Condensed Matter Physics, 2000, Vol. 3, No. 2(22), pp. 265–275

CONDENSED
MATTER
PHYSICS

Generalized condensation and the Bogoliubov theory of superfluidity

V.A.Zagrebnov*

Université de la Méditerranée and Centre de Physique Théorique-CNRS-Luminy-Case 907,
13288 Marseille, Cedex 09, France

Received March 20, 2000

In this paper we scrutinize the Bogoliubov theory of superfluidity on the level of the first Bogoliubov ansatz. This ansatz preconizes the Bogoliubov truncated Hamiltonian as a starting point of this theory. Since this ansatz is based on the hypothesis concerning the Bose-Einstein condensation in zero-mode as well as on its stability with respect to interactions, we present some rigorous results, which cast doubt on this hypothesis.

Key words: *Bogoliubov weakly imperfect gas, Bose condensation, generalized condensation*

PACS: *05.30.Jp, 02.70.Lq, 03.75.fi, 67.40.Db*

1. Generalized condensation

Recall that the starting point of the Bogoliubov theory is based on two hypothesis:
(a) there is a Bose-Einstein (BE) condensation in one mode;
(b) this condensation is *stable* with respect to a weak interaction between particles.

These hypothesis inspired Bogoliubov (see e.g. [1,2]) into his *first ansatz*: the *truncation* of the *full* Hamiltonian

$$H_\Lambda = \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1,k_2,q\in\Lambda^*} v\left(q\right) a^*_{k_1+q} a^*_{k_2-q} a_{k_2} a_{k_1}, \tag{1.1}$$

where all sums run over the set Λ^* defined by

$$\Lambda^* = \left\{ k\in\mathbb{R}^3 : \alpha = 1,2,3,\ k_\alpha = \frac{2\pi n_\alpha}{L} \text{ et } n_\alpha = 0,\pm1,\pm2,...... \right\}, \tag{1.2}$$

to the Hamiltonian of the *Weakly Imperfect Bose-Gas* (WIBG):

$$H_\Lambda^{\mathrm{B}} = T_\Lambda + U_\Lambda^{\mathrm{D}} + U_\Lambda^{\mathrm{ND}}, \tag{1.3}$$

*E-mail: zagrebnov@cpt.univ-mrs.fr

where

$$T_\Lambda = \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k, \tag{1.4}$$

$$U_\Lambda^{\mathrm{D}} = \frac{v(0)}{V} a_0^* a_0 \sum_{k\in\Lambda^*, k\neq 0} a_k^* a_k + \frac{1}{2V} \sum_{k\in\Lambda^*, k\neq 0} v(k)\, a_0^* a_0 \left(a_k^* a_k + a_{-k}^* a_{-k}\right) + \frac{v(0)}{2V} a_0^{*^2} a_0^2, \tag{1.5}$$

$$U_\Lambda^{\mathrm{ND}} = \frac{1}{2V} \sum_{k\in\Lambda^*, k\neq 0} v(k) \left(a_k^* a_{-k}^* a_0^2 + a_0^{*^2} a_k a_{-k}\right). \tag{1.6}$$

Here $\varepsilon_k = \hbar^2 k^2/2m$ is the kinetic energy, and $a_k^\# = \{a_k^*, a_k\}$ are usual boson creation and annihilation operators in the one-particle state $\psi_k(x) = V^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}kx}, k \in \Lambda^*$, $x \in \Lambda$; for example, $a_k^* \equiv a^*(\psi_k) = \int_\Lambda \mathrm{d}x\, \psi_k(x)\, a^*(x)$, where $a^\#(x)$ are boson-field operators in the Fock space $\mathcal{F}_\Lambda$ over $L^2(\Lambda)$, see[3].

In this section we scrutinize the (most innocent) hypothesis (a), which concerns essentially the Perfect Bose-Gas (PBG). As it is indicated in [4] , the conventional BE condensation is a *subtle* matter even for the PBG.

Let instead of the *cubic box* $\Lambda = L \times L \times L$, $|\Lambda| = V$, one takes a *prism* $\Lambda = L_1 \times L_2 \times L_3$ of the *same* volume with sides of length $L_j = V^{\alpha_j}$, $j = 1, 2, 3$, such that $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. We impose periodic boundary conditions on $\partial\Lambda$, and we consider a possibility of conventional BE condensation in this prism. This example is due to Casimir [5] .

Now let us rewrite the equation for the particle density in the form:

$$\begin{aligned}\rho \;=\; & \frac{1}{V}\frac{1}{\mathrm{e}^{-\beta\mu}-1} + \frac{1}{V} \sum_{k\in\{\Lambda^*: n_1\neq 0, n_2=n_3=0\}} \frac{1}{\mathrm{e}^{\beta(\varepsilon_k-\mu)}-1} \\ & + \frac{1}{V} \sum_{k\in\{\Lambda^*: n_j\neq 0, j=2 \text{ or } 3\}} \frac{1}{\mathrm{e}^{\beta(\varepsilon_k-\mu)}-1}.\end{aligned} \tag{1.7}$$

For $\rho > \rho_c^{\mathrm{P}}(\theta)$ (the critical density for the PBG) the equation (1.7) gives asymptotic behaviour of the solution $\mu_\Lambda^{\mathrm{P}}\left(\theta, \rho > \rho_c^{\mathrm{P}}(\theta)\right)$ and distribution of bosons in the mode $k = 0$ and in its vicinity.

