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Supersymmetric method for constructing quasi-exactly solvable potentials

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We propose a new method for constructing the quasi-exactly solvable (QES) potentials with two known eigenstates using supersymmetric quantum mechanics. General expression for QES potentials with explicitly known energy levels and wave functions of both ground and excited states are obtained. Examples of new QES potentials are considered.

Key words: *supersymmetry, quantum mechanics, quasi-exactly solvable potentials*

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1. Introduction

A potential is said to be quasi-exactly solvable (QES) if a finite number of eigenstates of corresponding Schrödinger operator can be found exactly in explicit form. The first examples of QES potentials were given in [1–4]. Subsequently several methods for generating QES potentials were worked out and as a result many QES potentials were found [5–13] (see also review book [14]). Three different methods based respectively on a polynomial ansatz for wave functions, point canonical transformation, supersymmetric (SUSY) quantum mechanics are described in [12].

Recently, an anti-isospectral transformation, also called a duality transformation, was introduced in [15]. This transformation relates the energy levels and wave functions of two QES potentials. In [16] a new QES potential was discovered using this anti-isospectral transformation.

The SUSY method for constructing the QES potentials was used in [10–12]. The starting point of this method is some initial QES potential with $n+1$ known eigenstates. Then applying the technique of SUSY quantum mechanics (for review of SUSY quantum mechanics see [17, 18]) one can calculate the supersymmetric partner of the QES potential. From the properties of the unbroken SUSY it follows that the supersymmetric partner is a new QES potential with n known eigenstates.

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In addition SUSY was used to develop some generalized method for constructing the so-called conditionally exactly solvable (CES) potentials in [19, 20]. The CES potential is the one for which the eigenvalues problem for the corresponding Hamiltonian is exactly solvable only when the potential parameters obey certain conditions. Such a class of potentials was first considered in [21].

In the present paper we develop a new SUSY method for generating QES potentials which unlike the previous papers [10–12] does not require to know the initial QES potential to generate a new one.

2. Supersymmetric quantum mechanics

In the Witten's model of supersymmetric quantum mechanics the SUSY partner Hamiltonians $H_\pm$ read

$$H_\pm = B^\mp B^\pm = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x), \tag{1}$$

where

$$B^\pm = \frac{1}{\sqrt{2}}\left(\mp\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right), \tag{2}$$

$$V_\pm(x) = \frac{1}{2}\Big(W^2(x) \pm W'(x)\Big), \quad W'(x) = \frac{\mathrm{d}W(x)}{\mathrm{d}x}, \tag{3}$$

$W(x)$ is referred to as a superpotential. In this paper we shall consider the systems on the full real line $-\infty < x < \infty$.

Consider the equation for the energy spectrum

$$H_\pm \psi_n^\pm(x) = E_n^\pm \psi_n^\pm(x), \quad n = 0, 1, 2, \dots . \tag{4}$$

As a consequence of SUSY the Hamiltonians H_+ and H_- have the same energy spectrum except for the zero energy ground state. The latter exists in the case of the unbroken SUSY. Only one of the Hamiltonians $H_\pm$ has a square integrable eigenfunction corresponding to zero energy. We shall use the convention that the zero energy eigenstate belongs to H_-. Due to the factorization of the Hamiltonians $H_\pm$ (see (1)) the ground state for H_- satisfies the equation $B^-\psi_0^-(x) = 0$ the solution of which is

$$\psi_0^-(x) = C_0^- \ \exp\left(-\int W(x)\mathrm{d}x\right), \tag{5}$$

C_0^- is the normalization constant. Hereinafter C denotes the normalization constant of the corresponding wave function. From the condition of square integrability of wave function $\psi_0^-(x)$ it follows that superpotential must satisfy the condition

$$\mathrm{sign}\Big(W(\pm\infty)\Big) = \pm 1. \tag{6}$$

Note that this is the condition of the existence of unbroken SUSY.

The eigenvalues and eigenfunctions of the Hamiltonians H_+ and H_- are related by SUSY transformations

$$E^-_{n+1} = E^+_n, \quad E^-_0 = 0, \tag{7}$$

$$\psi^-_{n+1}(x) = \frac{1}{\sqrt{E^+_n}} B^+ \psi^+_n(x), \quad \psi^+_n(x) = \frac{1}{\sqrt{E^-_{n+1}}} B^- \psi^-_{n+1}(x). \tag{8}$$

For a detailed description of SUSY quantum mechanics and its application for the exact calculation of eigenstates of Hamiltonians see reviews [17,18]. The properties of the unbroken SUSY quantum mechanics reflected in SUSY transformation (7), (8) are used to exactly calculate the energy spectrum and wave functions. In the present paper we use these properties to generate the QES potentials with the two known eigenstates.

