Condensed Matter Physics, 1998, Vol. 1, No. 3(15), p. 439–452

CONDENSED
MATTER
PHYSICS

Classical relativistic spin particles

J.Llosa

Departament de Física Fonamental, Universitat de Barcelona
Diagonal, 647; E-08028 - Barcelona, Spain

Received January 30, 1998; revised May 6, 1998

Relativistic spin particles are approached from the stand point of Hamiltonian actions of Poincaré group on itself. The several possible solutions are classified and realizations are given in terms of Dirac's constraints formalism.

Key words: *relativistic spin particle, Poincaré group, Hamiltonian action*

PACS: *03.20, 03.30*

1. Introduction

In [1] the dynamics of classical spin particles was approached on the basis of simple kinematical postulates that were some extension of rigid body kinematics. The role associated in other approaches like [3] with the so-called "inner space" of the classical particle was embodied in three mutually orthogonal spacelike axes belonging to the comoving space. So the phase space was a submanifold of the cotangent space of the Poincaré group that was defined by three nonholonomous constraints, which basically require the momentum of the particle to be proportional to the time column of a Lorentz matrix.

Intuitive as it may be, this formulation is too restrictive. Indeed, as a consequence of the constraints, only positive mass particles can fit into it and both massless particles and tachyons are rejected from the very beginning.

Furthermore, in [1] there was rather wide class of dynamics permitted by the requirement that a spin particle is described by a Poincaré invariant Hamiltonian system. It was only reduced by assuming the condition of spherical symmetry (which was introduced by hand) in such a way obtaining the expected result, namely, that the Hamiltonian should only depend on the two Casimir functions [5] of the Poincaré group.

In the present paper we shall change a little bit our perspective and not use constraints. The aim is to encompass also massless particles and tachyons. The underlying ideas, to be shaped in mathematical form, are the following:

(A1) The configuration space is Poincaré group, G. Each point consisting of to components (x, Λ), the first one gives the position of the particle as seen by some given inertial frame, $\mathcal{S}$, and the second (i.e., the Lorentz matrix Λ) gives the components of a spacetime orthonormal tetrad of vectors (the "body" axes) referred to the basis of $\mathcal{S}$.

Within this framework coordinate transformations from one inertial frame $\mathcal{S}$ to another, $\mathcal{S}'$, are represented by the left action of Poincaré group (the symmetry group of relativity) on itself (the configuration space). On the phase space T^*G the corresponding lift must be considered (see [1] for details). According to the principle of relativity, physical laws must be the same in all inertial frames,

(A2) the dynamics of a spin particle in the phase space T^*G should be invariant under the above left action.

Let us now consider the right action of the element $g = (a, L) \in G$ of Poincaré group on any point $(x, \Lambda) \in G$ of configuration space.

$$(x, \Lambda) \longrightarrow (x, \Lambda) \cdot (a, L) = (x + \Lambda a, \Lambda L)$$

It results in a shift of the point x in Minkowski spacetime to $x + \Lambda a$ and in a change of the "body" axes, from Λ to $\Lambda \cdot L$.

The right space rotation does not change ${\Lambda^\mu}_0$, the time "body" axis. Were the phase space defined by some constraints involving ${\Lambda^\mu}_0$, as it happened in [1], it would be invariant by right space rotations. Furthermore, if spherical symmetry is assumed, there is no preferred triad of body axes (i.e., all of them are dynamically equivalent). Hence,

(A3) the dynamical system describing a classical spin particle should be invariant under the right action of space rotations.

The right action of a boost does change the time axis of the "body" frame, ${\Lambda^\mu}_0$, and takes the point (x, Λ) off the phase space (in case the latter is defined by some constraints involving ${\Lambda^\mu}_0$).

(A3') The right action of boosts can be used to extend, in a right invariant way, the dynamical system from the phase manifold to the whole T^*G.

It is necessary to stress the different roles assigned here to left and right actions of G. Indeed, by left action of (a, L) we change the reference frame, whereas by right action we are changing the "axes attached to the body" when L is a rotation.

In consequently with the above kinematical arguments, the dynamics of a classical spin particle will be defined by

a Hamiltonian system on T^*G, with its canonical symplectic structure [2], that is invariant under both actions (namely, left and right) of Poincaré group G on T^*G.

The present contribution could be taken as an exercise to chapter 4 in [2]. The notation and most notions used here are largely defined and fully developed there. Sections 2 and 3 are devoted to the general study of Hamiltonian systems on the cotangent space of a Lie group which are invariant under both, the left action and the right action of the group on itself. Then the right action invariance is used to *reduce* the phase space. In section 4 the particular case of Poincaré group, which is relevant to relativistic spin particles, is considered and all possible dynamics are classified. The final results are given in the form of Dirac constrained Hamiltonian systems ([4]).

2. Hamiltonian systems on the cotangent space of a connected Lie group

G is a connected Lie group, $\mathcal{G}$ its Lie algebra and $\mathcal{G}^*$ its dual vector space.

2.1. Symplectic structure and group actions

T^*G is endowed with the canonical symplectic form $\Omega \in \Lambda^2(T^*G)$, that is obtained as the differential of the Liouville form $\theta \in \Lambda^1(T^*G)$:

$$\Omega = -\mathrm{d}\theta \ . \tag{1}$$

The Liouville form is related to the projection map $\pi : T^*G \to G$, by

$$\langle \theta_\alpha, X_\alpha \rangle = \langle \alpha, T\pi(X_\alpha) \rangle \tag{2}$$

where $\alpha \in T^*G$, $X_\alpha \in T_\alpha(T^*G)$ and $T\pi$ is the corresponding tangent map.