(**i**) Let $\alpha_1 < 1/2$. Then for $V \to \infty$ one gets:

$$\inf_{k\in\Lambda^*\setminus\{0\}} \varepsilon_k = O\left(V^{-2\alpha_1}\right). \tag{1.8}$$

This means that

$$\lim_\Lambda \sup_{k\in\Lambda^*\setminus\{0\}} \frac{1}{V}\frac{1}{\mathrm{e}^{\beta\varepsilon_k}-1} = 0.$$

Therefore, the Darboux sum (1.7) for $k \neq 0$ is well-defined and converges to the integral

$$\rho^{\mathrm{P}}(\beta, 0) = \lim_\Lambda \rho_\Lambda^{\mathrm{P}}\Big(\beta, \mu_\Lambda^{\mathrm{P}}\left(\theta, \rho > \rho_c^{\mathrm{P}}(\theta)\right)\Big).$$

Since $\rho > \rho_c^P(\theta) = \rho^P(\beta, 0)$ the equation (1.7) implies that

$$\mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right) \simeq \frac{1}{\beta V\left(\rho - \rho_c^P(\theta)\right)}, \; V \to \infty, \tag{1.9}$$

as in the case of cube $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$. Consequently, one gets a conventional BE condensation in *one* mode $k = 0$:

$$\lim_\Lambda \tfrac{1}{V} \langle N_{k=0} \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right)\right) = \rho_0^P(\theta) \equiv \rho - \rho_c^P(\theta) > 0,$$
$$\lim_\Lambda \tfrac{1}{V} \langle N_{k\neq 0} \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right)\right) = 0.$$

(**ii**) Now let $\alpha_1 = 1/2$. Then one gets:

$$\inf_{k \in \Lambda^* \setminus \{0\}} \varepsilon_k = \frac{(2\pi\hbar)^2}{2m} \frac{1}{V}, \tag{1.10}$$

which corresponds to the mode with $(n_1 = 1, n_2 = 0, n_3 = 0)$. Since $\alpha_2 < 1/2$ and $\alpha_3 < 1/2$, we get

$$\lim_\Lambda \sup_{k \in \{\Lambda^* : n_j \neq 0, j = 2 \text{ or } 3\}} \frac{1}{V} \frac{1}{e^{\beta \varepsilon_k} - 1} = 0.$$

Then the Darboux sum corresponding to the last term in (1.7) converges to the integral $\rho^P(\beta, \mu)$ for all $\mu < 0$. Since this integral attains its $\sup_{\mu<0} \rho^P(\beta, \mu) = \rho_c^P(\theta)$ for $\mu \to 0^-$, from (1.7) one obtains for $\rho > \rho_c^P(\theta)$ that

$$\lim_\Lambda \left\{ \frac{1}{V} \frac{1}{e^{-\beta \mu_\Lambda^P(\theta,\rho)} - 1} + \frac{1}{V} \sum_{k \in \{\Lambda^* : n_1 \neq 0, n_2 = n_3 = 0\}} \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda^P(\theta,\rho)\right)} - 1} \right\} = \rho - \rho_c^P(\theta). \tag{1.11}$$

By virtue of $\alpha_1 = 1/2$ the left-hand side of (1.11) implies:

$$\mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right) = \frac{A}{V} + o\left(\frac{1}{V}\right), \; A > 0, \tag{1.12}$$

for $V \to \infty$. Since for the asymptotics (1.12) we have

$$\lim_\Lambda \left\{ \sum_{k \in \{\Lambda^* : (n_1, 0, 0)\}} \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda^P(\theta,\rho)\right)} - 1} \right\} = \frac{1}{\beta} \sum_{n_1 = 0, \pm 1, \pm 2, \ldots} \left(\frac{(2\pi\hbar)^2}{2m} n_1^2 + A \right)^{-1}, \tag{1.13}$$

the (1.11) implies

$$\frac{1}{\beta} \sum_{n_1 = 0, \pm 1, \pm 2, \ldots} \left(\frac{(2\pi\hbar)^2}{2m} n_1^2 + A \right)^{-1} = \rho - \rho_c^P(\theta), \tag{1.14}$$

where $A = A(\theta, \rho)$ is a *unique* root of this equation. Moreover, the limit (1.13) shows that for $\rho > \rho_c^P(\theta)$:

$$\lim_{\Lambda} \frac{1}{V} \langle N_k \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^P(\theta, \rho)\right) = \\ = \begin{cases} \beta^{-1} \left(\frac{(2\pi\hbar)^2}{2m} n_1^2 + A \right)^{-1}, & k \in \{\Lambda^* : (n_1, 0, 0)\}, \\ 0 & , \quad k \in \{\Lambda^* : n_j \neq 0, j = 2 \text{ or } 3\}. \end{cases} \tag{1.15}$$

This means that for the *long* prism with $\alpha_1 = 1/2$ there is *macroscopic* occupation of an *infinite* number of low-lying levels $k \in \{\Lambda^* : (n_1, 0, 0)\}$ including the zero-mode $k = 0$.

