

For technical reasons, instead of studying equation (2.1), we consider the differential equation satisfied by $g(\varphi) = f'(\varphi)$ (i.e. the second derivative of the potential with respect to the field):

$$\dot{g} = \frac{K_d}{2}\left[\frac{g''}{1+g} - \frac{(g')^2}{(1+g)^2}\right] + \left(1 - \frac{d}{2}\right)\varphi g' + 2g. \tag{A.1}$$

Starting with a known initial function (at "time" $l = 0$), we follow its evolution in S by approximating the differential equation (A.1) by finite differences and a two dimensional grid with the uniform spacings $\mathrm{d}y = 0.01$ and $\mathrm{d}l = 0.000390625$. The finite difference formulas for the derivatives g'' and g' have been chosen with the accuracy $O(\mathrm{d}y^4)$:

$$\begin{aligned} g'(y) &= \frac{8}{12\mathrm{d}y}\left[g(y+\mathrm{d}y) - g(y-\mathrm{d}y) - g(y+2\mathrm{d}y) + g(y-2\mathrm{d}y)\right] \\ &\quad + O(\mathrm{d}y^4), \end{aligned} \tag{A.2}$$

$$\begin{aligned} g''(y) &= \frac{16}{12\mathrm{d}y^2}\left[g(y+\mathrm{d}y) + g(y-\mathrm{d}y) - 30g(y)\right. \\ &\quad \left. - g(y+2\mathrm{d}y) - g(y-2\mathrm{d}y)\right] + O(\mathrm{d}y^4) \end{aligned} \tag{A.3}$$

The evolutionary function $g(y, l)$ is known (calculated) at the discreet set of points $y_i = i \cdot \mathrm{d}y$ with $(i \geqslant 0)$ and a maximum value $i_{\max} = 82$. (This value is large enough to study the approach to the Wilson-Fisher fixed point with great accuracy but it is too small to precisely study the approach to the Gaussian fixed point.) At each time $l_k = k \cdot \mathrm{d}l$, the derivatives are estimated from $g(y_i, l_k) = g(y_i, l_{k-1}) + \dot{g}(y_i, l_{k-1}) \cdot \mathrm{d}l$ by using equations (A.2), (A.3) which apply only for $1 < i < i_{\max -1}$. For the marginal points $i = 0, 1$ we use the parity of $g(y)$ [by inserting $g(-n \cdot \mathrm{d}y) = g(n \cdot \mathrm{d}y)$ for $n = 1, 2$ in equations (A.2), (A.3)]. For the two other marginal points $i = i_{\max -1}, i_{\max}$ of the grid, there is no fixed solution and we shall alternately use the two following conditions [using the obvious abbreviation $g(i)$ instead of $g(i \cdot \mathrm{d}y)$]:

$$\begin{aligned} g'(i) &= g'(i-1), \\ g''(i) &= g''(i-1), \end{aligned}$$

$$g'(i) = \frac{1}{\mathrm{d}y}\left[\frac{25}{12}g(i) - 4g(i-1) + 3g(i-2) - \frac{4}{3}g(i-3) + \frac{1}{4}g(i-4) + O(\mathrm{d}y^4)\right], \quad \text{(A.4)}$$

$$g''(i) = \frac{1}{\mathrm{d}y^2}\left[\frac{915}{244}g(i) - \frac{77}{6}g(i-1) + \frac{107}{6}g(i-2) - 13g(i-3) + \frac{61}{12}g(i-4) - \frac{5}{6}g(i-5)\right] + O(\mathrm{d}y^4). \quad \text{(A.5)}$$

Condition 2 is more accurate than condition 1 but sometimes leads to strong instabilities which do not appear when we first use condition 1 and then condition 2 after some finite "time" l_0. The validity of the procedure is tested by, for example, trying to approach a given fixed point (see the main part of the paper).

A.1. Spurious fixed points appearing in approaching the Gaussian fixed point

In trying to determine the attractive tricritical trajectory (approaching the Gaussian fixed point), we have encountered a spurious twice unstable fixed point lying at some finite and non-negligible distance to the Gaussian fixed point. To understand the origin of this undesirable numerical effect, it is necessary to shortly discuss the solution of the fixed point equation $\dot{f} = 0$.

From (2.1) or (A.1), one sees that the fixed point equation is a second order non-linear differential equation and a solution would be parametrized by two arbitrary constants. One of these two constants may easily be determined: since $g^*(\varphi)$ is expected to be an even function of φ [O(1) symmetry] then $g^{*\prime}(0) = 0$ may be imposed. There remains one free parameter, thus a one-parameter family of (non-trivial) fixed points are the solutions to the differential equation. But there is no infinity of physically acceptable fixed points; all but a finite number of the solutions in the family are singular at some φ_c [15,18,27]. Formally, by requiring the physical fixed point to be defined for all φ, the acceptable fixed points are limited to the Gaussian fixed point and (for $d = 3$) to the Wilson-Fisher fixed point.

However, in our study, because we numerically consider the function $g(\varphi)$ in some finite range of values of φ (see above: $i_{\max} = 82$), it appears that in approaching the origin of S, infinitely many pseudo-fixed points exist which have there φ_c-singularity located outside the finite range explicitly considered and there is at least one of them which looks like a tricritical fixed point. When we enlarge the range of φ to $i_{\max} = 200$, the previously observed nontrivial tricritical fixed point disappears to the benefit of another one located closer to the origin. In conclusion, a larger and larger number of grid-points must be considered as one tries to come closer and closer to the Gaussian fixed point. This particularity together with the slowness of the approach along a marginal direction makes it extremely difficult to come very close to the Gaussian fixed point.

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Області ренормалізаційної групи скалярного гамільтоніану

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Отримано 9 березня 2000 р.

Використовуючи наближення локального потенціалу точного рівняння ренормалізаційної групи (РГ), ми показуємо різні області значень параметрів O(1) симетричного скалярного гамільтоніану. У трьох вимірах додатково до звичайної критичної поверхні $S_{\rm c}$ (область притягання фіксованої точки Вільсона-Фішера), ми явно показуємо існування області фазового переходу першого ряду $S_{\rm f}$, відокремленої від $S_{\rm c}$ трикритичною поверхнею $S_{\rm f}$ (область притягання гаусової фіксованої точки). $S_{\rm f}$ і $S_{\rm c}$ є дві різні області відштовхування для гаусової фіксованої точки, а $S_{\rm f}$ не є в ділянці притягання фіксованої точки. $S_{\rm f}$ характеризується нескінченою ренормалізованою траєкторією, яка повністю лежить в області негативних значень констант взаємодії φ^4. Ця ренормалізована траєкторія також існує в чотирьох вимірах, роблячи гаусову фіксовану точку в ультрафіолетовій області стабільною (і ренормалізовану теорію поля φ^4 асимптотично вільною, але з неправильним знаком ідеальної дії). Ми також показуємо, що дуже запізнений кросовер від класичної до ізінгівської поведінки може існувати у трьох вимірах (фактично нижче чотирьох вимірів). Це може бути поясненням для неочікуваної класичної критичної поведінки, яка спостерігається в деяких іонних системах.

Ключові слова: *ренормалізаційна група, критичний, трикритичний, фазовий перехід першого роду, кросовер*

PACS: *05.10.Cc, 05.70.Jk, 11.10.Hi, 61.20.Qg*