We now have two actions of G on T^*G, namely, *left translation*, Φ^{L}_g, and *right translation*, Φ^{R}_g, which are defined as the lifts to T^*G of the left translation and the right translation, respectively, on G:

$$L_g: \left.\begin{array}{ccc} G & \longrightarrow & G \\ h & \longrightarrow & gh \end{array}\right\} \qquad \Phi^{\mathrm{L}}_g: \left.\begin{array}{ccc} T^*_hG & \longrightarrow & T^*_{gh}G \\ \alpha_h & \longrightarrow & T^*L_g^{-1}(\alpha_h) \end{array}\right\}$$

$$R_{g^{-1}}: \left.\begin{array}{ccc} G & \longrightarrow & G \\ h & \longrightarrow & h\,g^{-1} \end{array}\right\} \qquad \Phi^{\mathrm{R}}_g: \left.\begin{array}{ccc} T^*_hG & \longrightarrow & T^*_{hg^{-1}}G \\ \alpha_h & \longrightarrow & T^*R_g(\alpha_h) \end{array}\right\}$$

where T^*L_g and T^*R_g mean the respective cotangent maps.

Since both are lifts of actions of G on the configuration space G, they both preserve the Liouville form and, in addition, are symplectic. Hence, for all $\xi \in \mathcal{G}$, the corresponding infinitesimal generator $\vec{X}^A_\xi$ of the action Φ^A (A=L,R) is Hamiltonian relatively to Ω, with an associated Hamiltonian function $J^A(\xi) \in \Lambda^0(T^*G)$ such that

$$\mathrm{d}J^A(\xi) = i(\vec{X}^A_\xi)\Omega = \mathrm{d}\langle \theta, \vec{X}^A_\xi \rangle - \pounds(\vec{X}^A_\xi)\theta \tag{3}$$

where $\pounds$ means Lie derivative.

The correspondence $\xi \to J^A(\xi)$ is called the *momentum mapping* for the action Φ^A (see [2], definition 4.2.1). Since θ is preserved by both actions, Φ^{L} and Φ^{R}, the last term in (3) vanishes. Hence, a momentum mapping for the action Φ^A is:

$$J^A(\xi) = \langle \theta, \vec{X}^A_\xi \rangle \,, \qquad A = \mathrm{L}, \mathrm{R}\,. \tag{4}$$

2.2. Body coordinates

The cotangent space T^*G of a Lie group is a trivial fibre bundle. Indeed, it admits a global chart:

$$\left. \begin{array}{rccc} \lambda: & T^*G & \longrightarrow & G \times \mathcal{G}^* \\ & \alpha_g & \longrightarrow & (g, T^*L_g(\alpha_g)) \end{array} \right\}$$

that is known as *body coordinates* (see [2], section 4.4) evoking an obvious similitude with rigid body kinematics.

Another global chart could also be used, corresponding to the so called *space coordinates*, that is defined by

$$\left. \begin{array}{rccc} \rho: & T^*G & \longrightarrow & G \times \mathcal{G}^* \\ & \alpha_g & \longrightarrow & (g, T^*R_g(\alpha_g)) \end{array} \right\}$$

However, owing to reasons that will later on become clear, throughout this paper we shall only use body coordinates.

Let us now write the actions Φ^A, A=L,R, in terms of body coordinates. To this end we define $\tilde{\Phi}^A$ by commutatively closing the following diagram:

$$\begin{array}{ccc} T^*G & \xrightarrow{\Phi^A} & T^*G \\ \lambda \Big\downarrow & & \Big\downarrow \lambda \\ G \times \mathcal{G}^* & \xrightarrow{\tilde{\Phi}^A} & G \times \mathcal{G}^* \end{array}$$

Thus,

$$\tilde{\Phi}^A \equiv \lambda \circ \Phi^A \circ \lambda^{-1} \,, \tag{5}$$

that, more explicitly, yields:

$$\tilde{\Phi}^{\mathrm{L}}_g(h,\mu) = (gh, \mu)\,, \qquad \forall (h,\mu) \in G \times \mathcal{G}^* \tag{6}$$

$$\tilde{\Phi}^{\mathrm{R}}_g(h,\mu) = (hg^{-1}, ad^*(g^{-1})\mu)\,, \tag{7}$$

where ad^* is the *coadjoint* representation of G. (See, for instance, [2], ex. 4.1.25.)

By means of a similar diagram we can translate the Liouville form into $\Lambda^1(G \times \mathcal{G}^*)$, so obtaining:

$$\theta^B \equiv T^*\lambda^{-1} \circ \theta \circ \lambda^{-1}\,. \tag{8}$$

In order to have an explicit expression for $\langle\theta^B, X_{(h,\mu)}\rangle$, we split $X_{(h,\mu)}\in T_{(h,\mu)}(G\times \mathcal{G}^*)$ into its two components $X_{(h,\mu)} = (v_h, \rho_\mu)$, with $v_h \in T_hG$ and $\rho_\mu \in T_\mu\mathcal{G}^* \simeq \mathcal{G}^*$. Then, combining (2) and (8), we have:

$$\langle\theta^B_{(h,\mu)}, (v_h, \rho_\mu)\rangle = \langle\mu, TL_h^{-1}(v_h)\rangle = \langle\mu, \omega_h(v_h)\rangle\,, \tag{9}$$

where $\omega \in \Lambda^1(G, \mathcal{G})$ is the Maurer-Cartan $\mathcal{G}$-valued form.