(**iii**) Let $\alpha_1 > 1/2$, i.e. we deal with a *highly anisotropic* prism in direction $j = 1$. Then we have

$$\inf_{k \in \Lambda^* \setminus \{0\}} \varepsilon_k = \frac{(2\pi\hbar)^2}{2m} \frac{1}{V^{2\alpha_1}}, \ 2\alpha_1 > 1, \tag{1.16}$$

which corresponds to the mode with $(n_1 = 1, n_2 = 0, n_3 = 0)$. Since for any $\mu < 0$ the right-hand side of (1.7) converges to the integral $\rho^P(\beta, \mu)$ monotonously increasing up to $\rho_c^P(\theta)$, when $\mu \to 0^-$, the solution $\mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right)$ of (1.7) has to have the asymptotics :

$$\mu_\Lambda^P\left(\theta, \rho > \rho_c^P(\theta)\right) = \frac{B}{V^\delta} + o\left(\frac{1}{V^\delta}\right), \ B < 0, \ \delta > 0, \tag{1.17}$$

for $V \to \infty$. To calculate B and δ, notice that the first *two* terms in (1.7) may be represented as

$$\frac{1}{V} \sum_{k \in \{\Lambda^* : (n_1, 0, 0)\}} \frac{1}{e^{\beta\left(\varepsilon_k - \mu_\Lambda^P(\theta, \rho)\right)} - 1} = \frac{1}{V} \sum_{k \in \{\Lambda^* : (n_1, 0, 0)\}} \sum_{s=1}^{+\infty} e^{-s\beta\left(\varepsilon_k - \mu_\Lambda^P(\theta, \rho)\right)} \\ = \frac{1}{V} \sum_{s=1}^{+\infty} e^{s\beta\mu_\Lambda^P(\theta, \rho)} \sum_{n_1 = 0, \pm 1, \pm 2, \ldots} e^{-s\beta\left(\frac{(2\pi\hbar)^2}{2m} \frac{n_1^2}{V^{2\alpha_1}}\right)}. \tag{1.18}$$

Notice that

$$\sum_{n_1 = 0, \pm 1, \pm 2, \ldots} e^{-\pi\lambda n_1^2} = \frac{1}{\sqrt{\lambda}} \sum_{\xi = 0, \pm 1, \pm 2, \ldots} e^{-\left(\pi\xi^2/\lambda\right)}, \tag{1.19}$$

where $\lambda = \beta \frac{2\pi\hbar^2}{m} s V^{-2\alpha_1}$. Taking into account (1.17)–(1.19) we find that only the term with $\xi = 0$ is important for (1.18) when $V \to \infty$:

$$\rho - \rho_c^P(\theta) = \lim_{\Lambda} \left\{ \left(\frac{2\pi\hbar^2}{m} \beta \right)^{-1/2} \left\{ \frac{V^{\alpha_1 - 1}}{V^{\delta/2}} . V^\delta \right\} \frac{1}{V^\delta} \left\{ \sum_{s=1}^{+\infty} e^{\beta B\left(s/V^\delta\right)} \left(\frac{s}{V^\delta} \right)^{-1/2} \right\} \right\}. \tag{1.20}$$

The limit in (1.20) is *nontrivial* only for

$$\delta = 2(1 - \alpha_1). \tag{1.21}$$

Then one gets

$$0 < \rho - \rho_{\rm c}^{\rm P}(\theta) = \left(\frac{2\pi\hbar^2}{m}\right)^{-1/2} \beta^{-1/2} \int\limits_0^{+\infty} \mathrm{d}\xi\, \mathrm{e}^{\beta B \xi} \xi^{-1/2}, \tag{1.22}$$

where $B = B\left(\theta = \beta^{-1}, \rho\right) < 0$ is the *unique* root of equation (1.22), or

$$\rho - \rho_{\rm c}^{\rm P}(\theta) = \left(\frac{m}{2\hbar^2}\right)^{1/2} \frac{\theta}{\sqrt{|B(\theta,\rho)|}}, \tag{1.23}$$

since

$$I = \int\limits_0^{+\infty} \mathrm{d}y\, \mathrm{e}^{-y} y^{-1/2} = \sqrt{\pi}.$$