3. QES potentials with the two known eigenstates

Suppose we study a Hamiltonian H_-, whose ground state is given by (5). Let us consider the SUSY partner of H_-, i.e. the Hamiltonian H_+. If we calculate the ground state of H_+ we immediately find the first excited state of H_- using the SUSY transformation (7), (8). In order to calculate the ground state of H_+ let us rewrite it in the following form

$$H_+ = H^{(1)}_- + \epsilon = B^+_1 B^-_1 + \epsilon, \quad \epsilon > 0, \tag{9}$$

which leads to the following relation between potential energies

$$V_+(x) = V^{(1)}_-(x) + \epsilon, \tag{10}$$

where ϵ is the energy of the ground state of H_+ since we suppose that $H^{(1)}_-$ has zero energy ground state, $B^\pm_1$ and $V^{(1)}_-(x)$ are given by (2) and (3) with the superpotential $W_1(x)$.

As we see from (9) the ground state wave function of H_+ is also the ground state wave function of $H^{(1)}_-$ and it satisfies the equation $B^-_1 \psi^+_0(x) = 0$. The solution of this equation is

$$\psi^+_0(x) = C^+_0 \exp\left(-\int W_1(x)\mathrm{d}x\right), \tag{11}$$

where for square integrability of this function the superpotential $W_1(x)$ satisfies the same condition as $W(x)$ (6). Using (7) and (8) we obtain the energy level $E^-_1 = \epsilon$ and the wave function of the first excited state $\psi^-_1(x)$ for H_-.

From (10) we obtain the following relation between $W(x)$ and $W_1(x)$

$$W^2(x) + W'(x) = W^2_1(x) - W'_1(x) + 2\epsilon. \tag{12}$$

Previously the same equation was used in the case of the so-called shape invariant potentials to obtain the exact solutions of Schrödinger equation [22] (see also reviews

[17,18]). We consider a more general case and do not restrict ourselves to the shape invariant potentials. Note, that (12) is the Riccati equation which can not generally be solved exactly with respect to $W(x)$ for a given $W_1(x)$ and vice versa.

The basic idea of this paper consists in finding such a pair of $W(x)$ and $W_1(x)$ that satisfies equation (12). Recently it has been suggested by us in [25]. For this purpose let us rewrite equation (12) in the following form

$$W'_+(x) = W_-(x)W_+(x) + 2\epsilon, \tag{13}$$

where

$$\begin{aligned} W_+(x) &= W_1(x) + W(x), \\ W_-(x) &= W_1(x) - W(x). \end{aligned} \tag{14}$$

This new equation (13) can be easily solved with respect to $W_-(x)$ for a given arbitrary function $W_+(x)$ or with respect to $W_+(x)$ for a given arbitrary function $W_-(x)$. Then from (14) we obtain superpotentials $W(x)$ and $W_1(x)$ which satisfy equation (12).

3.1. Solution with respect to $W_-(x)$

In this subsection we construct the QES potentials using the solution of equation (13) with respect to $W_-(x)$

$$W_-(x) = \left(W'_+(x) - 2\epsilon\right) \Big/ W_+(x), \tag{15}$$

where $W_+(x)$ is some function of x. Note, that the superpotentials $W(x)$ and $W_1(x)$ must satisfy condition (6). Then as one may see from (14) $W_+(x)$ must satisfy the same condition (6) as $W(x)$ and $W_1(x)$ do.

Let us consider continuous functions $W_+(x)$. Because $W_+(x)$ satisfies condition (6) the function $W_+(x)$ must pass through zeros. Then as we see from (15) $W_-(x)$, and thus $W(x)$, $W_1(x)$ have poles. In order to construct the superpotential free of singularities suppose that $W_+(x)$ has only one zero at $x = x_0$ with the following behaviour in the vicinity of x_0 $W_+(x) = W'_+(x_0)(x - x_0)$. In this case the pole of $W_-(x)$ at $x = x_0$ can be cancelled by choosing

$$\epsilon = W'_+(x_0)/2. \tag{16}$$

Then the superpotentials free of singularities are

$$\begin{aligned} W(x) &= \frac{1}{2}\left[W_+(x) - \left(W'_+(x) - W'_+(x_0)\right) \Big/ W_+(x)\right], \\ W_1(x) &= \frac{1}{2}\left[W_+(x) + \left(W'_+(x) - W'_+(x_0)\right) \Big/ W_+(x)\right]. \end{aligned} \tag{17}$$

Substituting the obtained result for $W(x)$ into (3) we obtain QES potential $V_-(x)$ with explicitly known wave function of ground state (5) and wave function of the first excited state. The latter can be calculated using (11) and (8)

$$\psi_1^-(x) = C_1^- \ W_+(x) \exp\left(-\int W_1(x)\mathrm{d}x\right). \tag{18}$$

It is indeed the wave function of the first excited state because $W_+(x)$ has one zero.