In the body coordinates, an explicit expression for $\Omega^B = -d\theta^B$ is then obtained by exterior differentiation of (9):

$$\forall X = (v_h, \rho),\ Y = (u_h, \sigma) \in T_{(h,\mu)}(G\times\mathcal{G}^*) \simeq (T_hG)\times\mathcal{G}^*$$

$$\Omega^B_{(h,\mu)}(X, Y) = \langle\sigma, \omega_h(v_h)\rangle - \langle\rho, \omega_h(u_h)\rangle + \langle\mu, [\omega_h(v_h), \omega_h(u_h)]\rangle \tag{10}$$

where the Maurer-Cartan structure equation

$$\mathrm{d}\omega = -\frac{1}{2}[\omega, \omega] \tag{11}$$

has been used.

Finally, in body coordinates, the momentum mappings $\tilde{J}^A$, $A = L, R$ are obtained from (4):

$$\forall\xi \in \mathcal{G}, \qquad \tilde{J}^A(\xi) \equiv J^A(\xi)\circ\lambda^{-1} \tag{12}$$

and their explicit expressions, respectively, are

$$\tilde{J}^{\mathrm{L}}(h, \mu) \;=\; ad^*(h^{-1})\mu\,, \tag{13}$$

$$\tilde{J}^{\mathrm{R}}(h, \mu) \;=\; \mu\,. \tag{14}$$

3. The reduced phase space

As we have already pointed out in section 1, left invariance and right invariance for the systems under consideration have neatly different meanings. Whereas left invariance has a physical interpretation in terms of relativistic invariance, right invariance has been introduced to enlarge the phase space and so to avoid the need of constraints.

We shall now consider a Φ^R invariant Hamiltonian system on $G\times\mathcal{G}^*$ and eliminate this 'unphysical' symmetry by reducing the phase space. The techniques are those developed in [2] (section 4.3) and the method basically consists in the use of the integrals of motion associated to the latter symmetry (that are in involution with respect to Poisson brackets) to eliminate some degrees of freedom.

Now, theorem 4.3.1 in [2] can be applied. Indeed, we have

> a symplectic manifold $(G\times\mathcal{G}^*, \Omega^B)$, on which the Lie group G acts symplectically — namely, $\tilde{\Phi}^R$ — and an ad^*-equivariant momentum mapping $\tilde{J}^R$. Furthermore, as it can be easily checked:
>
> 1) any $\mu \in \mathcal{G}^*$ is a *regular value* of $\tilde{J}^R$ (i.e., $T_{(h.\mu)}\tilde{J}^R$ is surjective),

2) the momentum mapping $\tilde{J}^R$ defined by (14) is ad^*-equivariant under Φ^R, that is, $\forall g \in G$,

$$\tilde{J}^{\mathrm{R}} \circ \tilde{\Phi}^{\mathrm{R}}_g = ad^*(g^{-1}) \circ J^{\mathrm{R}}$$

(see definition 4.2.6 in [2]).

3) $\tilde{J}^{R\,-1}(\mu) = \{(h,\mu);\, h \in G\}$ is diffeomorphic with G, and the isotropy group of μ, namely $G_\mu = \{g \in G;\, ad^*(g)\mu = \mu\}$, acts properly and freely on $\tilde{J}^{R\,-1}(\mu)$.

Hence, $P_\mu \equiv G/G_\mu$ has a unique symplectic form, $\Omega^{(\mu)}$, such that:

$$T^*\Pi_\mu(\Omega^{(\mu)}) = T^*i_\mu(\Omega^B) \tag{15}$$

where Π_μ and i_μ are, respectively, the *canonical projection* and the *inclusion*:

$$\left.\begin{array}{rccc} \Pi_\mu: & G & \longrightarrow & P_\mu \\ & h & \longrightarrow & hG_\mu \end{array}\right\} \qquad\qquad \left.\begin{array}{rccc} i_\mu: & G & \longrightarrow & G\times\mathcal{G}^* \\ & h & \longrightarrow & (h,\mu) \end{array}\right\}$$

Therefore, given any $\mu \in \mathcal{G}^*$ we have a *reduced phase space*, i.e. a symplectic manifold $(P_\mu, \Omega^{(\mu)})$.

3.1. A realization of P_μ

This reduced phase space P_μ can actually be realized as a submanifold of $\mathcal{G}^*$ (this is a consequence of the *Kirillov-Kostant-Souriau theorem*, see [2], example 4.3.4(v)). Indeed, let us consider the coadjoint representation of G, and define:

$$\left.\begin{array}{rccc} \Psi: & G\times\mathcal{G}^* & \longrightarrow & \mathcal{G}^* \\ & (g,\mu) & \longrightarrow & ad^*(g^{-1})\mu \equiv \psi_\mu(g) \equiv \psi_g(\mu) \end{array}\right\} \tag{16}$$

For any given $\mu \in \mathcal{G}^*$, we shall have an orbit: $\psi_\mu : G \longrightarrow \mathcal{G}^*$ and, for any given $\rho \in \psi_\mu(G) \subset \mathcal{G}^*$, $\psi_\mu^{-1}(\rho)$ is a coset in $P_\mu = G/G_\mu$. Therefore, P_μ is diffeomorphic with $\Gamma_\mu \equiv \psi_\mu(G)$, the orbit of $\mu \in \mathcal{G}^*$ by the coadjoint representation of G. The diffeomorphism is

$$\left.\begin{array}{rccc} \hat{\psi}_\mu: & P_\mu & \longrightarrow & \Gamma_\mu \\ & hG_\mu & \longrightarrow & ad^*(h^{-1})\mu \end{array}\right\} \tag{17}$$

Thus $\Gamma_\mu \subset \mathcal{G}^*$ is a realization of P_μ.