By virtue of (1.17), (1.21) and (1.23) we get that for $\rho > \rho_{\rm c}^{\rm P}(\theta)$ and $V \to \infty$:

$$\begin{aligned} &\varepsilon_k - \mu_\Lambda^{\rm P}(\theta,\rho) \simeq \\ &\simeq \begin{cases} \frac{\hbar^2}{2m}\left(\frac{2\pi n_1}{V^{\alpha_1}}\right)^2 + \frac{m/2\hbar^2}{\beta^2(\rho-\rho_{\rm c}^{\rm P}(\theta))^2 V^{2(1-\alpha_1)}}, & k \in \{\Lambda^* : (n_1,0,0)\}, \\ \frac{\hbar^2}{2m}\left[\left(\frac{2\pi n_2}{V^{\alpha_2}}\right)^2 + \left(\frac{2\pi n_3}{V^{\alpha_3}}\right)^2\right] + \frac{m/2\hbar^2}{\beta^2(\rho-\rho_{\rm c}^{\rm P}(\theta))^2 V^{2(1-\alpha_1)}}, & k \in \{\Lambda^* : n_{j=2 \text{ or } 3} \neq 0\}. \end{cases} \end{aligned} \tag{1.24}$$

Since $\alpha_1 > 1/2$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, the asymptotics (1.24) imply

$$\lim_\Lambda \frac{1}{V} \langle N_k \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^{\rm P}\left(\theta, \rho > \rho_{\rm c}^{\rm P}(\theta)\right)\right) = 0, \ k \in \Lambda^*, \tag{1.25}$$

while

$$\lim_\Lambda \frac{1}{V^{2(1-\alpha_1)}} \langle N_k \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^{\rm P}\left(\theta, \rho > \rho_{\rm c}^{\rm P}(\theta)\right)\right) = \frac{m/2\hbar^2}{\beta^2\left(\rho - \rho_{\rm c}^{\rm P}(\theta)\right)^2}, \ k \in \{\Lambda^* : (n_1,0,0)\}, \tag{1.26}$$

and (cf. (1.7), (1.18), (1.20))

$$\rho_0^{\rm P}(\theta) = \rho - \rho_{\rm c}^{\rm P}(\theta) = \lim_\Lambda \frac{1}{V} \sum_{k \in \{\Lambda^* : (n_1,0,0)\}} \langle N_k \rangle_{T_\Lambda} \left(\beta, \mu_\Lambda^{\rm P}(\theta,\rho)\right) > 0.$$

This means that for the *extremely long* prism with $\alpha_1 > 1/2$ for $\rho > \rho_{\rm c}^{\rm P}(\theta)$ there is conventional BE condensation with density $\rho_0^{\rm P}(\theta) > 0$, whereas there are no *macroscopically* occupied levels in Λ^*.

Cases $\alpha_1 = 1/2$ and $\alpha_1 > 1/2$ are provided examples of *generalized condensation* in the]irardeau sense [6]. This gives a motivation for the following van den Berg-Lewis-Pulè's *classification* of the generalized BE condensation [4,7,8]:

- the condensation is called the *type I* when a finite number of single-particle levels are macroscopically occupied;

- it is of *type II* when an infinite number of the levels are macroscopically occupied;
- it is called the *type III*, or the *non-extensive* condensation, when none of the levels is macroscopically occupied, whereas one has:

$$\lim_{\delta\to 0^+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*,0\leqslant\|k\|\leqslant\delta\}}\langle N_k\rangle=\rho-\rho_c(\theta).$$

Remark 1 *Returning back to the Bogoliubov theory we have to recognize that the hypothesis* (*a*) *about the one-mode BE condensation looses its grip if we pass to the PBG in a long prism. The latter is not an exotic situation if one considers superfluidity of extremely thin films of* $^4\mathrm{He}$, *see [9,10].*

2. The model

In [11,12] it was shown that the *type III* condensation can be caused in the PBG by a weak *external field* or by a specific choice of the boundary conditions and geometry [7,13]. But recently other examples of this *non-extensive* BE condensation were discovered for bosons in an *isotropic* box Λ with *repulsive* interaction which *spreads out* the conventional BE condensation of the *type I* into the *type III*, [14–16].

To avoid the influence of the shape on the BE condensation, let us take the *cubic* box $\Lambda=L\times L\times L$. Imposing the periodic boundary conditions on $\partial\Lambda$, we consider a simple interacting boson system with the Hamiltonian:

$$H_\Lambda^{\mathrm{I}}=\sum_{k\in\Lambda^*}\varepsilon_k a_k^*a_k+\frac{1}{2V}\sum_{k\in\Lambda^*}v(0)a_k^*a_k^*a_ka_k,\ \ v(0)>0. \tag{2.1}$$

Interaction in (2.1) results from a “severe” truncation of the full interaction H_Λ. Hereinafter we shall demonstrate that even this “small” part of the total interaction can drastically change the distribution of condensed particles spreading out the BE condensation in the one mode $k=0$ into non-extensive BE condensation of the type III.

First we show that the interaction in (2.1) is so gentle that it even does not modify the pressure of the perfect system.