We may choose various functions $W_+(x)$ and obtain as a result various QES potentials. The functions $W_+(x)$ must be such that $\psi_0^-(x)$ and $\psi_1^-(x)$ are square integrable. If the eigenfunctions $\psi_0^-(x)$ and $\psi_1^-(x)$ belong to the Hilbert space of square integrable functions in which the Hamiltonian is Hermitian then these functions must be orthogonal

$$< \psi_0^-|\psi_1^- >= -C_0^- C_1^- \left[\exp\left(-\int \mathrm{d}x W_+(x)\right)\right]\Big|_{-\infty}^{\infty} = 0. \tag{19}$$

The wave functions must also satisfy appropriate boundary conditions.

To conclude this subsection let us consider explicit examples. Choosing $W_+(x)= A\,(\sinh(\alpha x) - \sinh(\alpha x_0))$ we obtain the well known QES potential derived in [8,9] by the method elaborated in the quantum theory of spin systems. Note that the case $x_0 = 0$ corresponds to Razavy potential [3]. Following the proposed SUSY method this example is considered in details in our earlier paper [25].

Consider the function $W_+(x)$ in the polynomial form

$$W_+(x) = ax + bx^3, \quad a > 0,\ b > 0, \tag{20}$$

which gives a new QES potential

$$V_-(x) = \frac{1}{8}(a^2 - 12b)x^2 + \frac{ab}{4}x^4 + \frac{b^2}{8}x^6 + \frac{3ab}{8(a+bx^2)^2} + \frac{3b}{8(a+bx^2)} - \frac{a}{4}. \tag{21}$$

The energy levels of ground and first excited states are $E_0^- = 0$, $E_1^- = a/2$. Note, that two energy levels of this potential do not depend on the parameter b. The wave functions of those states read

$$\begin{aligned} \psi_0^-(x) &= C_0^-(a+bx^2)^{3/4}\mathrm{e}^{-x^2(2a+bx^2)/8}, & (22)\\ \psi_1^-(x) &= C_1^- x(a+bx^2)^{1/4}\mathrm{e}^{-x^2(2a+bx^2)/8}. & (23) \end{aligned}$$

It is worth stressing that the case $b = 0$ corresponds to linear harmonic oscillator.

3.2. Solution with respect to $W_+(x)$

Equation (13) is the first order differential equation with respect to $W_+(x)$. A general solution can be written in the following form

$$W_+(x) = \exp\left(\int \mathrm{d}x W_-(x)\right)\left[2\epsilon\int \mathrm{d}x \exp\left(-\int \mathrm{d}x W_-(x)\right) + \lambda\right], \tag{24}$$

here λ is the constant of integration.

In order to simplify solution (24) let us choose $W_-(x)$ to be of the form

$$W_-(x) = -\phi''(x)/\phi'(x), \tag{25}$$

and suppose that $\phi'(x) > 0$. Then

$$W_+(x) = (2\epsilon\phi(x) + \lambda)/\phi'(x). \tag{26}$$

Note that the constant λ can be included into the function $\phi(x)$ and thus for $W_+(x)$ we obtain

$$W_+(x) = 2\epsilon\phi(x)/\phi'(x). \tag{27}$$

Finally for superpotentials $W(x)$ and $W_1(x)$ we have

$$W(x) = \left(\frac{1}{2}\phi''(x) + \epsilon\phi(x)\right) \Big/ \phi'(x), \tag{28}$$

$$W_1(x) = \left(-\frac{1}{2}\phi''(x) + \epsilon\phi(x)\right) \Big/ \phi'(x). \tag{29}$$

Using this result for wave functions of the ground state with the energy $E_0^- = 0$ and excited state with $E_1^- = \epsilon$ we obtain

$$\begin{array}{rcll} \psi_0^-(x) & = & C_0^-(\phi'(x))^{-1/2}\exp\left(-\epsilon\int \mathrm{d}x\phi(x)/\phi'(x)\right), & E_0^- = 0, \\ \psi_1^-(x) & = & C_1^-\phi(x)(\phi'(x))^{-1/2}\exp\left(-\epsilon\int \mathrm{d}x\phi(x)/\phi'(x)\right), & E_1^- = \epsilon, \end{array} \tag{30}$$

where function $\phi(x)$ must be such that these wave functions are square integrable. The condition of orthogonality in this case can by written similarly to (19)

$$< \psi_0^-|\psi_1^- >= -C_0^-C_1^-\left[\exp\left(-\epsilon\int \mathrm{d}x\phi(x)/\phi'(x)\right)\right]\Big|_{-\infty}^{\infty} = 0. \tag{31}$$

QES potential $V_-(x)$ is given by (3) with superpotential (28). Choosing different $\phi(x)$ we obtain different QES potentials with explicitly known two eigenstates. We shall consider a nonsingular monotonic function $\phi(x)$ with one node. Then $\psi_1^-(x)$ has also got one node and thus corresponds to the first excited state.