The symplectic form on Γ_μ is obtained from $\Omega^{(\mu)}$ by inverse pullback:

$$\hat{\Omega}^{(\mu)} \equiv T^*\hat{\psi}_\mu^{-1}(\Omega^{(\mu)}) \tag{18}$$

As a consequence of (15) and the fact that $\psi_\mu = \hat{\psi}_\mu \circ \Pi_\mu$, we have that $\hat{\Omega}^{(\mu)}$ is the only symplectic form on Γ_μ such that:

$$T^*\psi_\mu(\hat{\Omega}^{(\mu)}_\rho) = T^*i_\mu(\Omega^B_{(h,\mu)}) \tag{19}$$

where $\rho = ad^*(h^{-1})\mu$.

3.2. The left action of G on Γ_μ

Since the following diagram

$$\begin{array}{ccccc} G\times\mathcal{G}^* & \xleftarrow{i_\mu} & G & \xrightarrow{\psi_\mu} & \Gamma_\mu \\ \tilde{\Phi}^L_g \Big\downarrow & & L_g \Big\downarrow & & \psi_g \Big\downarrow \\ G\times\mathcal{G}^* & \xleftarrow{i_\mu} & G & \xrightarrow{\psi_\mu} & \Gamma_\mu \end{array} \tag{20}$$

with ψ_g and ψ_μ defined in (16), is commutative, we have that the left action $\tilde{\Phi}^L$ of G on $G\times\mathcal{G}^*$ is "projected" onto the coadjoint action Ψ of G on Γ_μ:

$$\psi_g:\ ad^*(h^{-1})\mu \longrightarrow ad^*(g^{-1})ad^*(h^{-1})\mu = ad^*((gh)^{-1})\mu$$

By its very construction, Ψ acts transitively on Γ_μ. Hence, at any $\rho\in\Gamma_\mu$, the set of its infinitesimal generators spans the whole tangent space $T_\rho(\Gamma_\mu)$, that is:

$$T_\rho(\Gamma_\mu) = \{V_\rho(\xi),\ \xi\in\mathcal{G}\} \tag{21}$$

Since $TR_h(\xi)\in T_hG$ is an infinitesimal generator for the left action L, and the diagram (20) is commutative, we have that:

$$V_\rho(\xi) = T\psi_\mu\circ TR_h(\xi) \tag{22}$$

for any $h\in G$ and $\mu\in\mathcal{G}^*$ such that $\rho = ad^*(h^{-1})\mu$.

In particular, taking $h=e$ and $\mu=\rho$, we obtain:

$$V_\rho(\xi) = T\psi_\rho(\xi) = -Ad^*(\xi)\rho\in T_\rho(\Gamma_\mu) \tag{23}$$

where Ad^* is the coadjoint representation of $\mathcal{G}$ on $\mathcal{G}^*$.

From (19) and (23), after a little manipulation, we arrive at:

$$\hat{\Omega}^{(\mu)}_\rho\left(V_\rho(\xi),V_\rho(\eta)\right) = \langle\rho,[\xi,\eta]\rangle\ . \tag{24}$$

Now, in order to prove that Ψ is a symplectic action, we can either use the commutativity of diagram (20), together with the definition (19) and the uniqueness of $\hat{\Omega}^{(\mu)}$, or work directly (22) and (24). This second way goes as follows. Let $\rho' = \psi_{g^{-1}}\rho = ad^*(g)\rho\in\Gamma_\mu$, and let us evaluate, $\forall\xi,\eta\in\mathcal{G}$,

$$T^*\psi_g\left(\hat{\Omega}^{(\mu)}_\rho\right)\left(V_{\rho'}(\xi),V_{\rho'}(\eta)\right) = \hat{\Omega}^{(\mu)}_\rho\left(T\psi_g(V_{\rho'}(\xi)),T\psi_g(V_{\rho'}(\eta))\right)\ . \tag{25}$$

It is easy to show that

$$T\psi_g(V_{\rho'}(\xi)) = -ad^*(g^{-1})\left(Ad^*(\xi)\rho'\right) = -Ad^*\left(ad(g)\xi\right)\rho = V_\rho(ad(g)\xi) \tag{26}$$

substituted in the right hand side of (25), and taking (24) into account, yields:

$$\begin{aligned} T^*\psi_g\left(\hat{\Omega}^{(\mu)}_\rho\right)\left(V_{\rho'}(\xi),V_{\rho'}(\eta)\right) &= \langle\rho,[ad(g)\xi,ad(g)\eta]\rangle = \langle\rho',[\xi,\eta]\rangle \\ &= \hat{\Omega}^{(\mu)}_\rho\left(V_{\rho'}(\xi),V_{\rho'}(\eta)\right) \end{aligned}$$

That means

$$T^*\psi_g\left(\hat{\Omega}^{(\mu)}\right) = \hat{\Omega}^{(\mu)} \ . \tag{27}$$

Hence ψ_g is symplectic $\forall g \in G$. An associated momentum mapping can be derived by means of (23), (24) and definition 4.2.1 in [2]:

$$J(\rho) = \rho\,, \qquad\qquad \rho \in \Gamma_\mu \ . \tag{28}$$

3.3. The implicit equations for Γ_μ

Γ_μ is characterized by some functions f on $\mathcal{G}^*$ defining constraints that look like

$$f(\rho) = \text{constant on } \Gamma_\mu \ . \tag{29}$$

Now, since the infinitesimal generators (23) span the whole tangent space $T_\rho(\Gamma_\mu)$, the constraints (29) must satisfy:

$$\langle df, V_\rho(\xi)\rangle = 0\,, \qquad\qquad \rho \in \Gamma_\mu\,, \xi \in \mathcal{G} \tag{30}$$

which implies that

$$\langle Ad^*(\xi)\rho, (df)_\rho\rangle = \langle \rho, [\xi, (df)_\rho]\rangle = 0 \ . \tag{31}$$

(Note that $(df)_\rho \in T^*_\rho(\mathcal{G}^*) \simeq \mathcal{G}^{**} \simeq \mathcal{G}$).