Theorem 1 *For the model (2.1) the thermodynamic limit of the pressure coincides with the pressure of the PBG:*

$$\lim_{\Lambda}p_\Lambda\left[H_\Lambda^{\mathrm{I}}\right]=\ \ \lim_{\Lambda}p_\Lambda\left[T_\Lambda\right]=p^{\mathrm{P}}(\beta,\mu). \tag{2.2}$$

Proof. Notice that $a_k^*a_k^*a_ka_k=N_k^2-N_k$. Therefore, the grand partition function for the model (2.1) takes the form :

$$\Xi_\Lambda^{\mathrm{I}}(\beta,\mu)=\mathrm{Tr}_{\mathcal{F}_{\mathrm{B}}}\mathrm{e}^{-\beta\left(H_\Lambda^{\mathrm{I}}-\mu N_\Lambda\right)}=\prod_{k\in\Lambda^*}\sum_{n_k=0}^{+\infty}\mathrm{e}^{-\beta\left[(\varepsilon_k-\mu)n_k+\frac{v(0)}{2V}\left(n_k^2-n_k\right)\right]}, \tag{2.3}$$

which converges for all $\mu \in \mathbb{R}^1$. Then for the pressure in the finite volume we have the estimates:

$$p_\Lambda [T_\Lambda] \geqslant p_\Lambda \left[H^{\mathrm{I}}_\Lambda\right] \geqslant \frac{1}{\beta V} \sum_{k\in\Lambda^*} \ln \sum_{n_k=0}^{[\ln V]} \mathrm{e}^{-\beta\left[(\varepsilon_k-\mu)n_k+\frac{v(0)}{2V}\left(n_k^2-n_k\right)\right]}. \tag{2.4}$$

By virtue of $n_k^2 - n_k \leqslant [\ln V]^2$ (here $[x]$ denotes the integer part of x) one gets:

$$\begin{aligned}&\sum_{n_k=0}^{[\ln V]} \mathrm{e}^{-\beta\left[(\varepsilon_k-\mu)n_k+\frac{v(0)}{2V}\left(n_k^2-n_k\right)\right]} \geqslant \\ &\qquad \geqslant \sum_{n_k=0}^{[\ln V]} \mathrm{e}^{-\beta\left[(\varepsilon_k-\mu)n_k+\frac{v(0)}{2V}[\ln V]^2\right]} = \mathrm{e}^{-\beta\frac{v(0)}{2V}[\ln V]^2} \left\{\frac{1-\mathrm{e}^{-\beta(\varepsilon_k-\mu)([\ln V]-1)}}{1-\mathrm{e}^{-\beta(\varepsilon_k-\mu)}}\right\}. \end{aligned} \tag{2.5}$$

For $\mu < 0$ the left-hand side of (2.5) converges (uniformly in $k \in \Lambda^*$) to $\left(1-\mathrm{e}^{-\beta(\varepsilon_k-\mu)}\right)^{-1}$ when $V \to \infty$. Hence, in the thermodynamic limit, the lower estimate in (2.4) coincides with the upper estimate:

$$\begin{aligned}\lim_\Lambda \frac{1}{\beta V} \sum_{k\in\Lambda^*} \ln \left\{\mathrm{e}^{-\beta\frac{v(0)}{2V}[\ln V]^2}\frac{1-\mathrm{e}^{-\beta(\varepsilon_k-\mu)([\ln V]-1)}}{1-\mathrm{e}^{-\beta(\varepsilon_k-\mu)}}\right\} &= \\ = \lim_\Lambda \frac{1}{\beta V} \sum_{k\in\Lambda^*} \ln\left(1-\mathrm{e}^{-\beta(\varepsilon_k-\mu)}\right)^{-1} & \\ = \lim_\Lambda p_\Lambda [T_\Lambda], &\end{aligned}$$

which proves the equality (2.2). Since $\lim_{\mu\to 0^-}\lim_\Lambda \; p_\Lambda [T_\Lambda]$ is bounded, one extends $p^{\mathrm{P}}(\beta,\mu)$ to $\mu = 0$ by continuity. ■

As usual, the total density of particles for the model (2.1) is defined in the grand-canonical ensemble for a finite volume V by :

$$\rho_\Lambda(\beta,\mu) = \left\langle \frac{N_\Lambda}{V} \right\rangle_{H^{\mathrm{I}}_\Lambda} (\beta,\mu) = \frac{1}{V}\sum_{k\in\Lambda^*} \langle N_k\rangle_{H^{\mathrm{I}}_\Lambda}(\beta,\mu) = \partial_\mu p_\Lambda \left[H^{\mathrm{I}}_\Lambda\right]. \tag{2.6}$$