In conclusion of this subsection let us consider an explicit example. Let us put

$$\phi(x) = ax + bx^3/3, \quad a, b > 0. \tag{32}$$

Note, that the case $b = 0$ corresponds to a linear harmonic oscillator. The function (32) generates the following superpotentials

$$W(x) = \epsilon x/3 + (b + 2a\epsilon/3)\frac{x}{a + bx^2}, \tag{33}$$

$$W_1(x) = \epsilon x/3 + (-b + 2a\epsilon/3)\frac{x}{a + bx^2}, \tag{34}$$

which as we see satisfy condition (6).

Substituting $W(x)$ into (3) we obtain the following QES potential $V_-(x)$ and its SUSY partner $V_+(x)$

$$V_-(x) = \frac{A_-}{2}x^2 + \frac{B_-}{a + bx^2} + \frac{D_-}{(a + bx^2)^2} + R_-, \tag{35}$$

$$V_+(x) = \frac{A_+}{2}x^2 + \frac{D_+}{(a + bx^2)^2} + R_+, \tag{36}$$

where

$$A_- = A_+ = \epsilon^2/9, \quad B_- = b + \frac{2}{3}a\epsilon, \quad R_- = R_+ = \frac{\epsilon}{18b}(3b + 4a\epsilon),$$
$$D_- = -\frac{1}{18b}(27ab^2 + 24a^2b\epsilon + 4a^3\epsilon^2), \quad D_+ = \frac{1}{18b}(9ab^2 - 4a^3\epsilon^2).$$

Using (30) we obtain the wave functions of the ground and first excited states

$$\psi_0^-(x) = C_0^- \ (a + bx^2)^{-1/2 - a\epsilon/3b} \ \exp(-\epsilon x^2/6), \tag{37}$$
$$\psi_1^-(x) = C_1^- \ (ax + bx^3/3)(a + bx^2)^{-1/2 - a\epsilon/3b} \ \exp(-\epsilon x^2/6). \tag{38}$$

The same result for QES potential (35) and wave functions (37), (38) can be obtained using the method described in section 3.1 and taking the function $W_+(x)$ to be of the form (27).

It is interesting to stress that in the special case $\epsilon = 3b/2a$ the QES potential $V_-(x)$ reads

$$V_-(x) = \frac{b^2}{8a^2}x^2 + \frac{2b}{a + bx^2} - \frac{4ab}{(a + bx^2)^2} + \frac{3b}{4a} \tag{39}$$

and can be solved exactly. To see this, note that in this special case the superpotential $W_1(x) = \epsilon x/3$ corresponds to superpotential of a linear harmonic oscillator. Then $V_-^{(1)}(x)$ and, as a result of (10), $V_+(x)$ are the potential energies of the linear harmonic oscillator

$$V_+(x) = \frac{b^2}{8a^2}x^2 + \frac{5b}{4a}. \tag{40}$$

The fact that $V_+(x)$ corresponds to the linear harmonic oscillator follows also directly from (36) because the coefficient D_+ in the considered case is equal to zero. Therefore in this case H_+ is the Hamiltonian of the linear harmonic oscillator and we know all its eigenfunctions in explicit form. Using SUSY transformations (7), (8) we can easily calculate the energy levels and the wave functions of all the excited states of H_-. Note that in this special case $V_-(x)$ can be treated as CES potential and it corresponds to the one studied in [19,20,23,24].

As far as we know the potential in general form (35) has not been previously discussed. This potential is interesting from that point of view that in the case of $\epsilon = 3b/2a$ this potential is the CES potential for which the whole energy spectrum and the corresponding eigenfunctions can be calculated in the explicit form.

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Метод суперсиметрії для генерування квазі-точно розв'язуваних потенціалів

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Використовуючи суперсиметричну квантову механіку ми запропонували новий метод для генерування квазі-точно розв'язуваних (КТР) потенціалів з двома відомими станами. Отримані загальні вирази для КТР потенціалів з явно відомими енергетичними рівнями і хвильовими функціями основного і першого збудженого станів. Розглянуті приклади нових КТР потенціалів.

Ключові слова: *суперсиметрія, квантова механіка, квазі-точно розв'язувані потенціали.*

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