Writing the latter condition in terms of one basis of $\mathcal{G}$, (i.e., in a given parametrization of G) it reads:

$$\rho_i C^i_{jk}\frac{\partial f}{\partial \rho_j} = 0\,, \tag{32}$$

where C^i_{jk} are the structure constants of G in that parametrization. The solutions of equation (32) are related to the Casimir invariants of the group [6], [5].

4. The Poincaré group

To apply the above results to Poincaré group, it will be helpful to begin with a previous recall of some general results and notations.

$h = (x^\mu, \Lambda^\mu{}_\nu)$, $g = (y^\mu, L^\mu{}_\nu)$ denote any couple of elements in G, the Poincaré group. The group law is

$$hg = (x^\mu + \Lambda^\mu{}_\nu y^\nu, \Lambda^\mu{}_\nu L^\nu{}_\rho) \tag{33}$$

A general element in the Lie algebra $\mathcal{G}$ is denoted by

$$\xi = (b^\mu, V^\mu{}_\nu)\,, \qquad V_{\mu\nu} + V_{\nu\mu} = 0\,,$$

and a general element in the dual space $\mathcal{G}^*$ is written as

$$\mu = (a_\alpha, W^\alpha{}_\beta)\,, \qquad W_{\alpha\beta} + W_{\beta\alpha} = 0\,.$$

Indices are raised and lowered with Minkowski metrics $\eta = (-+++)$.

The commutation relations in $\mathcal{G}$ are those corresponding to the semidirect sum

$$[(b^\mu, V^\mu{}_\nu), (c^\mu, T^\mu{}_\nu)] = (V^\mu{}_\nu c^\nu - T^\mu{}_\nu b^\nu, V^\mu{}_\rho T^\rho{}_\nu - T^\mu{}_\rho V^\rho{}_\nu) \tag{34}$$

and the dual product is:

$$\langle \mu, \xi \rangle = \langle (a_\alpha, W^\alpha{}_\beta), (b^\mu, V^\mu{}_\nu) \rangle \equiv a_\alpha b^\alpha + \frac{1}{2} W^\mu{}_\nu V^\nu{}_\mu \ . \tag{35}$$

Other useful results are listed below: [1]

1. The coadjoint action:

$$\mu = (a_\alpha, W^\alpha{}_\beta) \longrightarrow ad^*(g^{-1})\mu = (a'_\alpha, W'^\alpha{}_\beta) \qquad \text{with}$$

$$a'_\alpha = \Lambda_\alpha{}^\beta a_\beta \qquad \text{and} \qquad W'_{\alpha\beta} = y_\alpha a'_\beta - y_\beta a'_\alpha + L_\alpha{}^\rho L_\beta{}^\sigma W_{\rho\sigma} \ . \tag{36}$$

2. The Liouville form:

$$\theta^B_{(h,\mu)} = a_\rho \Lambda_\sigma{}^\rho dx^\sigma + \frac{1}{2} W_{\rho\nu} \Lambda_\sigma{}^\nu d\Lambda^{\sigma\rho} \ . \tag{37}$$

3. The momentum mapping for the left action of G:

$$\tilde{J}^{\mathrm{L}}(h, \mu) = ad^*(h^{-1})\mu = (p_\rho, J_{\rho\sigma})$$

whose components correspond to

the linear momentum $\qquad p_\rho = a_\sigma \Lambda_\rho{}^\sigma \qquad (38)$

and the angular momentum $\qquad J_{\rho\sigma} = x_\rho p_\sigma - x_\sigma p_\rho + W_{\alpha\beta} \Lambda_\rho{}^\alpha \Lambda_\sigma{}^\beta \ . \qquad (39)$

4.1. The reduced phase space: explicit realizations

Poincaré group has two Casimir functions, namely, the square of mass and the square of Pauli-Lubanski 4-vector. According to subsection 3.3, given any couple of possible values (c_1, c_2), we shall have a submanifold $\Gamma_{(c_1,c_2)}$ and each connected component of it will be a realization of a Γ_μ, $\mu \in \mathcal{G}^*$. The implicit equations are:

$$c_1 = a_\mu a_\nu \eta^{\mu\nu} \tag{40}$$
$$c_2 = \eta_{\mu\nu\alpha\beta} a^\nu W^{\alpha\beta} \eta^{\mu\rho\tau\sigma} a_\rho W_{\tau\sigma} \tag{41}$$

Now, the point is to find the simplest representative $(a_\alpha, W_{\mu\nu})$ for a given couple (c_1, c_2). It is important to realize that c_1 and c_2 are the Minkowski squares of two mutually orthogonal 4-vectors, namely, a_μ and $\eta_{\mu\nu\alpha\beta} a^\nu W^{\alpha\beta}$. Hence, if one of them is either timelike or lightlike, the other is necessarily spacelike. That is,

$$c_1 < 0 \Rightarrow c_2 > 0 \qquad \text{and} \qquad c_2 < 0 \Rightarrow c_1 > 0 \, .$$

To find a simple representative $(a'_\rho, W'_{\alpha\beta})$ for $\Gamma_{(a,W)}$, we shall use that the skewsymmetric matrix $W_{\alpha\beta}$ can always be written as:

$$W_{\alpha\beta} = E_\alpha a_\beta - E_\beta a_\alpha + \frac{1}{2}\eta_{\alpha\beta\mu\nu} b^\mu \Sigma^\nu \tag{42}$$

where the formulae relating E_β, b^μ and Σ^ν to $W_{\alpha\beta}$ and a_β depend on the class of the latter relatively to the Minkowski metric.