Corollary 2 *Due to $v(0) > 0$ the pressure $p_\Lambda\left[H^{\mathrm{I}}_\Lambda\right]$ is defined for all chemical potentials $\mu \in \mathbb{R}^1$, see (2.3). Moreover it is obviously a convex function of this parameter. The limit function $p^{\mathrm{P}}(\beta,\mu)$ is defined only for $\mu \leqslant 0$ and is also a convex function, but only for $\mu < 0$. Since (2.2) is the statement regarding the convergence of convex functions of $\mu \in (-\infty, 0)$, by the Griffiths lemma we get convergence of the corresponding derivatives (2.6)):*

$$\lim_\Lambda \rho^{\mathrm{I}}_\Lambda(\beta,\mu) = \lim_\Lambda \partial_\mu p_\Lambda\left[H^{\mathrm{I}}_\Lambda\right] = \partial_\mu \lim_\Lambda p_\Lambda\left[H^{\mathrm{I}}_\Lambda\right] = \partial_\mu p^{\mathrm{P}}(\beta,\mu) = \rho^{\mathrm{P}}(\beta,\mu),$$

on the very same interval $\mu \in (-\infty, 0)$. So, in the thermodynamic limit the total particle density $\rho^{\mathrm{I}}(\beta,\mu)$ of the model (2.1) coincides with $\rho^{\mathrm{P}}(\beta,\mu)$. Since $\lim_{\mu\to 0^-} \rho^{\mathrm{P}}(\beta,\mu)$ is bounded for $d > 2$, one defines $\rho^{\mathrm{I}}(\beta,\mu = 0) = \rho^{\mathrm{P}}(\beta,\mu = 0)$ by continuity.

From this corollary one deduces that, as in the case of the PBG, $\rho^{\mathrm{I}}(\beta,\mu)$ attains its maximum $\rho_{\mathrm{c}}^{\mathrm{I}}(\theta) = \lim_{\mu\to 0^-} \rho^{\mathrm{I}}(\beta,\mu) = \rho_{\mathrm{c}}^{\mathrm{P}}(\theta)$ at $\mu = 0$, i.e. for densities $\rho > \rho_{\mathrm{c}}^{\mathrm{P}}(\theta)$ one *must* have a BE condensation. On the other hand, for the expectation of the ground-state occupation number for $\mu = 0$ we get from (2.1):

$$\left\langle \frac{N_0}{\sqrt{V}} \right\rangle_{H_\Lambda^{\mathrm{I}}} (\beta, \mu = 0) = \frac{\frac{1}{\sqrt{V}} \sum\limits_{n_0=0}^{+\infty} \frac{n_0}{\sqrt{V}} \mathrm{e}^{-\beta \frac{v(0)}{2}\left(\frac{n_0}{\sqrt{V}}\right)^2}}{\frac{1}{\sqrt{V}} \sum\limits_{n_0=0}^{+\infty} \mathrm{e}^{-\beta \frac{v(0)}{2}\left(\frac{n_0}{\sqrt{V}}\right)^2}}. \tag{2.7}$$

Since the numerator and the denominator in the right-hand side of (2.7) are nothing but Darboux sums for the corresponding integrals in the limit $V \to \infty$, we obtain that

$$\lim_{\Lambda} \frac{1}{\sqrt{V}} \langle N_0 \rangle_{H_\Lambda^{\mathrm{I}}} (\beta, \mu = 0) = \frac{\int \mathrm{d}x.x.\mathrm{e}^{-\beta\frac{v(0)}{2}x^2}}{\int \mathrm{d}x.\mathrm{e}^{-\beta\frac{v(0)}{2}x^2}} = \frac{1}{2}\sqrt{\frac{2\theta}{v(0)}}. \tag{2.8}$$

This means that there is *no macroscopic occupation* of the mode $k = 0$ because $\langle N_0 \rangle_{H_\Lambda^{\mathrm{I}}} (\beta, \mu = 0)$ growth only as $V^{1/2}$ for $V \to \infty$.

Theorem 3 *For any parameters $(\beta, \mu \leqslant 0)$ one has*

$$\lim_{\Lambda} \frac{1}{V} \langle N_k \rangle_{H_\Lambda^{\mathrm{I}}} (\beta, \mu) = 0, \ \forall k \in \Lambda^*, \tag{2.9}$$

i.e. there is no macroscopic occupation of any mode $k \in \Lambda^*$.

Proof. By the Bogoliubov convexity inequality for the pressure (see e.g. [17,18]), we get

$$\frac{v(0)}{2V^2} \left\langle \sum_{k\in\Lambda^*} \left(N_k^2 - N_k\right) \right\rangle_{H_\Lambda^{\mathrm{I}}} \leqslant p_\Lambda [T_\Lambda] - p_\Lambda \left[H_\Lambda^{\mathrm{I}}\right] \leqslant \frac{v(0)}{2V^2} \left\langle \sum_{k\in\Lambda^*} \left(N_k^2 - N_k\right) \right\rangle_{T_\Lambda}.$$

By virtue of $v(0) > 0$, $N_k^2 - N_k \geqslant (N_k - I)^2$ and theorem 1, one has

$$\lim_{\Lambda} \frac{1}{V^2} \sum_{k\in\Lambda^*} \left\langle (N_k - I)^2 \right\rangle_{H_\Lambda^{\mathrm{I}}} (\beta, \mu \leqslant 0) = 0. \tag{2.10}$$

Notice that by the Cauchy-Schwarz inequality for the Gibbs state we get

$$\langle N_k - I \rangle_{H_\Lambda^{\mathrm{I}}}^2 \leqslant \left\langle (N_k - I)^2 \right\rangle_{H_\Lambda^{\mathrm{I}}}.$$

Then

$$0 \leqslant \left(\frac{1}{V} \langle N_k - I \rangle_{H_\Lambda^{\mathrm{I}}}\right)^2 \leqslant \frac{1}{V^2} \left\langle (N_k - I)^2 \right\rangle_{H_\Lambda^{\mathrm{I}}} \leqslant \frac{1}{V^2} \sum_{q\in\Lambda^*} \left\langle (N_q - I)^2 \right\rangle_{H_\Lambda^{\mathrm{I}}}. \tag{2.11}$$

The estimates (2.10) and (2.11) give the assertion (2.9). ■

Resuming corollary 2 and theorem 3 we must admit that the model (2.1) gives an example of non-extensive (i.e. the type III) BE condensation $\rho_0(\theta) = \rho - \rho_c(\theta)$. Namely, for $\rho > \rho_c^{\mathrm{I}}(\theta) = \rho_c^{\mathrm{P}}(\theta)$ one has

$$\lim_{\Lambda} \frac{1}{V} \langle N_k \rangle_{H_\Lambda^{\mathrm{I}}} \Big(\beta, \mu_\Lambda(\theta, \rho > \rho_c(\theta))\Big) = 0, \;\; k \in \Lambda^*, \tag{2.12}$$

and at the same time

$$\begin{aligned} \rho &= \lim_{\Lambda} \rho_\Lambda(\beta, \mu_\Lambda(\theta, \rho > \rho_c(\theta))) \\ &= \rho_c(\theta) + \lim_{\delta \to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*:\, 0 \leqslant \|k\| \leqslant \delta\}} \langle N_k \rangle_{H_\Lambda^{\mathrm{I}}} \Big(\beta, \mu_\Lambda(\theta, \rho > \rho_c(\theta))\Big). \end{aligned} \tag{2.13}$$

Remark 2 *Returning back to the Bogoliubov theory we again recognize that there is a problem with the hypothesis* (*a*) *and* (*b*) *regarding the one-mode (i.e. type I) BE condensation. Even if it exists in PBG, a very gentle interaction (e.g. (2.1)), which does not change the pressure of the PBG, may drastically change the asymptotics of the occupation number expectations, cf. (2.8). In the model (2.1) the interaction spreads out the type I condensation of the PBG into the type III BE condensation, which means that the Bogoliubov hypothesis* (*b*) *is à priori doubtful.*

3. Conclusion

The perturbation of the model (2.1) is a function of only occupation-number operators, i.e. this interaction commutes with the kinetic-energy operator. This class of interactions is called *diagonal*. In the series of papers [19–23], van den Berg, Dorlas, Lewis and Pulè gave an exhausting analysis of a hole class of *diagonal models* (including the exact solution of the Huang-Yang-Luttinger model), showing how subtle and multiform is the BE condensation in presence of this type of interaction. Since the interaction is diagonal, their analysis is based on a refined Large Deviation Principle of the classical probability. For another approach to the Huang-Yang-Luttinger model see [24].

If one accepts the first Bogoliubov ansatz, one is faced with a *nondiagonal* interaction U_Λ^{ND} in (1.3), which does *not* commute with T_Λ. Therefore, the existence of the $k = 0$ mode BE condensation in the Bogoliubov WIBG H_Λ^{B} is screaming for a rigorous analysis, see [25,26]

Returning back to the theory of superfluidity we have to avow that the discussion above is rather far away from real systems. In the recent paper by Vakarchuk [27] one can find a discussion of the difficulties and of the methods going beyond the Bogoliubov theory of superfluidity. Developing some ideas by Bogoliubov, Zubarev and Yukhnovskii regarding the method of collective variables, he obtained new convincing results (see references quoted in [27]) to advance in this challenging unsolved problem of theoretical and mathematical physics.

The paper is dedicated to Professor Academician Yukhnovskii. I am very thankful to the journal of Condensed Matter Physics for the invitation to write this paper and for an occasion to express in this way my sincere respects to Igor Rafailovich.