If $a^\rho a_\rho \neq 0$, it follows obviously that:

$$E_\alpha = \frac{1}{a_\rho a^\rho} W_{\alpha\nu} a^\nu\,, \qquad \Sigma_\alpha = \frac{1}{a_\rho a^\rho}\eta_{\alpha\beta\mu\nu} a^\beta W^{\mu\nu}\,, \qquad \text{and} \quad b^\mu = a^\mu\,. \tag{43}$$

In the case $a^\rho a_\rho = 0$, a lightlike vector b^μ can be chosen such that $b^\mu a_\mu = 1$ and the vectors E_α and Σ_β are given by:

$$E_\alpha = W_{\alpha\nu} b^\nu\,, \qquad \Sigma_\alpha = \eta_{\alpha\beta\mu\nu} a^\beta W^{\mu\nu}\,. \tag{44}$$

In this second case the decomposition(42) is not unique.

It immediately follows from (42) and (36) that:

$$\Gamma_{(a,W)} = \Gamma_{(a,W')} \qquad \text{with} \qquad W'_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu} b^\mu \Sigma^\nu\,. \tag{45}$$

Hence, the 4-vectors a_μ and Σ_ν determine $\Gamma_{(a,W)}$.

According to the discussion above, these two 4-vectors are orthogonal to each other and are obviously connected with the two Casimir functions of Poincaré group. Hence, only the cases listed below are possible:

$c_1 \neq 0$, $\quad c_2 = c_1^2\,\Sigma_\mu \Sigma^\mu$				$c_1 = 0$, $\quad c_2 = \Sigma_\mu \Sigma^\mu$	
I	**II**	**III**	**IV**	**V**	**VI**
$c_1 < 0,\, c_2 > 0$	$c_1 > 0,\, c_2 > 0$	$c_1 > 0,\, c_2 < 0$	$c_1 > 0,\, c_2 = 0$	$c_1 = 0,\, c_2 = 0$	$c_1 = 0,\, c_2 > 0$

The different cases are analysed with detail in the appendices. As a summary, the several reduced phase spaces $\Gamma_{(a,W)}$ are realized as the submanifolds of $T^*(\mathbf{R}^8)$ defined by the constraints:

$$\left.\begin{array}{llll} \phi_1 \equiv p^\rho p_\rho - c_1 = 0, & & \phi_2 \equiv \pi^\rho \pi_\rho + K = 0, & \\ \psi_1 \equiv r^\rho r_\rho - L = 0, & \psi_2 \equiv r^\rho \pi_\rho = 0 & \psi_3 \equiv r^\rho p_\rho, & \psi_4 \equiv p^\rho \pi_\rho = 0 \end{array}\right\} \tag{46}$$

where:

$K = c_2/4c_1$ in cases **I, II, III**, and **IV**, $K = -1/4$ in the case **V** and $K = 0$ in the case **VI**, and

$L = 1$ in the cases **I, II, III**, and **V**, $L = c_1$ in the case **IV**, and $L = c_2$ in the case **VI**.

The Poisson bracket structure on each of these submanifolds is obtained as the Dirac bracket structure obtained from the canonical elementary Poisson brackets on $T^*(\mathbf{R}^8)$:

$$\{x^\nu, p_\mu\} = \delta^\nu_\mu\,, \qquad \{r^\nu, \pi_\mu\} = \delta^\nu_\mu \tag{47}$$

together with the constraints listed above. Among them, ψ_a, $a = 1,\ldots,4$, are second class constraints, whereas ϕ_1 and $\phi_2 + \dfrac{c_2}{2c_1}\psi_1$ are first class.

In terms of these variables, the generating functions of infinitesimal Poincaré transformations are:

$$\text{linear momentum} \qquad p_\mu \tag{48}$$

$$\text{angular momentum} \qquad J_{\rho\sigma} = x_\rho p_\sigma - x_\sigma p_\rho + r_\rho \pi_\sigma - r_\sigma \pi_\rho\ . \tag{49}$$

Appendix A: Cases I, II and III

The vectors a_ρ and Σ_μ can be complemented with a unitary and orthogonal vector R_ν, and with

$$L_\beta = \eta_{\beta\mu\nu\alpha} a^\mu R^\nu \Sigma^\alpha\,, \qquad\qquad L^\beta L_\beta = -a^\rho a_\rho \Sigma_\nu \Sigma^\nu = -\frac{c_2}{c_1}\ . \tag{50}$$

As a result, $\{a_\rho, \Sigma_\mu, R_\nu, L_\alpha\}$ is an orthogonal tetrad in terms of which the Lorentz matrix $\Lambda_\rho{}^\sigma$ can be written as

$$\Lambda_\rho{}^\sigma = \frac{l_\rho L^\sigma}{L^\alpha L_\alpha} + \frac{p_\rho a^\sigma}{p^\alpha p_\alpha} + \frac{s_\rho \Sigma^\sigma}{s^\alpha s_\alpha} + r_\rho R^\sigma \tag{51}$$

where: $l_\rho = \Lambda_\rho{}^\sigma L_\sigma$, $p_\rho = \Lambda_\rho{}^\sigma a_\sigma$, $s_\rho = \Lambda_\rho{}^\sigma \Sigma_\sigma$ and $r_\rho = \Lambda_\rho{}^\sigma R_\sigma$ is also an orthogonal tetrad.

Using (50) and (51), the Liouville form (37) becomes

$$\theta^B = p_\rho \mathrm{d}x^\rho + \frac{1}{2} l_\rho \mathrm{d}r^\rho\ . \tag{52}$$

Notice that the 16 variables $x^\sigma, r^\rho, l_\alpha, p_\mu$ can be used to coordinate (redundantly) the reduced phase space $\Gamma_{(a,W)}$ in case that $c_1 \neq 0$ and $c_2 \neq 0$. Indeed, given three non-null orthogonal vectors r^ρ, l_α, p_μ, the fourth vector in the tetrad is obtained from $l_\beta = \eta_{\beta\mu\nu\alpha} p^\mu r^\nu s^\alpha$. The point in $\Gamma_{(a,W)}$ corresponding to $ad^*(h^{-1})(a,W)$ is coordinated (also redundantly) by $h = (x^\nu, \Lambda_\rho{}^\sigma)$ with the Lorentz matrix given by (51).

Those 16 variables are, however, redundant in $\Gamma_{(a,W)}$ and are constrained by the following relationships:

$$\left.\begin{array}{llll} \phi_1 \equiv p_\rho p^\rho - c_1 = 0, & c_1 \neq 0, & 4\,\phi_2 \equiv l_\rho l^\rho + c_2/c_1 = 0, & c_2 \neq 0 \\ \psi_1 \equiv r_\rho r^\rho - 1 = 0, & \psi_2 \equiv r^\rho l_\rho = 0, & \psi_3 \equiv r^\rho p^\rho = 0, & \psi_4 \equiv l_\rho p^\rho = 0 \end{array}\right\} \tag{53}$$

In terms of these variables, the linear and angular momenta, (38) and (39), respectively, are

$$p_\mu \quad \text{and} \quad J_{\rho\sigma} = x_\rho p_\sigma - x_\sigma p_\rho + \frac{1}{2}(r_\rho l_\sigma - r_\sigma l_\rho) \ . \tag{54}$$

Now writing $\pi_\rho = \frac{1}{2} l_\rho$ in equations (54) and (52), the expressions (49), the elementary Poisson brackets (47) and the constraints (46) follow immediately.

Appendix B: Case IV

In this case the tetrad $a_\rho, R_\nu, \Sigma_\mu, L_\beta$ is completed so that

$$L^\mu L_\mu = L^\mu a_\mu = L^\mu R_\mu = R^\mu a_\mu = R^\mu \Sigma_\mu = 0\,, \qquad R^\mu R_\mu = c_1\,, \qquad L^\mu \Sigma_\mu = 1 \tag{55}$$

and $\eta_{\beta\mu\nu\alpha} a^\beta \Sigma^\mu L^\nu R^\alpha = -c_1$.

In terms of this tetrad the Lorentz matrix $\Lambda_\rho{}^\sigma$ can be written as

$$\Lambda_\rho{}^\sigma = l_\rho \Sigma^\sigma + s_\rho L^\sigma + \frac{p_\rho a^\sigma + r_\rho R^\sigma}{p^\alpha p_\alpha} \tag{56}$$

with

$$l_\rho = \Lambda_\rho{}^\sigma L_\sigma, p_\rho = \Lambda_\rho{}^\sigma a_\sigma, s_\rho = \Lambda_\rho{}^\sigma \Sigma_\sigma \qquad \text{and} \qquad r_\rho = \Lambda_\rho{}^\sigma R_\sigma .$$

Similarly as in appendix A the Liouville form is:

$$\theta^B = p_\rho \mathrm{d}x^\rho + \frac{1}{2} s_\rho \mathrm{d}r^\rho\,, \tag{57}$$

and the linear and angular momenta are:

$$p_\mu \quad \text{and} \quad J_{\rho\sigma} = x_\rho p_\sigma - x_\sigma p_\rho + \frac{1}{2}(r_\rho s_\sigma - r_\sigma s_\rho) \ . \tag{58}$$

The same comments as in appendix A do hold and the constraints are:

$$\left.\begin{array}{llll} \phi_1 \equiv p_\rho p^\rho - c_1 = 0, & c_1 \neq 0, & 4\,\phi_2 \equiv s_\rho s^\rho = 0, & c_2 = 0 \\ \psi_1 \equiv r_\rho r^\rho - c_1 = 0, & \psi_2 \equiv r^\rho s_\rho = 0, & \psi_3 \equiv r^\rho p^\rho = 0, & \psi_4 \equiv s_\rho p^\rho = 0 \end{array}\right\} \tag{59}$$

Finally, writing $\pi_\rho = \frac{1}{2} s_\rho$ in equations (58) and (57), the expressions (49), the elementary Poisson brackets (47), and the constraints (46) follow immediately.

Appendix C: Case V

Since a_α and $\Sigma_\alpha \equiv \eta_{\alpha\beta\mu\nu} a^\beta W^{\mu\nu}$ are orthogonal null vectors, they are linearly dependent. By a Lorentz transformation, a representative of $\Gamma_{(a,W)}$ can be chosen such that $a_\mu = \Sigma_\mu$.

Consider now a tetrad $a_\rho, L_\beta, R_\nu, S_\mu$ such that

$$L^\mu L_\mu = L^\mu R_\mu = L^\mu S_\mu = R^\mu a_\mu = R^\mu S_\mu = 0\,, \qquad R^\mu R_\mu = S^\mu S_\mu = 1\,, \qquad L^\mu a_\mu = 1 \tag{60}$$

and $\eta_{\beta\mu\nu\alpha}a^\beta L^\mu S^\nu R^\alpha = 1$.