References

1. Bogoliubov N.N. About the theory of superfluidity. // Izv. Akad. Nauk USSR, 1947, vol. 11, p. 77–90.
2. Bogoliubov N.N. Lectures on Quantum Statistics. – In: Collection of papers. vol. 2. Kiev, Naukova Dumka, 1970, p. 287–493.
3. Petrina D.Ya. Mathematical Foundations of Quantum Statistical Mechanics. Kluwer Academic Publishers, 1995.
4. van den Berg M., Lewis J.T. On generalized condensation in the free boson gas. // Physica A, 1982, vol. 110, p. 550–564.
5. Casimir H.B.G. On Bose-Einstein condensation. – In: Fundamental Problems in Statistical Mechanics III, ed. E.G.D. Cohen. Amsterdam, North-Holland Publishing Company, 1968, p. 188–196.
6. Girardeau M. Relationship between systems of impenetrable bosons and fermions in one dimension. // J. Math. Phys., 1960, vol. 1, p. 516–523.
7. van den Berg M. On boson condensation into an infinite number of low-lying levels. // J. Math. Phys., 1982, vol. 23, p. 1159–1161.
8. van den Berg M., Lewis J.T., Pulè J.V. A general theory of Bose-Einstein condensation. // Helv. Phys. Acta., 1986, vol. 59, p. 1271–128.
9. Khalatnikov I.M. An Introduction to the Theory of Superfluidity. New York, Benjamin, 1965.
10. Wilks J. An Introduction to Liquid Helium. Oxford, Claredon, 1970.
11. van den Berg M., Lewis J.T. On the free boson gas in a weak external potential. // Commun. Math. Phys., 1981, vol. 81, p. 475–494.
12. Pulè J.V. The free boson gas in a weak external potential. // J. Math. Phys., 1983, vol. 24, p. 138–142.
13. van den Berg M. On condensation in the free-bosons gas and the spectrum of the Laplacian. // J. Stat. Phys., 1983, vol. 31, p. 623–637.
14. Michoel T., Verbeure A. Nonextensive Bose-Einstein condensation model. // J. Math. Phys., 1999, vol. 40, p. 1268–1279.
15. Bru J.-B., Zagrebnov V.A. Exactly soluble model with two kinds of Bose-Einstein condensations. // Physica A, 1999, vol. 268, p. 309–325.
16. Bru J.-B., Zagrebnov V.A. A model with coexistence of two kinds of Bose condensations. // J. Phys. A: Math. Gen., 2000, vol. 33, p. 449–464.
17. Bogoliubov N.N. (Jr) A Method of Studying Model Hamiltonians. Oxford, Pergamon, 1977.
18. Bogoliubov N.N. (Jr), Brankov J.G., Zagrebnov V.A., Kurbatov A.M., Tonchev N.S. The Approximating Hamiltonian method in Statistical Physics. Sofia, Publ. Bulgarian Acad. Sciences, 1981.
19. van den Berg M., Lewis J.T., Pulè J.V. The large deviation principle and some models of an interacting boson gas. // Commun. Math. Phys., 1988, vol. 118, p. 61–85.
20. van den Berg M., Dorlas T.C., Lewis J.T., Pulè J.V. A perturbed meanfield model of an interacting boson gas and the large deviation principle. // Commun. Math. Phys.,

1990, vol. 127, p. 41–69.
21. van den Berg M., Dorlas T.C., Lewis J.T., Pulè J.V. The pressure in the Huang-Yang-Luttinger model of an interacting boson gas. // Commun. Math. Phys., 1990, vol. 128, p. 231–245.
22. Dorlas T.C., Lewis J.T., Pulè J.V. The full diagonal model of a Bose gas. // Commun. Math. Phys., 1993, vol. 156, p. 37–65.
23. Dorlas T.C., Lewis J.T., Pulè J.V. Condensation in some perturbed meanfield models of a Bose gas. // Helv. Phys. Acta., 1991, vol. 64, p. 1200–1224.
24. Petrina D.Ya., Gredzhuk V.I. On two models of interacting Bose gas: Bogolyubov model of superfluidity and Huang-Yang-Lattinger model. // Mat. Fizika. Anal. Geometr., 1996, vol. 3, p. 370–405.
25. Bru J.-B., Zagrebnov V.A. Exact solution of the Bogoliubov Hamiltonian for Weakly Imperfect Bose gas. // J. Phys. A: Math.Gen., 1998, vol. 31, p. 9377–9404.
26. Zagrebnov V.A. The Bogoliubov weakly imperfect Bose-gas. // J. Physical Studies, 1999, vol. 3, p. 239–251.
27. Vakarchuk I.O. Statistical operator of a system of identical interacting particles in coordinate representation. // J. Physical Studies, 1996, vol. 1, p. 25–38.

Узагальнена конденсація і теорія надплинності Боголюбова

В.А.Загребнов

Середземноморський університет і центр теоретичної фізики, 13288 Марсель, Франція

Отримано 20 березня 2000 р.

У цій статті ми детально досліджуємо теорію надплинності Боголюбова на рівні першого анзацу Боголюбова. Цей анзац стверджує обрізаний гамільтоніан Боголюбова як стартову точку цієї теорії. Оскільки цей анзац ґрунтується на гіпотезі стосовно Бозе-Ейнштейна конденсації в нульовій моді і на її стабільності по відношенню до взаємодії, ми представляємо строгі результати, які піддають сумніву цю гіпотезу.

Ключові слова: *слабо неідеальний газ Боголюбова, Бозе конденсація, узагальнена конденсація*

PACS: *05.30.Jp, 02.70.Lq, 03.75.fi, 67.40.Db*