In terms of which the Lorentz matrix $\Lambda_\rho^{\ \ \sigma}$ can be written as

$$\Lambda_\rho^{\ \ \sigma} = l_\rho a^\sigma + p_\rho L^\sigma + r_\rho R^\sigma + s_\rho S^\sigma \tag{61}$$

with:

$$l_\rho = \Lambda_\rho^{\ \ \sigma} L_\sigma, p_\rho = \Lambda_\rho^{\ \ \sigma} a_\sigma, s_\rho = \Lambda_\rho^{\ \ \sigma} S_\sigma \qquad \text{and} \qquad r_\rho = \Lambda_\rho^{\ \ \sigma} R_\sigma.$$

The Liouville form can be obtained as in appendix A and it yields:

$$\theta^B = p_\rho \mathrm{d}x^\rho + \frac{1}{2} s_\rho \mathrm{d}r^\rho\,, \tag{62}$$

and the linear and angular momenta are:

$$p_\mu \quad \text{and} \quad J_{\rho\sigma} = x_\rho p_\sigma - x_\sigma p_\rho + \frac{1}{2}(r_\rho s_\sigma - r_\sigma s_\rho)\ . \tag{63}$$

The same comments as in appendix A also hold and the constraints are:

$$\left.\begin{array}{llll} \phi_1 \equiv p_\rho p^\rho - c_1 = 0, & c_1 = 0, & 4\,\phi_2 \equiv s_\rho s^\rho - 1 = 0, & c_2 = 0 \\ \psi_1 \equiv r_\rho r^\rho - 1 = 0, & \psi_2 \equiv r^\rho s_\rho = 0, & \psi_3 \equiv s^\rho p^\rho = 0, & \psi_4 \equiv r_\rho p^\rho = 0 \end{array}\right\} \tag{64}$$

Finally, writing $\pi_\rho = \frac{1}{2}s_\rho$ in equations (63) and (62), the expressions (49), the elementary Poisson brackets (47), and the constraints (46) follow immediately.

Appendix D: Case VI

Complete a_ρ and Σ_ν to a tetrad $a_\rho, L_\beta, R_\nu, \Sigma_\mu$ such that

$$L^\mu L_\mu = L^\mu R_\mu = L^\mu \Sigma_\mu = R^\mu a_\mu = R^\mu \Sigma_\mu = 0\,, \qquad R^\mu R_\mu = c_2\,, \qquad L^\mu a_\mu = 1 \tag{65}$$

and $\eta_{\beta\mu\nu\alpha}a^\beta L^\mu R^\nu \Sigma^\alpha = c_2$.

In terms of which the Lorentz matrix $\Lambda_\rho^{\ \ \sigma}$ can be written as

$$\Lambda_\rho^\sigma = l_\rho a^\sigma + p_\rho L^\sigma + \frac{r_\rho R^\sigma + s_\rho \Sigma^\sigma}{c_2} \tag{66}$$

with

$$l_\rho = \Lambda_\rho^{\ \ \sigma} L_\sigma, p_\rho = \Lambda_\rho^{\ \ \sigma} a_\sigma, s_\rho = \Lambda_\rho^{\ \ \sigma} \Sigma_\sigma \quad \text{and} \quad r_\rho = \Lambda_\rho^{\ \ \sigma} R_\sigma.$$

The Liouville form can be obtained as in appendix A and it yields:

$$\theta^B = p_\rho \mathrm{d}x^\rho + \frac{1}{2} l_\rho \mathrm{d}r^\rho\,, \tag{67}$$

and the linear and angular momenta are:

$$p_\mu \quad \text{and} \quad J_{\rho\sigma} = x_\rho p_\sigma - x_\sigma p_\rho + \frac{1}{2}(r_\rho l_\sigma - r_\sigma l_\rho) \ . \tag{68}$$

The same comments as in appendix A also hold and the constraints are:

$$\left.\begin{array}{llll} \phi_1 \equiv p_\rho p^\rho - c_1 = 0, & c_1 = 0, & 4\,\phi_2 \equiv s_\rho s^\rho = 0, & c_2 \neq 0 \\ \psi_1 \equiv r_\rho r^\rho - c_2 = 0, & \psi_2 \equiv r^\rho s_\rho = 0, & \psi_3 \equiv s^\rho p^\rho = 0, & \psi_4 \equiv r_\rho p^\rho = 0 \end{array}\right\} \tag{69}$$

Finally, writing $\pi_\rho = \frac{1}{2} l_\rho$ in equations (67) and (68), the expressions (49), the elementary Poisson brackets (47), and the constraints (46) follow immediately.

References

1. Blázquez, C. and Llosa, J., *Rep. Math. Phys.*, **30**, 2, pp. 131-48 (1991)
2. Abraham, R. and Marsden, J. E., *Foundations of Mechanics* (second edition), Benjamin (Reading, 1978)
3. Atre, M. V. and Mukunda, N., *J. Math. Phys.*, **27**, 12, pp. 2908-29 (1986)
4. Dirac, P.A.M., *Lectures on Quantum Mechanics*, Yeshiva University, Belfer Graduate School of Science, (New York, 1964)
5. Beltrameti, E. G. and Blasi, A., *Phys. Lett.*, **20**, 1, pp. 62-4 (1966)
6. Abellanas, L. and Martínez Alonso, L., *J. Math. Phys.*, **16**, 8, pp. 1580-4 (1974)

Класичні релятивістичні частинки зі спіном

Й.Льоза

Барселонський університет, кафедра теоретичної фізики,
Барселона, Іспанія

Отримано 30 січня 1998 р.; після доопрацювання – 6 травня 1998 р.

Релятивістичні частинки зі спіном розглядаються з точки зору гамільтонової дії групи Пуанкаре на себе. Прокласифіковано низку можливих розв'язків та побудовано реалізації у термінах Діракового формалізму з в'язями.

Ключові слова: *релятивістична частинка зі спіном, група Пуанкаре, гамільтонова дія*

PACS: *03.20, 03.30*